## Contents

1 Functions on Euclidean Space .................................................. 1  
  1.1 Norm and Inner Product .................................................. 1  
  1.2 Subsets of Euclidean Space ............................................. 6  
  1.3 Functions and Continuity ............................................... 9  

2 Differentiation ................................................................. 13  
  2.1 Basic Definitions ...................................................... 13  
  2.2 Basic Theorems ............................................................ 18  
  2.3 Partial Derivatives ....................................................... 26  
  2.4 Derivatives ................................................................. 34  
  2.5 Inverse Functions ....................................................... 38  
  2.6 Implicit Functions ....................................................... 40  

3 Integration ............................................................... 45  
  3.1 Basic Definitions ...................................................... 45  
  3.2 Measure Zero and Content Zero ....................................... 51  
  3.3 Fubini's Theorem ......................................................... 51  

4 Integration on Chains ........................................................... 55  
  4.1 Algebraic Preliminaries .................................................. 55  

References ................................................................. 57  

Index ................................................................. 59
1

FUNCTIONS ON EUCLIDEAN SPACE

1.1 Norm and Inner Product

Exercise 1 (1-1). Prove that \( \|x\| \leq \sum_{i=1}^{n} |x^i| \).

Proof. Let \( x = (x^1, \ldots, x^n) \). Then
\[
\left( \sum_{i=1}^{n} |x^i| \right)^2 = \sum_{i=1}^{n} (x^i)^2 + \sum_{i \neq j} |x^i x^j| \geq \sum_{i=1}^{n} (x^i)^2 = \|x\|^2.
\]
Taking the square root of both sides gives the result. \( \square \)

Exercise 2 (1-2). When does equality hold in Theorem 1-1 (3) \( \|x + y\| \leq \|x\| + \|y\| \)?

Proof. We reprove that \( |\langle x, y \rangle| \leq \|x\| \cdot \|y\| \) for every \( x, y \in \mathbb{R}^n \). Obviously, if \( x = 0 \) or \( y = 0 \), then \( \langle x, y \rangle = \|x\| \cdot \|y\| = 0 \). So we assume that \( x \neq 0 \) and \( y \neq 0 \).

We first find some \( w \in \mathbb{R}^n \) and \( \alpha \in \mathbb{R} \) such that \( \langle w, \alpha y \rangle = 0 \). Write \( w = x - \alpha y \). Then
\[
0 = \langle w, \alpha y \rangle = \langle x - \alpha y, \alpha y \rangle = \alpha \langle x, y \rangle - \alpha^2 \|y\|^2
\]
implies that
\[
\alpha = \langle x, y \rangle / \|y\|^2.
\]
Then
\[
\|x\|^2 = \|w\|^2 + \|\alpha y\|^2 \geq \|\alpha y\|^2 = \left( \frac{\langle x, y \rangle}{\|y\|} \right)^2.
\]
Hence, \( |\langle x, y \rangle| \leq \|x\| \cdot \|y\| \). Particularly, the above display holds with equality if and only if \( \|w\| = 0 \), if and only if \( w = 0 \), if and only if \( x - \alpha y = 0 \), if and only if \( x = \alpha y \).

Since
\[
\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + \|y\|^2 + 2 \langle x, y \rangle \leq \|x\|^2 + \|y\|^2 + 2 \|x\| \cdot \|y\|
\]
\[
= (\|x\| + \|y\|)^2,
\]

equality holds precisely when \( \langle x, y \rangle = \|x\| \cdot \|y\| \), i.e., when one is a nonnegative multiple of the other.

**Exercise 3 (1-3).** Prove that \( \|x - y\| \leq \|x\| + \|y\| \). When does equality hold?

**Proof.** By Theorem 1-1 (3) we have \( \|x - y\| = \|x + (-y)\| \leq \|x\| + \|-y\| = \|x\| + \|y\| \). The equality holds precisely when one vector is a non-positive multiple of the other.

**Exercise 4 (1-4).** Prove that \( \|x - y\| \leq \|x\| - \|y\| \).

**Proof.** We have \( \|x - y\|^2 = \sum_{i=1}^{n} (x_i - y_i)^2 = \|x\|^2 + \|y\|^2 - 2 \sum_{i=1}^{n} x_i y_i \geq \|x\|^2 + \|y\|^2 - 2 \|x\| \|y\| = (\|x\| - \|y\|)^2 \). Taking the square root of both sides gives the result.

**Exercise 5 (1-5).** The quantity \( \|y - x\| \) is called the distance between \( x \) and \( y \). Prove and interpret geometrically the “triangle inequality”: \( \|z - x\| \leq \|z - y\| + \|y - x\| \).

**Proof.** The inequality follows from Theorem 1-1 (3):

\[
\|z - x\| = \|(z - y) + (y - x)\| \leq \|z - y\| + \|y - x\|.
\]

Geometrically, if \( x, y, \) and \( z \) are the vertices of a triangle, then the inequality says that the length of a side is no larger than the sum of the lengths of the other two sides.

**Exercise 6 (1-6).** If \( f \) and \( g \) be integrable on \([a, b]\).

a. Prove that \( \left| \int_{a}^{b} f \cdot g \right| \leq \left( \int_{a}^{b} f^2 \right)^{\frac{1}{2}} \cdot \left( \int_{a}^{b} g^2 \right)^{\frac{1}{2}} \).

b. If equality holds, must \( f = \lambda g \) for some \( \lambda \in \mathbb{R} \)? What if \( f \) and \( g \) are continuous?

c. Show that Theorem 1-1 (2) is a special case of (a).

**Proof.**

a. Theorem 1-1 (2) implies the inequality of Riemann sums:

\[
\left| \sum_{i} f(x_i) g(x_i) \Delta x_i \right| \leq \left( \sum_{i} f(x_i)^2 \Delta x_i \right)^{\frac{1}{2}} \cdot \left( \sum_{i} g(x_i)^2 \Delta x_i \right)^{\frac{1}{2}}.
\]

Taking the limit as the mesh approaches 0, one gets the desired inequality.

b. No. We could, for example, vary \( f \) at discrete points without changing the values of the integrals. If \( f \) and \( g \) are continuous, then the assertion is true. In fact, suppose that for each \( \lambda \in \mathbb{R} \), there is an \( x \in [a, b] \) with
\[ f(x) - \lambda g(x) \leq 0. \] Then the inequality holds true in an open neighborhood of \( x \) since \( f \) and \( g \) are continuous. So \( f(b) - \lambda g(x) > 0 \) since the integrand is always non-negative and is positive on some subinterval of \([a,b]\). Expanding out gives \( f(b)^2 - 2\lambda f(a) + \lambda^2 g^2 > 0 \) for all \( \lambda \). Since the quadratic has no solutions, it must be that its discriminant is negative.

c. Let \( a = 0, b = n, f(x) = x_i \) and \( g(x) = y_i \) for all \( x \in [i-1,i) \) for \( i = 1, \ldots, n \).
Then part (a) gives the inequality of Theorem 1-1 (2). Note, however, that the equality condition does not follow from (a).

**Exercise 7 (1-7).** A linear transformation \( M : \mathbb{R}^n \to \mathbb{R}^n \) is called norm preserving if \( \|Mx\| = \|x\| \), and inner product preserving if \( \langle Mx, My \rangle = \langle x, y \rangle \).

a. Prove that \( M \) is norm preserving if and only if \( M \) is inner product preserving.

b. Prove that such a linear transformation \( M \) is 1-1 and \( M^{-1} \) is of the same sort.

**Proof.**

(a) If \( M \) is norm preserving, then the polarization identity together with the linearity of \( M \) give:
\[
\langle Mx, My \rangle = \frac{\|Mx + My\|^2 - \|Mx - My\|^2}{4} = \frac{\|M(x + y)\|^2 - \|M(x - y)\|^2}{4} = \frac{\|x + y\|^2 - \|x - y\|^2}{4} = \langle x, y \rangle.
\]

If \( M \) is inner product preserving, then one has by Theorem 1-1 (4):
\[
\|Mx\| = \sqrt{\langle Mx, Mx \rangle} = \sqrt{\langle x, x \rangle} = \|x\|.
\]

(b) Take any \( Mx, My \in \mathbb{R}^n \) with \( Mx = My \). Then \( Mx - My = 0 \) and so
\[
0 = \langle Mx - My, Mx - My \rangle = \langle x - y, x - y \rangle;
\]
but the above equality forces \( x = y \); that is, \( M \) is 1-1.

Since \( M \in \mathcal{L}(\mathbb{R}^n) \) and \( M \) is injective, it is invertible; see Axler (1997, Theorem 3.21). Hence, \( M^{-1} \in \mathcal{L}(\mathbb{R}^n) \) exists. For every \( x, y \in \mathbb{R}^n \), we have
\[
\|M^{-1}x\| = \|M(M^{-1}x)\| = \|x\|,
\]
and
\[
\langle M^{-1}x, M^{-1}y \rangle = \langle M(M^{-1}x), M(M^{-1}y) \rangle = \langle x, y \rangle.
\]
Therefore, \( M^{-1} \) is also norm preserving and inner product preserving. \( \square \)
Exercise 8 (1-8). If \(x, y \in \mathbb{R}^n\) are non-zero, the angle between \(x\) and \(y\), denoted \(\angle (x, y)\), is defined as \(\arccos \left(\frac{x \cdot y}{\|x\| \cdot \|y\|}\right)\), which makes sense by Theorem 1-1 (2). The linear transformation \(T\) is angle preserving if \(T\) is 1-1, and for \(x, y \neq 0\) we have \(\angle (Tx, Ty) = \angle (x, y)\).

a. Prove that if \(T\) is norm preserving, then \(T\) is angle preserving.

b. If there is a basis \((x_1, \ldots, x_n)\) of \(\mathbb{R}^n\) and numbers \(\lambda_1, \ldots, \lambda_n\) such that \(Tx_i = \lambda_i x_i\), prove that \(T\) is angle preserving if and only if all \(|\lambda_i|\) are equal.

c. What are all angle preserving \(T: \mathbb{R}^n \to \mathbb{R}^n\)?

Proof.

(a) If \(T\) is norm preserving, then \(T\) is inner product preserving by the previous exercise. Hence, for \(x, y \neq 0\),

\[
\angle (Tx, Ty) = \arccos \left(\frac{\langle Tx, Ty \rangle}{\|Tx\| \cdot \|Ty\|}\right) = \arccos \left(\frac{x \cdot y}{\|x\| \cdot \|y\|}\right) = \angle (x, y).
\]

(b) We first suppose that \(T\) is angle preserving. Since \((x_1, \ldots, x_n)\) is a basis of \(\mathbb{R}^n\), all \(x_i\)'s are nonzero. Since

\[
\angle (Tx_i, Tx_j) = \arccos \left(\frac{\langle Tx_i, Tx_j \rangle}{\|Tx_i\| \cdot \|Tx_j\|}\right) = \arccos \left(\frac{\langle \lambda_i x_i, \lambda_j x_j \rangle}{\|\lambda_i x_i\| \cdot \|\lambda_j x_j\|}\right)
= \arccos \left(\frac{\lambda_i \lambda_j \langle x_i, x_j \rangle}{|\lambda_i| \cdot |\lambda_j| \cdot \|x_i\| \cdot \|x_j\|}\right)
= \angle (x_i, x_j),
\]

it must be the case that

\[|\lambda_i| \cdot |\lambda_j| = |\lambda_i| \cdot |\lambda_j|\].

Then \(\lambda_i\) and \(\lambda_j\) have the same signs. \(\square\)

Exercise 9 (1-9). If \(0 \leq \theta < \pi\), let \(T: \mathbb{R}^2 \to \mathbb{R}^2\) have the matrix

\[
A = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}.
\]

Show that \(T\) is angle preserving and if \(x \neq 0\), then \(\angle (x, Tx) = \theta\).

Proof. For every \((x, y) \in \mathbb{R}^2\), we have

\[
T(x, y) = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
x \\
y
\end{pmatrix} = \begin{pmatrix}
x \cos \theta + y \sin \theta \\
-x \sin \theta + y \cos \theta
\end{pmatrix}.
\]

Therefore,

\[
\|T(x, y)\|^2 = x^2 + y^2 = \|(x, y)\|^2,
\]

that is, \(T\) is norm preserving. Then it is angle preserving by Exercise 8 (a).
Let \( x = (a, b) \neq 0 \). We first have
\[
\langle x, Tx \rangle = a (a \cos \theta + b \sin \theta) + b (-a \sin \theta + b \cos \theta) = \left( a^2 + b^2 \right) \cos \theta.
\]
Hence,
\[
\angle (x, Tx) = \arccos \left( \frac{\langle x, Tx \rangle}{\|x\| \cdot \|Tx\|} \right) = \arccos \left( \frac{(a^2 + b^2) \cos \theta}{a^2 + b^2} \right) = \theta.
\]

**Exercise 10 (1-10*).** If \( M : \mathbb{R}^m \to \mathbb{R}^n \) is a linear transformation, show that there is a number \( M \) such that \( \|Mh\| \leq M \|h\| \) for \( h \in \mathbb{R}^m \).

**Proof.** Let \( M \)'s matrix be
\[
A = \begin{pmatrix}
a_{11} & \cdots & a_{1m} \\
\vdots & \ddots & \vdots \\
a_{n1} & \cdots & a_{nm}
\end{pmatrix} = \begin{pmatrix}
a^1 \\
\vdots \\
a^n
\end{pmatrix}.
\]
Then
\[
Mh = Ah = \begin{pmatrix}
(a^1, h) \\
\vdots \\
(a^n, h)
\end{pmatrix},
\]
and so
\[
\|Mh\|^2 = \sum_{i=1}^n (a^i, h)^2 \leq \sum_{i=1}^n (\|a^i\| \cdot \|h\|)^2 = \left( \sum_{i=1}^n \|a^i\|^2 \right) \cdot \|h\|^2,
\]
that is,
\[
\|Mh\| \leq \left( \sqrt{\sum_{i=1}^n \|a^i\|^2} \right) \cdot \|h\|.
\]
Let \( M = \sqrt{\sum_{i=1}^n \|a^i\|^2} \) and we get the result.

**Exercise 11 (1-11).** If \( x, y \in \mathbb{R}^n \) and \( z, w \in \mathbb{R}^m \), show that \( \langle (x, z), (y, w) \rangle = \langle x, y \rangle + \langle z, w \rangle \) and \( \|(x, z)\| = \sqrt{\|x\|^2 + \|z\|^2} \).

**Proof.** We have \( (x, z), (y, w) \in \mathbb{R}^{n+m} \). Then
\[
\langle (x, z), (y, w) \rangle = \sum_{i=1}^n x_i y_i + \sum_{j=1}^m z_j w_j = \langle x, y \rangle + \langle z, w \rangle,
\]
and
\[
\|(x, z)\|^2 = \langle (x, z), (x, z) \rangle = \langle x, x \rangle + \langle z, z \rangle = \|x\|^2 + \|z\|^2.
\]
CHAPTER 1  FUNCTIONS ON EUCLIDEAN SPACE

Exercise 12 (1-12*). Let $(\mathbb{R}^n)^*$ denote the dual space of the vector space $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, define $\varphi_x \in (\mathbb{R}^n)^*$ by $\varphi_x(y) = \langle x, y \rangle$. Define $M : \mathbb{R}^n \to (\mathbb{R}^n)^*$ by $Mx = \varphi_x$. Show that $M$ is a 1-1 linear transformation and conclude that every $\varphi \in (\mathbb{R}^n)^*$ is $\varphi_x$ for a unique $x \in \mathbb{R}^n$.

Proof. We first show $M$ is linear. Take any $x, y \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$. Then

$$M(ax + by) = \varphi_{ax + by} = a\varphi_x + b\varphi_y = aMx + bMy,$$

where the second equality holds since for every $z \in \mathbb{R}^n$,

$$\varphi_{ax + by}(z) = \langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle = a\varphi_x(z) + b\varphi_y(z).$$

To see $M$ is 1-1, we need only to show that $\text{null set of } M = \{0\}$, where $\text{null set of } M$ is the null set of $M$. But this is clear and so $M$ is 1-1. Since $\dim (\mathbb{R}^n)^* = \dim \mathbb{R}^n$, $M$ is also onto. This proves the last claim.

Exercise 13 (1-13*). If $x, y \in \mathbb{R}^n$, then $x$ and $y$ are called perpendicular (or orthogonal) if $\langle x, y \rangle = 0$. If $x$ and $y$ are perpendicular, prove that $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Proof. If $\langle x, y \rangle = 0$, we have

$$\|x + y\|^2 = \langle x + y, x + y \rangle = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2 = \|x\|^2 + \|y\|^2.$$

1.2 SUBSETS OF EUCLIDEAN SPACE

Exercise 14 (1-14*). Simple. Omitted.

Exercise 15 (1-15). Prove that $\{x \in \mathbb{R}^n : \|x - a\| < r\}$ is open.

Proof. For any $y \in \{x \in \mathbb{R}^n : \|x - a\| < r\} =: B(a; r)$, let $\varepsilon = r - \|a, y\|$. We show that $B(y; \varepsilon) \subseteq B(a; r)$. Take any $z \in B(y; \varepsilon)$. Then

$$\|a, z\| \leq \|a, y\| + \|y, z\| < \|a, y\| + \varepsilon = r.$$


Exercise 17 (1-17). Omitted.
Exercise 18 (1-18). If \( A \subset [0, 1] \) is the union of open intervals \((a_i, b_i)\) such that each rational number in \((0,1)\) is contained in some \((a_i, b_i)\), show that \( \partial A = [0, 1] \setminus A \).

**Proof.** Let \( X := [0, 1] \). Obviously, \( A \) is open since \( A = \bigcup_i (a_i, b_i) \). Then \( X \setminus A \) is closed in \( X \) and so \( X \setminus \overline{A} = X \setminus A \). Since \( \partial A = \overline{A} \cap X \setminus A = \overline{A} \cap (X \setminus A) \), it suffices to show that

\[
X \setminus A \subset \overline{A} \tag{1.1}
\]

But (1.1) holds if and only if \( \overline{A} = X \). Now take any \( x \in X \) and any open nhood \( U \) of \( x \) in \( X \). Since \( \mathbb{Q} \) is dense, there exists \( y \in U \). Since there exists some \( i \) such that \( y \in (a_i, b_i) \), we know that \( U \cap (a_i, b_i) \neq \emptyset \), which means that \( U \cap A \neq \emptyset \), which means that \( x \in \overline{A} \). Hence, \( X = \overline{A} \), i.e., \( A \) is dense in \( X \). \( \square \)

Exercise 19 (1-19*). If \( A \) is a closed set that contains every rational number \( r \in (0, 1) \), show that \( [0, 1] \subset A \).

**Proof.** Take any \( r \in (0, 1) \) and any open interval \( r \in I \subset (0, 1) \). Then there exists \( q \in \mathbb{Q} \cap (0, 1) \) such that \( q \in I \). Since \( q \in A \), we know that \( r \in \overline{A} = A \). Since \( 0, 1 \in A \), the claim holds. \( \square \)

Exercise 20 (1-20). Prove the converse of Corollary 1-7: A compact subset of \( \mathbb{R}^n \) is closed and bounded.

**Proof.** To show \( A \) is closed, we prove that \( A^c \) is open. Assume that \( x \notin A \), and let \( G_m = \{ y \in \mathbb{R}^n : \|x - y\| > 1/m \} \), \( m = 1, 2, \ldots \). If \( y \in A \), then \( x \neq y \); hence, \( \|x - y\| > 1/m \) for some \( m \); therefore \( y \in G_m \) (see Figure 1.1). Thus, \( A \subseteq \bigcup_{m=1}^{\infty} G_m \), and by compactness we have a finite subcovering. Now observe that the \( G_m \) for an increasing sequence of sets: \( G_1 \subseteq G_2 \subseteq \cdots \); therefore, a finite union of some of the \( G_m \) is equal to the set with the highest index. Thus, \( K \subseteq G_s \) for some \( s \), and it follows that \( \mathbb{B}(x; 1/s) \subseteq A^c \). Therefore, \( A^c \) is open.

![Figure 1.1. A compact set is closed](image)

Let \( A \) be compact. We first show that \( A \) is bounded. Let
an open cover of $A$. Then there is a finite subcover $\{(-i, i)^n : i \in \mathbb{N}\}$ of $A$. Let $i' = \max \{i_1, \ldots, i_m\}$. Hence, $A \subset (-i', i')$, that is, $A$ is bounded. □

**Exercise 21 (1-21).**

a. If $A$ is closed and $x \notin A$, prove that there is a number $d > 0$ such that $\|y - x\| \geq d$ for all $y \in A$.

b. If $A$ is closed, $B$ is compact, and $A \cap B = \emptyset$, prove that there is $d > 0$ such that $\|y - x\| \geq d$ for all $y \in A$ and $x \in B$.

c. Give a counterexample in $\mathbb{R}^2$ if $A$ and $B$ are closed but neither is compact.

**Proof.**

(a) $A$ is closed implies that $A^c$ is open. Since $x \in A^c$, there exists an open ball $B(x; d)$ with $d > 0$ such that $x \in B(x; d) \subset A^c$. Then $\|y - x\| \geq d$ for all $y \in A$.

(b) For every $x \in B$, there exists $d_x > 0$ such that $x \in B(x; d_x/2) \subset A^c$ and $\|y - x\| \geq d_x$ for all $y \in A$. Then the family $\{B(x; d_x/2) : x \in B\}$ is an open cover of $B$. Since $B$ is compact, there is a finite set $\{x_1, \ldots, x_n\}$ such that $\{B(x_1; d_{x_1}/2), \ldots, B(x_n; d_{x_n}/2)\}$ covers $B$ as well. Now let

$$d = \min \{d_{x_1}/2, \ldots, d_{x_n}/2\}/2.$$

Then for any $x \in B$, there is an open ball $B(x; x_i/2)$ containing $x$ and $\|y - x_i\| \geq d_i$. Hence,

$$\|y - x\| \geq \|y - x_i\| - \|x_i - x\| \geq d_i - d_i/2 = d_i/2 \geq d.$$

(c) See Figure 1.2.

**Figure 1.2.**
EXERCISE 22 (1-22*). If $U$ is open and $C \subset U$ is compact, show that there is a compact set $D$ such that $C \subset D$ and $D \subset U$.

PROOF.

1.3 Functions and Continuity

EXERCISE 23 (1-23). If $f : A \to \mathbb{R}^m$ and $a \in A$, show that $\lim_{x \to a} f(x) = b$ if and only if $\lim_{x \to a} f^i(x) = b^i$ for $i = 1, \ldots, m$.

PROOF. Let $f : A \to \mathbb{R}^m$ and $a \in A$.

If: Assume that $\lim_{x \to a} f^i(x) = b^i$ for $i = 1, \ldots, m$. Then for every $\varepsilon > 0$, there is a number $\delta_i > 0$ such that $\|f^i(x) - b^i\| < \varepsilon/\sqrt{m}$ for all $x \in A$ which satisfy $0 < \|x - a\| < \delta_i$, for every $i = 1, \ldots, m$. Put

$$\delta = \min\{\delta_1, \ldots, \delta_m\}.$$ 

Then for all $x \in A$ satisfying $0 < \|x - a\| < \delta$,

$$\|f^i(x) - b^i\| < \frac{\varepsilon}{\sqrt{m}}, \quad i = 1, \ldots, m.$$ 

Therefore, for every $x \in A$ which satisfy $0 < \|x - a\| < \delta$,

$$\|f(x) - b\| = \sqrt{\sum_{i=1}^{m} (f^i(x) - b_i)^2} < \sqrt{\sum_{i=1}^{m} \left(\varepsilon^2/m\right)} = \varepsilon;$$ 

that is, $\lim_{x \to a} f(x) = b$.

Only if: Now suppose that $\lim_{x \to a} f(x) = b$. Then for every number $\varepsilon > 0$ there is a number $\delta > 0$ such that $\|f(x) - b\| < \varepsilon$ for all $x \in A$ which satisfy $0 < \|x - a\| < \delta$. But then for every $i = 1, \ldots, m$,

$$\|f^i(x) - b^i\| \leq \|f(x) - b\| < \varepsilon,$$

i.e. $\lim_{x \to a} f^i(x) = b^i$. □

EXERCISE 24 (1-24). Prove that $f : A \to \mathbb{R}^m$ is continuous at $a$ if and only if each $f^i$ is.

PROOF. By definition, $f$ is continuous at $a$ if and only if $\lim_{x \to a} f(x) = f(a)$; it follows from Exercise 23 that $\lim_{x \to a} f(x) = f(a)$ if and only if $\lim_{x \to a} f^i(x) = f^i(a)$ for every $i = 1, \ldots, m$; that is, if and only if $f^i$ is continuous at $a$ for each $i = 1, \ldots, m$. □

EXERCISE 25 (1-25). Prove that a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is continuous.
PROOF. Take any \( \mathbf{a} \in \mathbb{R}^n \). Then, by Exercise 10 (1-10), there exists \( M > 0 \) such that
\[
T \mathbf{x} - T \mathbf{a} = T ( \mathbf{x} - \mathbf{a} ) \leq M \| \mathbf{x} - \mathbf{a} \|.
\]
Hence, for every \( \varepsilon > 0 \), let \( \delta = \varepsilon / M \). Then \( T \mathbf{x} - T \mathbf{a} < \varepsilon \) when \( \mathbf{x} \in \mathbb{R}^n \) and \( 0 < \| \mathbf{x} - \mathbf{a} \| < \delta = \varepsilon / M \); that is, \( \lim_{\mathbf{x} \to \mathbf{a}} T \mathbf{x} = T \mathbf{a} \), and so \( T \) is continuous. \( \square \)

\textbf{Exercise 26 (1-26).} Let \( A = \{ (x, y) \in \mathbb{R}^2 : x > 0 \text{ and } 0 < y < x^2 \} \).

\textbf{a.} Show that every straight line through \((0, 0)\) contains an interval around \((0, 0)\) which is in \( \mathbb{R}^2 \setminus A \).

\textbf{b.} Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by \( f (x) = 0 \) if \( x \notin A \) and \( f (x) = 1 \) if \( x \in A \). For \( \mathbf{h} \in \mathbb{R}^2 \) define \( g_\mathbf{h} : \mathbb{R} \to \mathbb{R} \) by \( g_\mathbf{h} (t) = f (t \mathbf{h}) \). Show that each \( g_\mathbf{h} \) is continuous at \( 0 \), but \( f \) is not continuous at \((0, 0)\).

\textbf{Proof.}

\textbf{(a)} Let the line through \((0, 0)\) be \( y = ax \). If \( a \leq 0 \), then the whole line is in \( \mathbb{R}^2 \setminus A \). If \( a > 0 \), then \( ax \) intersects \( x^2 \) at \((a, a^2)\) and \((0, 0)\) and nowhere else; see Figure 1.3.

\textbf{(b)} We first show that \( f \) is not continuous at \( 0 \). Clearly, \( f (0) = 0 \) since \( 0 \notin A \). For every \( \delta > 0 \), there exists \( x \in A \) satisfying \( 0 < \| x \| < \delta \), but \( |f (x) - f (0)| = 1 \).

We next show \( g_\mathbf{h} (t) = f (t \mathbf{h}) \) is continuous at \( 0 \) for every \( \mathbf{h} \in \mathbb{R}^2 \). If \( \mathbf{h} = 0 \), then \( g_0 (t) = f (0) = 0 \) and so is continuous. So we now assume that \( \mathbf{h} \neq 0 \). It is clear that
\[
g_\mathbf{h} (0) = f (0) = 0.
\]
The result is now from (a) immediately. \( \square \)
EXERCISE 27 (1-27). Prove that \( \{ x \in \mathbb{R}^n : \| x - a \| < r \} \) is open by considering the function \( f : \mathbb{R}^n \to \mathbb{R} \) with \( f(x) = \| x - a \| \).

Proof. We first show that \( f \) is continuous. Take a point \( b \in \mathbb{R}^n \). For any \( \varepsilon > 0 \), let \( \delta = \varepsilon \). Then for every \( x \) satisfying \( \| x - b \| < \delta \), we have
\[
| f(x) - f(b) | = \| x - a \| - \| b - a \| \leq \| x - a \| - \| b - a \| \leq \| x - b \| < \delta = \varepsilon.
\]
Hence, \( \{ x \in \mathbb{R}^n : \| x - a \| < r \} = f^{-1}(-\infty, r) \) is open in \( \mathbb{R}^n \). \( \square \)

EXERCISE 28 (1-28). If \( A \subset \mathbb{R}^n \) is not closed, show that there is a continuous function \( f : A \to \mathbb{R} \) which is unbounded.

Proof. Take any \( x \in \partial A \). Let \( f(y) = 1/\| y - x \| \) for all \( y \in A \). \( \square \)


EXERCISE 30 (1-30). Let \( f : [a, b] \to \mathbb{R} \) be an increasing function. If \( x_1, \ldots, x_n \in [a, b] \) are distinct, show that \( \sum_{i=1}^n f(x_i) < f(b) - f(a) \).

Proof. \( \square \)
2

Differentiation

2.1 Basic Definitions

Exercise 31 (2-1*). Prove that if \( f : \mathbb{R}^n \to \mathbb{R}^m \) is differentiable at \( a \in \mathbb{R}^n \), then it is continuous at \( a \).

Proof. Let \( f \) be differentiable at \( a \in \mathbb{R}^n \); then there exists a linear map \( \lambda : \mathbb{R}^n \to \mathbb{R}^m \) such that

\[
\lim_{h \to 0} \frac{f(a + h) - f(a) - \lambda(h)}{\|h\|} = 0,
\]

or equivalently,

\[
f(a + h) - f(a) = \lambda(h) + r(h),
\]  

where the remainder \( r(h) \) satisfies

\[
\lim_{h \to 0} \frac{\|r(h)\|}{\|h\|} = 0.
\]

Let \( h \to 0 \) in (2.1). The error term \( r(h) \to 0 \) by (2.2); the linear term \( \lambda(h) \) also tends to \( 0 \) because if \( h = \sum_{i=1}^n h_i e_i \), where \( e_1, \ldots, e_n \) is the standard basis of \( \mathbb{R}^n \), then by linearity we have \( \lambda(h) = \sum_{i=1}^n h_i \lambda(e_i) \), and each term on the right tends to \( 0 \) as \( h \to 0 \). Hence,

\[
\lim_{h \to 0} [f(a + h) - f(a)] = 0;
\]

that is, \( \lim_{h \to 0} f(a + h) = f(a) \). Thus, \( f \) is continuous at \( a \). \( \square \)

Exercise 32 (2-2). A function \( f : \mathbb{R}^2 \to \mathbb{R} \) is independent of the second variable if for each \( x \in \mathbb{R} \) we have \( f(x, y_1) = f(x, y_2) \) for all \( y_1, y_2 \in \mathbb{R} \). Show that \( f \) is independent of the second variable if and only if there is a function \( g : \mathbb{R} \to \mathbb{R} \) such that \( f(x, y) = g(x) \). What is \( f'(a, b) \) in terms of \( g' \)?

Proof. The first assertion is trivial: if \( f \) is independent of the second variable, we can let \( g \) be defined by \( g(x) = f(x, 0) \). Conversely, if \( f(x, y) = g(x) \), then \( f(x, y_1) = g(x) = f(x, y_2) \).

If \( f \) is independent of the second variable, then
CHAPTER 2  DIFFERENTIATION

\[
\lim_{\substack{h,k \to 0}} \frac{|f(a + h, b + k) - f(a, b) - g'(a)h|}{\| (h, k) \|} = \lim_{\substack{h,k \to 0}} \frac{|g(a + h) - g(a) - g'(a)h|}{\| (h, k) \|} \\
\leq \lim_{h \to 0} \frac{|g(a + h) - g(a) - g'(a)h|}{|h|} \\
= 0;
\]

hence, \( f'(a, b) = (g'(a), 0) \). \hfill \Box

**Exercise 33 (2-3).** Define when a function \( f : \mathbb{R}^2 \to \mathbb{R} \) is independent of the first variable and find \( f'(a, b) \) for such \( f \). Which functions are independent of the first variable and also of the second variable?

**Proof.** We have \( f'(a, b) = (0, g'(b)) \) with a similar argument as in Exercise 32. If \( f \) is independent of the first and second variable, then for any \((x_1, y_1), (x_2, y_2) \in \mathbb{R}^2\), we have \( f(x_1, y_1) = f(x_2, y_1) = f(x_2, y_2) \); that is, \( f \) is constant. \hfill \Box

**Exercise 34 (2-4).** Let \( g \) be a continuous real-valued function on the unit circle \( \{x \in \mathbb{R}^2 : \|x\| = 1\} \) such that \( g(0, 1) = g(1, 0) = 0 \) and \( g(-x) = -g(x) \). Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by

\[
f(x) = \begin{cases} 
\|x\| \cdot g \left( \frac{x}{\|x\|} \right) & \text{if } x \neq 0, \\
0 & \text{if } x = 0.
\end{cases}
\]

**a.** If \( x \in \mathbb{R}^2 \) and \( h : \mathbb{R} \to \mathbb{R} \) is defined by \( h(t) = f(tx) \), show that \( h \) is differentiable.

**b.** Show that \( f \) is not differentiable at \((0, 0)\) unless \( g = 0 \).

**Proof.** (a) If \( x = 0 \) or \( t = 0 \), then \( h(t) = f(0) = 0 \); if \( x \neq 0 \) and \( t > 0 \),

\[
h(t) = f(tx) = t \|x\| \cdot g \left( \frac{tx}{t \|x\|} \right) = \left[ \|x\| \cdot g \left( \frac{x}{\|x\|} \right) \right] \cdot t = f(x)t;
\]

finally, if \( x \neq 0 \) and \( t < 0 \),

\[
h(t) = f(tx) = -t \|x\| \cdot g \left( \frac{tx}{-t \|x\|} \right) = -t \|x\| \cdot g \left( \frac{-x}{\|x\|} \right) = f(x)t.
\]

Therefore, \( h(t) = f(x)t \) for every given \( x \in \mathbb{R}^2 \), and so is differentiable: \( \mathbb{D}h = h \).

(b) Since \( g(1, 0) = 0 \) and \( g(-x) = -g(x) \), we have \( g(-1, 0) = g(-1, 0) = -g(1, 0) = 0 \). If \( f \) is differentiable at \((0, 0)\), there exists a matrix \((a, b)\) such that \( \mathbb{D}f(0, 0)(h, k) = ah + bk \). First consider any sequence \((h, b) \to (0, 0)\). Then
0 = \lim_{h \to 0} \frac{|f(h, 0) - f(0, 0) - ah|}{|h|} = \lim_{h \to 0} \frac{|h| \cdot g(h/|h| \cdot 0) - ah|}{|h|} = \lim_{h \to 0} \frac{|h| \cdot g(\pm 1, 0) - ah|}{|h|} = |a|

implies that \( a = 0 \). Next let us consider \((0, k) \to (0, 0)\). Then

\[
0 = \lim_{k \to 0} \frac{|f(0, k) - f(0, 0) - bk|}{|k|} = \lim_{k \to 0} \frac{|k| \cdot g(0, k/|k|) - bk|}{|k|} = |b|
\]

forces that \( b = 0 \). Therefore, \( f'(0, 0) = (0, 0) \) and \( \partial f(0, 0)(x, y) = 0 \). If \( g(x) \neq 0 \), then

\[
\lim_{x \to 0} \frac{|f(x) - f(0) - 0|}{\|x\|} = \lim_{x \to 0} \frac{\|x\| \cdot g(x/\|x\|)}{\|x\|} = \lim_{x \to 0} \frac{|g(x/\|x\|)|}{\|x\|} \neq 0,
\]

and so \( f \) is not differentiable.

Of course, if \( g(x) = 0 \), then \( f(x) = 0 \) and is differentiable. \( \square \)

**Exercise 35 (2-5).** Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by

\[
f(x, y) = \begin{cases} 
  \frac{|x|}{\sqrt{x^2 + y^2}} & \text{if } (x, y) \neq 0, \\
  0 & \text{if } (x, y) = 0.
\end{cases}
\]

Show that \( f \) is a function of the kind considered in Exercise 34, so that \( f \) is not differentiable at \((0, 0)\).

**Proof.** If \( (x, y) \neq 0 \), we can rewrite \( f(x, y) \) as

\[
f(x, y) = \frac{x \cdot |y|}{\sqrt{x^2 + y^2}} = \frac{x \cdot |y|}{\|x, y\|} = \|x, y\| \cdot \left( \frac{x}{\|x, y\|} \cdot \frac{|y|}{\|x, y\|} \right).
\]

If we let \( g : \{x \in \mathbb{R}^2 : \|x\| = 1\} \to \mathbb{R} \) be defined as \( g(x, y) = x \cdot |y| \), then (2.3) can be rewritten as

\[
f(x, y) = \|x, y\| \cdot g((x, y)/\|x, y\|).
\]

It is easy to see that

\[
g(0, 1) = g(1, 0) = 0, \quad \text{and} \quad g(-x, -y) = -x \cdot |y| = -x|y| = -f(x, y);
\]

that is, \( g \) satisfies all of the properties listed in Exercise 34. Since \( g(x) \neq 0 \) unless \( x = 0 \) or \( y = 0 \), we know that \( f \) is not differentiable at \( 0 \). A direct proof can be found in Berkovitz (2002, Section 1.11). \( \square \)

**Exercise 36 (2-6).** Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( f(x, y) = \sqrt{|xy|} \). Show that \( f \) is not differentiable at \((0, 0)\).
CHAPTER 2 DIFFERENTIATION

Proof. It is clear that
\[
\lim_{h \to 0} \frac{|f(h, 0)|}{|h|} = 0 = \lim_{k \to 0} \frac{|f(0, k)|}{|k|};
\]
hence, if \( f \) is differentiable at \((0, 0)\), it must be that \( \nabla f(0, 0)(x, y) = 0 \) since derivative is unique if it exists. However, if we let \( h = k > 0 \), and take a sequence \( \{(h, h)\} \to (0, 0) \), we have
\[
\lim_{(h,k) \to (0,0)} \frac{|f(h, h) - f(0, 0) - \sqrt{2}h|}{\|(h, h)\|} = \lim_{(h,k) \to (0,0)} \frac{\sqrt{2}h}{\|(h, h)\|} = \frac{1}{\sqrt{2}} \neq 0.
\]
Therefore, \( f \) is not differentiable. \( \square \)

Exercise 37 (2-7). Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a function such that \( |f(x)| \leq \|x\|^2 \). Show that \( f \) is differentiable at \( 0 \).

Proof. \( |f(0)| \leq \|0\|^2 = 0 \) implies that \( f(0) = 0 \). Since
\[
\lim_{x \to 0} \frac{|f(x) - f(0)|}{\|x\|} = \lim_{x \to 0} \frac{|f(x)|}{\|x\|} \leq \lim_{x \to 0} \|x\| = 0,
\]
\( \nabla f(0)(x, y) = 0 \). \( \square \)

Exercise 38 (2-8). Let \( f : \mathbb{R} \to \mathbb{R}^2 \). Prove that \( f \) is differentiable at \( a \in \mathbb{R} \) if and only if \( f^1 \) and \( f^2 \) are, and that in this case
\[
f'(a) = \left( \begin{array}{c} (f^1)'(a) \\ (f^2)'(a) \end{array} \right).
\]

Proof. Suppose that \( f \) is differentiable at \( a \) with \( f'(a) = \left( \begin{array}{c} c^1 \\ c^2 \end{array} \right) \). Then for \( i = 1, 2 \),
\[
0 \leq \lim_{h \to 0} \frac{|f^i(a + h) - f^i(a) - c^i \cdot h|}{|h|} \leq \lim_{h \to 0} \frac{\|f(a + h) - f(a) - \nabla f(a)(h)\|}{|h|} = 0
\]
implies that \( f^i \) is differentiable at \( a \) with \( (f^i)'(a) = c^i \).

Now suppose that both \( f^1 \) and \( f^2 \) are differentiable at \( a \), then by Exercise 1,
\[
0 \leq \frac{\|f(a + h) - f(a) - \nabla f(a)(h)\|}{|h|} \leq \sum_{i=1}^{2} \frac{|f^i(a + h) - f^i(a) - (f^i)'(a) \cdot h|}{|h|}
\]
implies that \( f \) is differentiable at \( a \) with \( f'(a) = \left( \begin{array}{c} (f^1)'(a) \\ (f^2)'(a) \end{array} \right) \). \( \square \)

Exercise 39 (2-9). Two functions \( f, g : \mathbb{R} \to \mathbb{R} \) are equal up to \( n \)-th order at \( a \) if
SECTION 2.1  BASIC DEFINITIONS

\[
\lim_{h \to 0} \frac{f(a + h) - g(a + h)}{h} = 0.
\]

a. Show that \( f \) is differentiable at \( a \) if and only if there is a function \( g \) of the form \( g(x) = a_0 + a_1(x - a) \) such that \( f \) and \( g \) are equal up to first order at \( a \).

b. If \( f'(a), \ldots, f^{(n)}(a) \) exist, show that \( f \) and the function \( g \) defined by

\[
g(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x - a)^i
\]

are equal up to \( n \)-th order at \( a \).

Proof. (a) If \( f \) is differentiable at \( a \), then by definition,

\[
\lim_{h \to 0} \frac{f(a + h) - f(a) - f'(a) \cdot h}{h} = 0,
\]

so we can let \( g(x) = f(a) + f'(a) \cdot (x - a) \).

On the other hand, if there exists a function \( g(x) = a_0 + a_1(x - a) \) such that

\[
\lim_{h \to 0} \frac{f(a + h) - a_0 - a_1h}{h} = 0,
\]

then \( a_0 = f(a) \), and so \( f \) is differentiable at \( a \) with \( f'(a) = a_1 \).

(b) By Taylor’s Theorem\(^1\) we rewrite \( f \) as

\[
f(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x - a)^i + \frac{f^{(n)}(y)}{n!} (x - a)^n,
\]

where \( y \) is between \( a \) and \( x \). Thus,

\[
\lim_{x \to a} \frac{f(x) - g(x)}{(x - a)^n} = \lim_{x \to a} \frac{f^{(n)}(y)(x - a)^n - f^{(n)}(a)(x - a)^n}{(x - a)^n}
\]

\[
= \lim_{x \to a} \frac{f^{(n)}(y) - f^{(n)}(a)}{n!} = 0.
\]

\(^1\) (Rudin, 1976, Theorem 5.15) Suppose \( f \) is a real function on \([a, b]\), \( n \) is a positive integer, \( f^{(n-1)} \) is continuous on \([a, b]\), \( f^{(n)} \) exists for every \( t \in (a, b) \). Let \( \alpha, \beta \) be distinct points of \([a, b]\), and define

\[
P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!} (t - \alpha)^k.
\]

Then there exists a point \( x \) between \( \alpha \) and \( \beta \) such that

\[
f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!} (\beta - \alpha)^n.
\]
2.2 Basic Theorems

Exercise 40 (2-10). Use the theorems of this section to find $f'$ for the following:

a. $f(x, y, z) = x^y$.

b. $f(x, y, z) = (x^y, z)$.

c. $f(x, y) = \sin(x \sin y)$.

d. $f(x, y, z) = \sin(x \sin(y \sin z))$.

e. $f(x, y, z) = x^{yz}$.

f. $f(x, y, z) = x^{y+z}$.

g. $f(x, y, z) = (x + y)^z$.

h. $f(x, y) = \sin(xy)$.

i. $f(x, y) = \left[\sin(xy)\right]^{\cos^3}$.

j. $f(x, y) = \left(\sin(xy), \sin(x \sin y) \cdot x^y\right)$.

Solution. Compare this with Exercise 47.

(a) We have $f(x, y, z) = x^y = e^{\ln x^y} = e^{y \ln x} = \exp \circ (\pi^2 \cdot \ln \pi^1)(x, y, z)$. It follows from the Chain Rule that

$$f'(a, b, c) = \exp'\left[(\pi^2 \ln \pi^1)(a, b, c) \cdot \left(\pi^2 \ln \pi^1\right)'(a, b, c)\right] = \exp(b \ln a) \cdot \left[(\ln \pi^1)(\pi^2)' + \pi^2(\ln \pi^1)'\right](a, b, c) = a^b \cdot \left[(0, \ln a, 0) + (b/a, 0, 0)\right] = \left(a^{b-1}b \quad a^b \ln a \quad 0\right)$.

(b) By (a) and Theorem 2-3(3), we have

$$f'(a, b, c) = \left(a^{b-1}b \quad a^b \ln a \quad 0\right)$$.

(c) We have $f(x, y) = \sin \circ (\pi^1 \sin(\pi^2))$. Then, by the chain rule,

$$f'(a, b) = \sin'\left[(\pi^1 \sin(\pi^2))(a, b)\right] \cdot \left[\pi^1 \sin(\pi^2)'\right](a, b) = \cos(a \sin b) \cdot \left[(\sin \pi^2)(\pi^1)' + \pi^1(\sin \pi^2)'\right](a, b) = \cos(a \sin b) \cdot \left[\sin(1, 0) + a(0, \cos b)\right] = \left(\cos(a \sin b) \cdot \sin b \quad a \cdot \cos(a \sin b) \cdot \cos b\right)$.

(d) Let $g(y, z) = \sin(y \sin z)$. Then
\[ f(x, y, z) = \sin(x \cdot g(y, z)) = \sin(\pi^1 \cdot g(\pi^2, \pi^3)). \]

Hence,
\[
f'(a, b, c) = \sin'(ag(b, c)) \cdot (\pi^1 \cdot g(\pi^2, \pi^3))'(a, b, c)
= \cos(ag(b, c)) \cdot \left[ g(b, c)(\pi^1)' + ag'(\pi^2, \pi^3) \right](a, b, c)
= \cos(ag(b, c)) \cdot \left[ (g(b, c), 0, 0) + ag'(\pi^2, \pi^3)(a, b, c) \right].
\]

It follows from (c) that
\[
g'(\pi^2, \pi^3)(a, b, c) = \begin{pmatrix} 0 & \cos(b \sin c) \cdot \sin c & b \cdot \cos(b \sin c) \cdot \cos c \end{pmatrix}.
\]

Therefore,
\[
f'(a, b, c) = \cos(a \sin(b \sin c)) \begin{pmatrix} \sin(b \sin c) & a \cos(b \sin c) \sin c & ab \cos(b \sin c) \cos c \end{pmatrix}.
\]

(e) Let \( g(x, y) = x^y \). Then
\[
f(x, y, z) = x^{g(y, z)} = g(x, g(y, z)) = g(\pi^1, g(\pi^2, \pi^3)).
\]

Then
\[
\mathbb{D} f(a, b, c) = \mathbb{D} g(a, g(b, c)) \circ \left[ \mathbb{D} \pi^1, \mathbb{D} g(\pi^2, \pi^3) \right](a, b, c).
\]

By (a),
\[
\mathbb{D} g(a, g(b, c))(x, y, z) = \begin{pmatrix} a^{g(b,c)} g(b, c)/a & a^{g(b,c)} \ln a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}
= \frac{a^{bc}}{a} x + \left( a^{bc} \ln a \right) y,
\]
\[
\mathbb{D} \pi^1(a, b, c)(x, y, z) = x,
\]
and
\[
\mathbb{D} g(\pi^2, \pi^3)(a, b, c)(x, y, z) = \mathbb{D} g(b, c) \circ \left[ \mathbb{D} \pi^2, \mathbb{D} \pi^3 \right](a, b, c)(x, y, z)
= \frac{b^c}{b} y + \left( b^c \ln b \right) z.
\]

Hence,
\[
\mathbb{D} f(a, b, c)(x, y, z) = \frac{a^{bc} b^c}{a} x + \left( a^{bc} \ln a \right) \left[ \frac{b^c}{b} y + \left( b^c \ln b \right) z \right],
\]
and
20 \hspace{1cm} \text{CHAPTER 2 DIFFERENTIATION} \\

\[ f'(a, b, c) = \left( a^{b^c/a} - a^{b^c} \ln a/b \right) \]

(f) Let \( g(x, y) = x^y \). Then \( f(x, y, z) = x^{y+z} = g(x, y + z) = g(\pi^1, \pi^2 + \pi^3) \).

Hence,

\[
D f(a, b, c)(x, y, z) = D g(a, b + c) \circ \left( D \pi^1, D \pi^2 + D \pi^3 \right)(a, b, c)(x, y, z) \\
= D g(a, b + c) \circ (x, y + z) \\
= a^{b+c} (b + c) \ln a \left( y + z \right).
\]

and

\[
f'(a, b, c) = \left( \frac{a^{b+c}(b+c)}{a} \right) a^{b+c} \ln a - a^{b+c} \ln a.
\]

(g) Let \( g(x, y) = x^y \). Then

\[
f(x, y, z) = (x + y)^z = g(x + y, z) = g(\pi^1 + \pi^2, \pi^3).
\]

Hence,

\[
D f(a, b, c)(x, y, z) = D g(a + b, c) \circ \left[ D \pi^1 + D \pi^2, D \pi^3 \right](a, b, c)(x, y, z) \\
= D g(a + b, c) \circ (x + y, z) \\
= (a + b)^c c \ln (a + b) + ((a + b)^c \ln (a + b)) z,
\]

and

\[
f'(a, b, c) = \left( \frac{(a+b)^c c}{a+b} \right) (a+b)^c \ln (a+b).
\]

(h) We have \( f(x, y) = \sin(xy) = \sin \circ (\pi^1 \pi^2) \). Hence,

\[
f'(a, b) = (\sin)'(ab) \left[ b(\pi^1)'(a, b) + a(\pi^2)'(a, b) \right] \\
= \cos(ab) \cdot \left[ b(1, 0) + a(0, 1) \right] \\
= \cos(ab) \cdot (b, a) \\
= \left( b \cdot \cos(ab) \right) a \cdot \cos(ab).
\]

(i) Straightforward.

(j) By Theorem 2-3 (3), we have

\[
f'(a, b, c) = \begin{pmatrix}
\left[ \sin(xy) \right]'(a, b, c) \\
\sin(x \sin y)'(a, b, c) \\
x'y'(a, b, c)
\end{pmatrix} \\
= \begin{pmatrix}
b \cdot \cos(ab) & a \cdot \cos(ab) \\
cos(a \sin b) \cdot \sin b & a \cdot \cos(a \sin b) \cdot \cos b \\
a^{b-1} & a^b \ln a
\end{pmatrix}.
\]

\[\Box\]

\(\triangleright\) \text{EXERCISE 41 (2-11). Find } f' \text{ for the following (where } g : \mathbb{R} \rightarrow \mathbb{R} \text{ is continuous):} \]
**Section 2.2 Basic Theorems**

a. \( f(x, y) = f_a^{x+y} g. \)

b. \( f(x, y) = f_a^{xy} g. \)

c. \( f(x, y, z) = \int_{xy} \sin(x \sin(y \sin z)) g. \)

**Solution.** (a) Let \( h(t) = f_a^t g. \) Then \( f(x, y) = \int_a^t \left[ h \circ (\pi_1 + \pi_2) \right](x, y), \) and so

\[
f'(a, b) = h' (a + b) \cdot \left[ (\pi_1 + \pi_2)'(a, b) \right]
= g (a + b) \cdot (1, 1)
= \left( g (a + b) \ g (a + b) \right).
\]

(b) Let \( h(t) = f_a^t g. \) Then \( f(x, y) = f_a^{xy} g = h(xy) = \left[ h \circ (\pi_1 \cdot \pi_2) \right](x, y). \) Hence,

\[
f'(a, b) = h' (ab) \cdot \left[ b \cdot (\pi_1)'(a, b) + a \cdot (\pi_2)'(a, b) \right]
= g (ab) \cdot (b, a)
= \left( b \cdot g (ab) \ a \cdot g (ab) \right).
\]

(c) We can rewrite \( f(x, y, z) \) as

\[
f(x, y, z) = \int_a^z g + \int_a^z \int_a^y g = \int_a^z \int_a^y g - \int_a^x g.
\]

Let \( \gamma(x, y, z) = \sin(x \sin(y \sin z)), k(x, y, z) = \int_a^{xy} g, \) and \( h(x, y, z) = \int_a^{xy} g. \) Then \( f(x, y, z) = k(x, y, z) - h(x, y, z), \) and so

\[
f'(a, b, c) = k'(a, b, c) - h'(a, b, c).
\]

It follows from Exercise 40 (d) that

\[
k'(a, b, c) = k' (\gamma(a, b, c)) \cdot \gamma'(a, b, c).
\]

The other parts are easy. \( \Box \)

**Exercise 42 (2-12).** A function \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p \) is bilinear if for \( x, x_1, x_2 \in \mathbb{R}^n, y, y_1, y_2 \in \mathbb{R}^m, \) and \( a \in \mathbb{R} \) we have

\[
f (ax, y) = af(x, y) = f (x, ay),
f (x_1 + x_2, y) = f (x_1, y) + f (x_2, y),
f (x, y_1 + y_2) = f (x, y_1) + f (x, y_2).
\]

a. Prove that if \( f \) is bilinear, then

\[
\lim_{{(h,k)\to 0}} \frac{\|f(h,k)\|}{\|(h,k)\|} = 0.
\]
b. Prove that $D f(a, b)(x, y) = f(a, y) + f(x, b)$.

c. Show that the formula for $D p(a, b)$ in Theorem 2-3 is a special case of (b).

PROOF. (a) Let $(e_1^n, \ldots, e_n^n)$ and $(e_1^m, \ldots, e_m^m)$ be the standard bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then for any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$, we have

$$x = \sum_{i=1}^n x_i e_i^n, \quad \text{and} \quad y = \sum_{j=1}^m y_j e_j^m.$$ 

Therefore,

$$f(x, y) = f \left( \sum_{i=1}^n x_i e_i^n, \sum_{j=1}^m y_j e_j^m \right) = \sum_{i=1}^n f \left( x_i e_i^n, \sum_{j=1}^m y_j e_j^m \right)$$

$$= \sum_{i=1}^n \sum_{j=1}^m f(x_i e_i^n, y_j e_j^m)$$

$$= \sum_{i=1}^n \sum_{j=1}^m x_i y_j f(e_i^n, e_j^m).$$

Then, by letting $M = \sum_{i,j} \left\| f(e_i^n, e_j^m) \right\|$, we have

$$\| f(x, y) \| = \left\| \sum_{i,j} x_i y_j f(e_i^n, e_j^m) \right\| \leq \sum_{i,j} |x_i y_j| \left\| f(e_i^n, e_j^m) \right\|$$

$$\leq M \left[ \max_i \left\{ |x_i| \right\} \max_j \left\{ |y_j| \right\} \right]$$

$$\leq M \| x \| \| y \| .$$

Hence,

$$\lim_{(h,k) \to 0} \frac{\| f(h,k) \|}{\| (h,k) \|} \leq \lim_{(h,k) \to 0} \frac{M \| h \| \| k \|}{\| (h,k) \|}$$

$$= \lim_{(h,k) \to 0} \frac{M \| h \| \| k \|}{\sqrt{\sum_{i,j} \left[ (h_i) + (k_j) \right]^2}}$$

$$= \lim_{(h,k) \to 0} \frac{M \| h \| \| k \|}{\sqrt{\| h \|^2 + \| k \|^2}}.$$

Now

$$\| h \| \| k \| \leq \left\{ \begin{array}{ll} \| h \|^2 & \text{if } \| k \| \leq \| h \| \\
\| k \|^2 & \text{if } \| h \| \leq \| k \|. \end{array} \right.$$ 

Hence $\| h \| \| k \| \leq \| h \|^2 + \| k \|^2$, and so
Exhibit a differentiable function

(a) It is evident that IP is bilinear; hence, by Exercise 42 (b), we have

\[ \lim_{(h,k) \to 0} M \|h\| \langle k \rangle \leq \lim_{(h,k) \to 0} M \|h\|^2 + \|k\|^2 = 0. \]

(b) We have

\[ \lim_{(h,k) \to 0} \frac{\|f(a + h, b + k) - f(a, b) - f(a, k) - f(h, b)\|}{\|h\| \langle k \rangle} = \lim_{(h,k) \to 0} \frac{\|f(a, b) + f(a, k) + f(h, b) - f(a, b) - f(a, k) - f(h, b)\|}{\|h\| \langle k \rangle} = \lim_{(h,k) \to 0} \frac{\|f(h, k)\|}{\|h\| \langle k \rangle} = 0 \]

by (a); hence, \( Df(a, b)(x, y) = f(a, y) + f(x, b) \).

(c) It is easy to check that \( p : \mathbb{R}^2 \to \mathbb{R} \) defined by \( p(x, y) = xy \) is bilinear. Hence, by (b), we have

\[ Dp(a, b)(x, y) = p(a, y) + p(x, b) = ay + xb. \]

\[ \square \]

\[ \textbf{Exercise 43 (2-13). Define IP: } \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \text{ by IP}(x, y) = (x, y). \]

a. Find \( D(\text{IP})(a, b) \) and \( (\text{IP})'(a, b) \).

b. If \( f, g : \mathbb{R} \to \mathbb{R}^n \) are differentiable and \( h : \mathbb{R} \to \mathbb{R} \) is defined by \( h(t) = \langle f(t), g(t) \rangle \), show that

\[ h'(a) = \left\langle f'(a)^T, g(a) \right\rangle + \left\langle f(a), g'(a)^T \right\rangle. \]

c. If \( f : \mathbb{R} \to \mathbb{R}^n \) is differentiable and \( \|f(t)\| = 1 \) for all \( t \), show that \( \left\langle f'(t)^T, f(t) \right\rangle = 0. \)

d. Exhibit a differentiable function \( f : \mathbb{R} \to \mathbb{R} \) such that the function \( |f| \) defined by \( |f|(t) = |f(t)| \) is not differentiable.

\[ \text{PROOF.} \quad (a) \text{ It is evident that IP is bilinear; hence, by Exercise 42 (b), we have} \]

\[ D(\text{IP})(a, b)(x, y) = \text{IP}(a, y) + \text{IP}(x, b) \]

\[ = \langle a, y \rangle + \langle x, b \rangle \]

\[ = \langle b, x \rangle + \langle a, y \rangle. \]

and so \((\text{IP})'(a, b) = (b, a).\)

(b) Since \( h(t) = \text{IP} \circ (f, g)(t) \), by the chain rule, we have

\[ Dh(a)(x) = D(\text{IP}) \left( f(a), g(a) \right) \circ (Df(a)(x), Dg(a)(x)) \]

\[ = \left\langle g(a), Df(a)(x) \right\rangle + \left\langle f(a), Dg(a)(x) \right\rangle \]

\[ = \left\langle g(a), f'(a) \right\rangle x + \left\langle f(a), g'(a) \right\rangle x. \]

(c) Let \( h(t) = (f(t), f(t)) \) with \( \|f(t)\| = 1 \) for all \( t \in \mathbb{R} \). Then
is constant, and so \( h'(a) = 0 \); that is,

\[ 0 = \left( f'(a)^\top, f(a) \right) + \left( f(a), f'(a)^\top \right) = 2 \left( f'(a)^\top, f(a) \right), \]

and so \( \left( f'(a)^\top, f(a) \right) = 0 \).

(d) Let \( f(t) = t \). Then \( f \) is linear and so is differentiable: \( Df = t \). However,

\[
\lim_{t \to 0^+} \frac{|t|}{t} = 1, \quad \lim_{t \to 0^-} \frac{|t|}{t} = -1;
\]

that is, \( |f| \) is not differentiable at 0.

\[ \square \]

Exercise 44 (2-14). Let \( \mathbb{E}_i, i = 1, \ldots, k \) be Euclidean spaces of various dimensions. A function \( f: \mathbb{E}_1 \times \cdots \times \mathbb{E}_k \to \mathbb{R}^p \) is called multilinear if for each choice of \( x_j \in \mathbb{E}_j, j \neq i \) the function \( g: \mathbb{E}_i \to \mathbb{R}^p \) defined by \( g(x) = f(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_k) \) is a linear transformation.

a. If \( f \) is multilinear and \( i \neq j \), show that for \( h = (h_1, \ldots, h_k) \), with \( h_\ell \in \mathbb{E}_\ell \), we have

\[
\lim_{h \to 0} \frac{\|f(a_1, \ldots, h_i, \ldots, h_j, \ldots, a_k)\|}{\|h\|} = 0.
\]

b. Prove that

\[ Df(a_1, \ldots, a_k)(x_1, \ldots, x_k) = \sum_{i=1}^k f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_k). \]

**Proof.**

(a) To lighten notation, define

\[ a_{i-j} := (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{j-1}, a_{j+1}, \ldots, a_k). \]

Let \( g: \mathbb{E}_i \times \mathbb{E}_j \to \mathbb{R}^p \) be defined as \( g(x, x_j) = f(a_{i-j}, x, x_j) \). Then \( g \) is bilinear and so

\[
\lim_{h \to 0} \frac{\|g(a_{i-j}, h_i, h_j)\|}{\|h\|} \leq \lim_{h \to 0} \frac{\|g(a_{i-j}, h_i, h_j)\|}{\|h_i\|, h_j\|} = 0
\]

by Exercise 42 (a).

(b) It follows from Exercise 42 (b) immediately. \[ \square \]

Exercise 45 (2-15). Regard an \( n \times n \) matrix as a point in the \( n \)-fold product \( \mathbb{R}^n \times \cdots \times \mathbb{R}^n \) by considering each row as a member of \( \mathbb{R}^n \).

a. Prove that \( \operatorname{det}: \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R} \) is differentiable and
\[ \det(a_1, \ldots, a_n)(x_1, \ldots, x_n) = \sum_{i=1}^n \det x_i. \]

b. If \( a_{ij} : \mathbb{R} \to \mathbb{R} \) are differentiable and \( f(t) = \det(a_{ij}(t)) \), show that
\[
f'(t) = \sum_{j=1}^n \det \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{j1}'(t) & \cdots & a_{jn}'(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}. \]

c. If \( \det(a_{ij}(t)) \neq 0 \) for all \( t \) and \( b_1, \ldots, b_n : \mathbb{R} \to \mathbb{R} \) are differentiable, let \( s_1, \ldots, s_n : \mathbb{R} \to \mathbb{R} \) be the functions such that \( s_1(t), \ldots, s_n(t) \) are the solutions of the equations
\[
\sum_{j=1}^n a_{ji}(t)s_j(t) = b_i(t), \quad i = 1, \ldots, n. \]

Show that \( s_i \) is differentiable and find \( s_i'(t) \).

**Proof.**

(a) It is easy to see that \( \det : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R} \) is multilinear; hence, the conclusion follows from Exercise 44.

(b) By (a) and the chain rule,
\[
f'(t) = (\det)'(a_{ij}(t)) \circ [a_{ij}'(t), \ldots, a_{ij}'(t)] \\
= \sum_{j=1}^n \det \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{j1}'(t) & \cdots & a_{jn}'(t) \\ \vdots & \ddots & \vdots \\ a_{n1}(t) & \cdots & a_{nn}(t) \end{pmatrix}. \]

(c) Let
\[
A = \begin{pmatrix} a_{11}(t) & \cdots & a_{1n}(t) \\ \vdots & \ddots & \vdots \\ a_{1n}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \quad s = \begin{pmatrix} s_1(t) \\ \vdots \\ s_n(t) \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} b_1(t) \\ \vdots \\ b_n(t) \end{pmatrix}. \]

Then
\[ As = b. \]
and so
\[ s_i(t) = \frac{\det(B_i)}{\det(A)}. \]
where \( B_i \) is obtained from \( A \) by replacing the \( i \)-th column with the \( b \). It follows from (b) that \( s_i(t) \) is differentiable. Define \( f(t) = \det(A) \) and \( g_i(t) = \det(B_i) \).

Then
\[ f'(t) = \sum_{j=1}^{n} \det \begin{pmatrix} a_{11}(t) & \cdots & a_{i-1,j}(t) & b_j(t) & a_{i+1,j}(t) & \cdots & a_{nj}(t) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{ij}'(t) & \cdots & a_{i-1,j}'(t) & b_j'(t) & a_{i+1,j}'(t) & \cdots & a_{nj}'(t) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n}(t) & \cdots & a_{i-1,n}(t) & b_n(t) & a_{i+1,n}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \]
and
\[ g_i'(t) = \sum_{j=1}^{n} \begin{pmatrix} a_{11}(t) & \cdots & a_{i-1,1}(t) & b_1(t) & a_{i+1,1}(t) & \cdots & a_{nj}(t) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{ij}'(t) & \cdots & a_{i-1,j}'(t) & b_j'(t) & a_{i+1,j}'(t) & \cdots & a_{nj}'(t) \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{1n}(t) & \cdots & a_{i-1,n}(t) & b_n(t) & a_{i+1,n}(t) & \cdots & a_{nn}(t) \end{pmatrix}. \]

Therefore,
\[ s_i'(t) = \frac{f'(t)g_i'(t) - f(t)g_i'(t)}{f^2(t)}. \]

**Exercise 46 (2-16).** Suppose \( f: \mathbb{R}^n \to \mathbb{R}^n \) is differentiable and has a differentiable inverse \( f^{-1}: \mathbb{R}^n \to \mathbb{R}^n \). Show that \( (f^{-1})'(a) = \left[ f'(f^{-1}(a)) \right]^{-1} \).

**Proof.** We have \( f \circ f^{-1}(x) = x \). On the one hand \( \mathcal{D} \left( f \circ f^{-1} \right)(a)(x) = x \) since \( f \circ f^{-1} \) is linear; on the other hand,
\[ \mathcal{D} \left( f \circ f^{-1} \right)(a)(x) = \left[ \mathcal{D} f \left( f^{-1}(a) \right) \circ \mathcal{D} f^{-1}(a) \right](x). \]
Therefore, \( \mathcal{D} f^{-1}(a) = \left[ \mathcal{D} f \left( f^{-1}(a) \right) \right]^{-1} \).

**2.3 Partial Derivatives**

**Exercise 47 (2-17).** Find the partial derivatives of the following functions:

a. \( f(x, y, z) = x^y \).

b. \( f(x, y, z) = z \).

c. \( f(x, y) = \sin(x \sin y) \).

d. \( f(x, y, z) = \sin(x \sin(y \sin z)) \).
e. \( f(x, y, z) = x^y z^z \).

f. \( f(x, y, z) = x^{y+z} \).

g. \( f(x, y, z) = (x + y)^2 \).

h. \( f(x, y) = \sin(xy) \).

i. \( f(x, y) = \left[\sin(xy)\right]^{\cos 3} \).

**SOLUTION.** Compare this with Exercise 40.

(a) \( \frac{\partial}{\partial x} f(x, y, z) = yx^{y-1}, \quad \frac{\partial}{\partial y} f(x, y, z) = x^y \ln x, \) and \( \frac{\partial}{\partial z} f(x, y, z) = 0. \)

(b) \( \frac{\partial}{\partial x} f(x, y, z) = \frac{\partial}{\partial y} f(x, y, z) = 0, \) and \( \frac{\partial}{\partial z} f(x, y, z) = 1. \)

c. \( \frac{\partial}{\partial x} f(x, y) = (\sin y) \cos(x \sin y), \) and \( \frac{\partial}{\partial y} f(x, y) = x \cos y \cos(x \sin y). \)

d. \( \frac{\partial}{\partial x} f(x, y, z) = \sin(y \sin z) \cos(x \sin(y \sin z)), \quad \frac{\partial}{\partial y} f(x, y, z) = \cos(x \sin(y \sin z)) \cos(y \sin z) \sin z, \) and \( \frac{\partial}{\partial z} f(x, y, z) = \cos(x \sin(y \sin z)) \cos(y \sin z) y \cos z. \)

e. \( \frac{\partial}{\partial x} f(x, y, z) = y^2 x y z^{z-1}, \quad \frac{\partial}{\partial y} f(x, y, z) = x^y z y^{z-1} \ln x, \) and \( \frac{\partial}{\partial z} f(x, y, z) = y^2 \ln y \left(x y^{z-1} \ln x\right). \)

(f) \( \frac{\partial}{\partial x} f(x, y, z) = (y + z) x y^{y+z-1}, \) and \( \frac{\partial}{\partial y} f(x, y, z) \frac{\partial}{\partial z} f(x, y, z) = x^{y+z} \ln x. \)

(g) \( \frac{\partial}{\partial x} f(x, y, z) = \frac{\partial}{\partial y} f(x, y, z) = (x + y)^{y-1}, \) and \( \frac{\partial}{\partial z} f(x, y, z) = (x + y)^2 \ln(x + y). \)

(h) \( \frac{\partial}{\partial x} f(x, y) = y \cos(xy), \) and \( \frac{\partial}{\partial y} f(x, y) = x \cos(xy). \)

(i) \( \frac{\partial}{\partial x} f(x, y) = \cos\left[\sin(xy)\right]^{\cos 3-1} y \cos(xy), \) and \( \frac{\partial}{\partial y} f(x, y) = \cos\left[\sin(xy)\right]^{\cos 3-1} x \cos(xy). \)

**Exercise 48 (2-18).** Find the partial derivatives of the following functions (where \( g : \mathbb{R} \rightarrow \mathbb{R} \) is continuous):

a. \( f(x, y) = f_a^{x+y} g. \)

b. \( f(x, y) = f_y^x g. \)

c. \( f(x, y) = f_a^{xy} g. \)

d. \( f(x, y) = f_a^{(\ln s)} g. \)

**SOLUTION.**

(a) \( \frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial y} f(x, y) = g(x + y). \)

(b) \( \frac{\partial}{\partial x} f(x, y) = g(x), \) and \( \frac{\partial}{\partial y} f(x, y) = -g(y). \)

c. \( \frac{\partial}{\partial x} f(x, y) = yg(xy), \) and \( \frac{\partial}{\partial y} f(x, y) = xg(xy). \)
\( \textbf{(d)} \quad \frac{\partial}{\partial x} f(x, y) = 0, \quad \text{and} \quad \frac{\partial}{\partial y} f(x, y) = g'(y) \cdot g \left( \int_b^y g \right). \)

**Exercise 49 (2-19).** If
\[
f(x, y) = x^{x+y} + \left( \ln x \right) \left( \arctan \left( \arctan \left( \sin (\cos xy) - \ln(x + y) \right) \right) \right)
\]
find \( \frac{\partial^2}{\partial x \partial y} f(1, y) \).

**Solution.** Putting \( x = 1 \) into \( f(x, y) \), we get \( f(1, y) = 1 \). Then \( \frac{\partial^2}{\partial x \partial y} f(1, y) = 0 \).

**Exercise 50 (2-20).** Find the partial derivatives of \( f \) in terms of the derivatives of \( g \) and \( h \) if
\begin{align*}
\text{a.} & \quad f(x, y) = g(x)h(y) \\
\text{b.} & \quad f(x, y) = g(x)h'(y) \\
\text{c.} & \quad f(x, y) = g(x) \\
\text{d.} & \quad f(x, y) = g(y) \\
\text{e.} & \quad f(x, y) = g(x + y).
\end{align*}

**Solution.**
\begin{align*}
\text{(a)} & \quad \frac{\partial}{\partial x} f(x, y) = g'(x)h(y), \quad \text{and} \quad \frac{\partial}{\partial y} f(x, y) = g(x)h'(y).
\text{(b)} & \quad \frac{\partial}{\partial x} f(x, y) = h(y)g(x)h'(y) - g'(x)h(y), \quad \text{and} \quad \frac{\partial}{\partial y} f(x, y) = h'(y)g(x)h'(y) \ln g(x).
\text{(c)} & \quad \frac{\partial}{\partial x} f(x, y) = g'(x), \quad \text{and} \quad \frac{\partial}{\partial y} f(x, y) = 0.
\text{(d)} & \quad \frac{\partial}{\partial x} f(x, y) = 0, \quad \text{and} \quad \frac{\partial}{\partial y} f(x, y) = g'(y).
\text{(e)} & \quad \frac{\partial}{\partial x} f(x, y) = \frac{\partial}{\partial y} f(x, y) = g'(x + y).
\end{align*}

**Exercise 51 (2-21*).** Let \( g_1, g_2 : \mathbb{R}^2 \to \mathbb{R} \) be continuous. Define \( f : \mathbb{R}^2 \to \mathbb{R} \) by
\[
f(x, y) = \int_0^x g_1(t, 0) \, dt + \int_0^y g_2(x, t) \, dt.
\]

\begin{enumerate}
\item Show that \( \frac{\partial}{\partial x} f(x, y) = g_2(x, y) \).
\item How should \( f \) be defined so that \( \frac{\partial}{\partial x} f(x, y) = g_1(x, y) \)?
\item Find a function \( f : \mathbb{R}^2 \to \mathbb{R} \) such that \( \frac{\partial}{\partial x} f(x, y) = x \) and \( \frac{\partial}{\partial y} f(x, y) = y \). Find one such that \( \frac{\partial}{\partial x} f(x, y) = y \) and \( \frac{\partial}{\partial y} f(x, y) = x \).
\end{enumerate}

**Proof.**
(a) $\mathbb{D}_2 f(x, y) = 0 + g_2(x, y) = g_2(x, y)$.

(b) We should let

\[
f(x, y) = \int_0^x g_1(t, y) \, dt + \int_0^y g_2(a, t) \, dt,
\]

where $t \in \mathbb{R}$ is a constant.

(c) Let

- $f(x, y) = (x^2 + y^2)/2$.
- $f(x, y) = xy$.

Near EXERCISE 52 (2-22*). If $f : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbb{D}_2 f = 0$, show that $f$ is independent of the second variable. If $\mathbb{D}_1 f = \mathbb{D}_2 f = 0$, show that $f$ is constant.

PROOF. Fix any $x \in \mathbb{R}$. By the mean-value theorem, for any $y_1, y_2 \in \mathbb{R}$, there exists a point $y^* \in (y_1, y_2)$ such that

\[
f(x, y_2) - f(x, y_1) = \mathbb{D}_2 f(x, y^*)(y_2 - y_1) = 0.
\]

Hence, $f(x, y_1) = f(x, y_2)$; that is, $f$ is independent of $y$.

Similarly, if $\mathbb{D}_1 f = 0$, then $f$ is independent of $x$. The second claim is then proved immediately. \hfill \Box

Near EXERCISE 53 (2-23*). Let $A = \{(x, y) \in \mathbb{R}^2 : x < 0, \text{ or } x > 0 \text{ and } y \neq 0\}$.

a. If $f : A \to \mathbb{R}$ and $\mathbb{D}_1 f = \mathbb{D}_2 f = 0$, show that $f$ is constant.

b. Find a function $f : A \to \mathbb{R}$ such that $\mathbb{D}_2 f = 0$ but $f$ is not independent of the second variable.

PROOF.

(a) As in Figure 2.1, for any $(a, b), (c, d) \in \mathbb{R}^2$, we have

\[
f(a, b) = f(-1, b) = f(-1, d) = f(c, d).
\]

(b) For example, we can let

\[
f(x, y) = \begin{cases} 
0 & \text{if } x < 0 \text{ or } y < 0 \\
x & \text{otherwise}.
\end{cases}
\]

Near EXERCISE 54 (2-24). Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

\[
f(x, y) = \begin{cases} 
xy \frac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq 0, \\
0 & (x, y) = 0.
\end{cases}
\]

a. Show that $\mathbb{D}_2 f(x, 0) = x$ for all $x$ and $\mathbb{D}_1 f(0, y) = -y$ for all $y$. 


b. Show that $\mathcal{D}_{1,2} f(0,0) \neq \mathcal{D}_{2,1} f(0,0)$.

Proof. 

(a) We have

$$\mathcal{D}_2 f(x, y) = \begin{cases} \frac{x(x^4-y^4-4x^2y^2)}{(x^2+y^2)^2} & (x, y) \neq 0, \\ 0 & (x, y) = 0, \end{cases}$$

and

$$\mathcal{D}_1 f(x, y) = \begin{cases} \frac{-y(x^4-y^4-4x^2y^2)}{(x^2+y^2)^2} & (x, y) \neq 0, \\ 0 & (x, y) = 0. \end{cases}$$

Hence, $\mathcal{D}_2 f(x, 0) = x$ and $\mathcal{D}_1 f(0, y) = -y$.

(b) By (a), we have $\mathcal{D}_{1,2} f(0, 0) = \mathcal{D}_2 \left( \mathcal{D}_1 f(0, y) \right)(0) = -1$; but $\mathcal{D}_{2,1} f(0, 0) = \mathcal{D}_1 \left( \mathcal{D}_2 (x, 0) \right)(0) = 1$. 

**Exercise 55 (2-25*).** Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-x^2} & x \neq 0 \\ 0 & x = 0. \end{cases}$$

Show that $f$ is a $C^\infty$ function, and $f^{(i)}(0) = 0$ for all $i$.

Proof. Figure 2.2 depicts $f(x)$. We first show that $f \in C^\infty$.

Let $p_n(y)$ be a polynomial with degree $n$ with respect to $y$. For $x \neq 0$ and $k \in \mathbb{N}$, we show that $f^{(k)}(x) = p_{3k}(x^{-1}) e^{-x^2}$. We do this by induction.

**Step 1** Clearly, $f'(x) = 2x^{-3} e^{-x^2}$.

**Step 2** Suppose that $f^{(k)}(x) = p_{3k}(x^{-1}) e^{-x^2}$.

**Step 3** Then by the chain rule,
SECTION 2.3  PARTIAL DERIVATIVES

\[ f^{(k+1)}(x) = \left[f^{(k)}(x)\right]' = p_{3k}'(x^{-1}) \cdot (-x^{-2}) \cdot e^{-x^{-2}} + p_{3k}(x^{-1}) \cdot 2x^{-3} \cdot e^{-x^{-2}} \]

\[ = \left[p_{3k}'(x^{-1}) \cdot (-x^{-2}) + p_{3k}(x^{-1}) \cdot 2x^{-3}\right] \cdot e^{-x^{-2}} \]

\[ = \left[q_{3k+1}(x^{-1}) + q_{3k+3}(x^{-1})\right] \cdot e^{-x^{-2}} \]

\[ = p_{3(k+1)}(x^{-1}) \cdot e^{-x^{-2}}. \]

where \(q_{3k+1}\) and \(q_{3k+3}\) are polynomials.

Therefore, \(f(x) \in C^\infty\) for all \(x \neq 0\). It remains to show that \(f^{(k)}(x)\) is defined and continuous at \(x = 0\) for all \(k\).

**Step 1**  Obviously,

\[ f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{e^{-x^{-2}}}{x} = \lim_{x \to 0} \frac{2x^{-3}e^{-x^{-2}}}{x} = 0 \]

by L'Hôpital's rule.

**Step 2**  Suppose that \(f^{(k)}(0) = 0\).

**Step 3**  Then,

\[ f^{(k+1)}(0) = \lim_{x \to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x} = \lim_{x \to 0} \frac{p_{3k+1}(x^{-1})e^{-x^{-2}}}{x} = \lim_{x \to 0} \frac{p_{3k+1}(x^{-1})}{e^{x^{-2}}}. \]

Hence, if we use L'Hôpital's rule \(3k + 1\) times, we get \(f^{(k+1)}(0) = 0\).

A similar computation shows that \(f^{(k)}(x)\) is continuous at \(x = 0\).  \(\Box\)

**Exercise 56 (2-26\textsuperscript{*}).** Let

\[ f(x) = \begin{cases} 
  e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}} & x \in (-1, 1), \\
  0 & x \notin (-1, 1). 
\end{cases} \]
a. Show that \( f: \mathbb{R} \to \mathbb{R} \) is a \( C^\infty \) function which is positive on \((-1, 1)\) and 0 elsewhere.

b. Show that there is a \( C^\infty \) function \( g: \mathbb{R} \to [0, 1] \) such that \( g(x) = 0 \) for \( x \leq 0 \) and \( g(x) = 1 \) for \( x \geq \varepsilon \).

c. If \( a \in \mathbb{R}^n \), define \( g: \mathbb{R}^n \to \mathbb{R} \) by

\[
g(x) = f \left( \frac{x^1 - a^1}{\varepsilon} \right) \cdots f \left( \frac{x^n - a^n}{\varepsilon} \right).
\]

Show that \( g \) is a \( C^\infty \) function which is positive on

\[
(a^1 - \varepsilon, a^1 + \varepsilon) \times \cdots \times (a^n - \varepsilon, a^n + \varepsilon)
\]

and zero elsewhere.

d. If \( A \subseteq \mathbb{R}^n \) is open and \( C \subseteq A \) is compact, show that there is a non-negative \( C^\infty \) function \( f: A \to \mathbb{R} \) such that \( f(x) > 0 \) for \( x \in C \) and \( f = 0 \) outside of some closed set contained in \( A \).

e. Show that we can choose such an \( f \) so that \( f: A \to [0, 1] \) and \( f(x) = 1 \) for \( x \in C \).

Proof.

(a) If \( x \in (-1, 1) \), then \( x - 1 \neq 0 \) and \( x + 1 \neq 0 \). It follows from Exercise 55 that \( e^{-(x-1)^2} \in C^\infty \) and \( e^{-(x+1)^2} \in C^\infty \). Then it is straightforward to check that \( f \in C^\infty \). See Figure 2.3

![Figure 2.3](image)

(b) By letting \( z = x + 1 \), we derive a new function \( j: \mathbb{R} \to \mathbb{R} \) from \( f \) as follows:

\[
j(z) = \begin{cases} e^{-(z-2)^2} \cdot e^{-z^2} & z \in (0, 2), \\ 0 & z \notin (0, 2). \end{cases}
\]

By letting \( w = \varepsilon z / 2 \), we derive a function \( k: \mathbb{R} \to \mathbb{R} \) from \( j \) as follows:

\[
k(w) = \begin{cases} e^{-(2w/\varepsilon - 2)^2} \cdot e^{-(2w/\varepsilon)^2} & w \in (0, \varepsilon), \\ 0 & w \notin (0, \varepsilon). \end{cases}
\]
It is easy to see that $k \in C^\infty$, which is positive on $(0, \varepsilon)$ and 0 elsewhere. Now let
\[ g(x) = \left( \int_0^x k(x) \right) / \left( \int_0^\varepsilon k(x) \right). \]
Then $g \in C^\infty$; it is 0 for $x \leq 0$, increasing on $(0, \varepsilon)$, and 1 for $x > \varepsilon$.

(c) It follows from (a) immediately.

(d) For every $x \in C$, let $R_x := ((-\varepsilon, \varepsilon))^n$ be a rectangle containing $x$, and $\overline{R_x}$ is contained in $A$ (we can pick such a rectangle since $A$ is open and $C \subset A$).
Then $\{R_x : x \in C\}$ is an open cover of $C$. Since $C$ is compact, there exists $\{x_1, \ldots, x_m\} \subset C$ such that $\{R_{x_1}, \ldots, R_{x_m}\}$ covers $C$. For every $x_i, i = 1, \ldots, n$, we define a function $g_i : R_{x_i} \to \mathbb{R}$ as
\[ g_i(x) = f \left( \frac{x_i^1 - a_i^1}{\varepsilon} \right) \cdots f \left( \frac{x_i^n - a_i^n}{\varepsilon} \right), \]
where $(a_1^1, \ldots, a_n^n) \in \mathbb{R}^n$ is the middle point of $R_{x_i}$.

Finally, we define $g : R_{x_1} \cup \cdots \cup R_{x_m} \to \mathbb{R}$ as follows:
\[ g(x) = \sum_{i=1}^m g_i(x). \]
Then $g \in C^\infty$; it is positive on $C$, and 0 outside $\overline{R_{x_1}} \cup \cdots \cup \overline{R_{x_m}}$.

(e) Follows the hints. \qed

\textbf{Exercise 57 (2-27).} Define $g, h : \{x \in \mathbb{R}^2: \|x\| \leq 1\} \to \mathbb{R}^3$ by
\[ g(x, y) = (x, y, \sqrt{1 - x^2 - y^2}), \quad h(x, y) = (x, y, -\sqrt{1 - x^2 - y^2}). \]
Show that the maximum of $f$ on $\{x \in \mathbb{R}^3: \|x\| = 1\}$ is either the maximum of $f \circ g$ or the maximum of $f \circ h$ on $\{x \in \mathbb{R}^2: \|x\| \leq 1\}$.

\textbf{Proof.} Let $A := \{x \in \mathbb{R}^2: \|x\| \leq 1\}$ and $B := \{x \in \mathbb{R}^3: \|x\| = 1\}$. Then $B = g(A) \cup h(A)$. \qed
2.4 DERIVATIVES

- **Exercise 58 (2-28).** Find expressions for the partial derivatives of the following functions:
  
  a. \( F(x, y) = f \left( g(x)k(y), g(x) + h(y) \right) \).
  
  b. \( F(x, y, z) = f \left( g(x + y), h(y + z) \right) \).
  
  c. \( F(x, y, z) = f \left( x^y, y^z, z^x \right) \).
  
  d. \( F(x, y) = f \left( x, g(x), h(x, y) \right) \).

**Proof.**

(a) Letting \( a := g(x)k(y), g(x) + h(y) \), we have

\[
\begin{align*}
\mathbb{D}_1 F(x, y) &= \mathbb{D}_1 f(a) \cdot g'(x) \cdot k(y) + \mathbb{D}_2 f(a) \cdot g'(x), \\
\mathbb{D}_2 F(x, y) &= \mathbb{D}_1 f(a) \cdot g(x) \cdot k'(y) + \mathbb{D}_1 f(a) \cdot h'(y).
\end{align*}
\]

(b) Letting \( a := g(x + y), h(y + z) \), we have

\[
\begin{align*}
\mathbb{D}_1 F(x, y, z) &= \mathbb{D}_1 f(a) \cdot g'(x + y), \\
\mathbb{D}_2 F(x, y, z) &= \mathbb{D}_1 f(a) \cdot g'(x + y) + \mathbb{D}_2 f(a) \cdot h'(y + z), \\
\mathbb{D}_3 F(x, y, z) &= \mathbb{D}_2 f(a) \cdot h'(y + z).
\end{align*}
\]

(c) Letting \( a := x^y, y^z, z^x \), we have

\[
\begin{align*}
\mathbb{D}_1 F(x, y, z) &= \mathbb{D}_1 f(a) \cdot y x^{y-1} + \mathbb{D}_3 f(a) \cdot z^x \ln z, \\
\mathbb{D}_2 F(x, y, z) &= \mathbb{D}_1 f(a) \cdot x^y \ln x + \mathbb{D}_2 f(a) \cdot z^x y^{x-1}, \\
\mathbb{D}_3 F(x, y, z) &= \mathbb{D}_2 f(a) \cdot y^z \ln y + \mathbb{D}_3 f(a) \cdot x z^{x-1}.
\end{align*}
\]

(d) Letting \( a := x, g(x), h(x, y) \), we have

\[
\begin{align*}
\mathbb{D}_1 F(x, y) &= \mathbb{D}_1 f(a) + \mathbb{D}_2 f(a) \cdot g'(x) + \mathbb{D}_3 f(a) \cdot \mathbb{D}_1 h(x, y), \\
\mathbb{D}_2 F(x, y) &= \mathbb{D}_3 f(a) \cdot \mathbb{D}_2 h(x, y).
\end{align*}
\]

**Exercise 59 (2-29).** Let \( f : \mathbb{R}^n \to \mathbb{R} \). For \( x \in \mathbb{R}^n \), the limit

\[
\lim_{t \to 0} \frac{f(a + tx) - f(a)}{t},
\]

if it exists, is denoted \( \mathbb{D}_x f(a) \), and called the directional derivative of \( f \) at \( a \), in the direction \( x \).

a. Show that \( \mathbb{D}_{e_i} f(a) = \mathbb{D}_i f(a) \).

b. Show that \( \mathbb{D}_{tx} f(a) = t \mathbb{D}_x f(a) \).
c. If $f$ is differentiable at $a$, show that $D_x f(a) = \mathbb{D} f(a)(x)$ and therefore $D_{x+y} f(a) = D_x f(a) + D_y f(a)$.

**Proof.**

(a) For $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, we have

$$
D_{e_i} f(a) = \lim_{t \to 0} \frac{f(a + te_i) - f(a)}{t} = \lim_{t \to 0} \frac{f(a_1, \ldots, a_{i-1}, a_i + t, a_{i+1}, \ldots, a_n) - f(a)}{t} = D_i f(a)
$$

by definition.

(b) We have

$$
D_{tx} f(a) = \lim_{s \to 0} \frac{f(a + stx) - f(a)}{s} = \lim_{st \to 0} \frac{f(a + stx) - f(a)}{st} = t D_x f(a).
$$

(c) If $f$ is differentiable at $a$, then for any $x \neq 0$ we have

$$
0 = \lim_{t \to 0} \frac{f(a + tx) - f(a) - D f(a)(t x)}{\|t x\|} = \lim_{t \to 0} \frac{|f(a + tx) - f(a) - t \cdot D f(a)(x)|}{|t|} \cdot \frac{1}{\|x\|}
$$

and so

$$
D_x f(a) = \lim_{t \to 0} \frac{f(a + tx) - f(a)}{t} = D f(a)(x).
$$

The case of $x = 0$ is trivial. Therefore,

$$
D_{x+y} f(a) = D f(a)(x + y) = D f(a)(x) + D f(a)(y) = D_x f(a) + D_y f(a).
$$

**Exercise 60 (2-30).** Let $f$ be defined as in Exercise 34. Show that $D_x f(0,0)$ exists for all $x$, but if $g \neq 0$, then $D_{x+y} f(0,0) \neq D_x f(0,0) + D_y f(0,0)$ for all $x, y$.

**Proof.** Take any $x \in \mathbb{R}^2$.

$$
\lim_{t \to 0} \frac{f(tx) - f(0,0)}{t} = \lim_{t \to 0} \frac{|t| \cdot \|x\| \cdot g \left( t x / \|t \cdot \|x\| \right)}{t}.
$$

Therefore, $D_x f(0,0)$ exists for any $x$.

Now let $g \neq 0$; then, $D_{(0,1)} f(0,0) = D_{(1,0)} f(0,0) = 0$, but $D_{(1,0) + (0,1)} f(0,0) = D_{(1,1)} f(0,0) \neq 0$. 

Exercise 61 (2-31). Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined as in Exercise 26. Show that \( \nabla_x f(0,0) \) exists for all \( x \), although \( f \) is not even continuous at \((0,0)\).

**Proof.** For any \( x \in \mathbb{R}^2 \), we have

\[
\lim_{t \to 0} \frac{f(tx) - f(0)}{t} = \lim_{t \to 0} \frac{f(tx)}{t} = 0
\]

by Exercise 26 (a).

Exercise 62 (2-32).

(a) Let \( f : \mathbb{R} \to \mathbb{R} \) be defined by

\[
f(x) = \begin{cases} 
x^2 \sin \frac{1}{x} & x \neq 0 \\
0 & x = 0.
\end{cases}
\]

Show that \( f \) is differentiable at 0 but \( f' \) is not continuous at 0.

(b) Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by

\[
f(x, y) = \begin{cases} 
(x^2 + y^2) \sin \frac{1}{\sqrt{x^2 + y^2}} & (x, y) \neq 0 \\
0 & (x, y) = 0.
\end{cases}
\]

Show that \( f \) is differentiable at \((0,0)\) but \( \mathbb{D}_i f \) is not continuous at \((0,0)\).

**Proof.**

(a) We have

\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{x^2 \sin \frac{1}{x}}{x} = \lim_{x \to 0} x \sin \frac{1}{x} = 0.
\]

Hence, \( f'(0) = 0 \). Further, for any \( x \neq 0 \), we have

\[
f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}.
\]

It is clear that \( \lim_{x \to 0} f'(x) \) does not exist. Therefore, \( f' \) is not continuous at 0.

(b) Since

\[
\lim_{(x,y) \to (0,0)} \frac{(x^2 + y^2) \sin \frac{1}{\sqrt{x^2 + y^2}}}{\sqrt{x^2 + y^2}} = \lim_{(x,y) \to (0,0)} \sqrt{x^2 + y^2} \sin \frac{1}{\sqrt{x^2 + y^2}} = 0,
\]

we know that \( f'(0,0) = (0,0) \). Now take any \((x, y) \neq (0,0)\). Then

\[
\mathbb{D}_1 f(x, y) = 2x \sin \frac{1}{\sqrt{x^2 + y^2}} - 2x \cos \frac{1}{\sqrt{x^2 + y^2}}.
\]
As in (a), \( \lim_{x \to 0} D_1 f(x, 0) \) does not exist. Similarly for \( D_2 f \).

\[ \square \]

**Exercise 63 (2-33)**. Show that the continuity of \( D_1 f \) at \( a \) may be eliminated from the hypothesis of Theorem 2-8.

**Proof.** It suffices to see that for the first term in the sum, we have, by letting \( (a^2, \ldots, a^n) =: a_{-1}, \)

\[ \lim_{h \to 0} \frac{|f(a^1 + h^1, a_{-1}) - f(a) - D_1 f(a) \cdot h^1|}{\|h\|} \]

\[ \leq \lim_{h^1 \to 0} \frac{|f(a^1 + h^1, a_{-1}) - f(a) - D_1 f(a) \cdot h^1|}{|h^1|} = 0. \]

See also Apostol (1974, Theorem 12.11).

**Exercise 64 (2-34)**. A function \( f : \mathbb{R}^n \to \mathbb{R} \) is homogeneous of degree \( m \) if \( f(tx) = t^m f(x) \) for all \( x \). If \( f \) is also differentiable, show that

\[ \sum_{i=1}^{n} x^i D_i f(x) = m f(x). \]

**Proof.** Let \( g(t) = f(tx) \). Then, by Theorem 2-9,

\[ g'(t) = \sum_{i=1}^{n} D_i f(tx) \cdot x^i. \]  \hspace{1cm} (2.4)

On the other hand, \( g(t) = f(tx) = t^m f(x) \); then

\[ g'(t) = m t^{m-1} f(x). \]  \hspace{1cm} (2.5)

Combining (2.4) and (2.5), and letting \( t = 1 \), we then get the result.

**Exercise 65 (2-35)**. If \( f : \mathbb{R}^n \to \mathbb{R} \) is differentiable and \( f(0) = 0 \), prove that there exist \( g_i : \mathbb{R}^n \to \mathbb{R} \) such that
\[ f(x) = \sum_{i=1}^{n} x^i g_i(x). \]

**Proof.** Let \( h_x(t) = f(tx) \). Then

\[
\int_0^1 h'_x(t) \, dt = h_x(1) - h_x(0) = f(x) - f(0) = f(x).
\]

Hence,

\[
f(x) = \int_0^1 h'_x(t) \, dt = \int_0^1 f'(tx) \, dt = \int_0^1 \left[ \sum_{i=1}^{n} x^i \partial_i f(tx) \right] \, dt
\]

\[
= \sum_{i=1}^{n} x^i \int_0^1 \partial_i f(tx) \, dt
\]

\[
= \sum_{i=1}^{n} x^i g_i(x).
\]

where \( g_i(x) = \int_0^1 \partial_i f(tx) \, dt \). \( \square \)

### 2.5 Inverse Functions

For this section, Rudin (1976, Section 9.3 and 9.4) is a good reference.

**Exercise 66 (2-36).** Let \( A \subset \mathbb{R}^n \) be an open set and \( f : A \to \mathbb{R}^n \) a continuously differentiable 1-1 function such that \( \det (f'(x)) \neq 0 \) for all \( x \). Show that \( f(A) \) is an open set and \( f^{-1} : f(A) \to A \) is differentiable. Show also that \( f(B) \) is open for any open set \( B \subset A \).

**Proof.** For every \( y \in f(A) \), there exists \( x \in A \) such that \( f(x) = y \). Since \( f \in \mathcal{C}'(A) \) and \( \det (f'(x)) \neq 0 \), it follows from the Inverse Function Theorem that there is an open set \( V \subset A \) containing \( x \) and an open set \( W \subset \mathbb{R}^n \) containing \( y \) such that \( W = f(V) \). This proves that \( f(A) \) is open.

Since \( f : V \to W \) has a continuous inverse \( f^{-1} : W \to V \) which is differentiable, it follows that \( f^{-1} \) is differentiable at \( y \); since \( y \) is chosen arbitrary, it follows that \( f^{-1} : f(A) \to A \) is differentiable.

Take any open set \( B \subset A \). Since \( f \upharpoonright B \in \mathcal{C}'(B) \) and \( \det \left( (f \upharpoonright B)'(x) \right) \neq 0 \) for all \( x \in B \subset A \), it follows that \( f(B) \) is open. \( \square \)

**Exercise 67 (2-37).**

a. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a continuously differentiable function. Show that \( f \) is not 1-1.
b. Generalize this result to the case of a continuously differentiable function

\[ f : \mathbb{R}^n \to \mathbb{R}^m \text{ with } m < n. \]

PROOF.

(a) Let \( f \in C' \). Then both \( D_1 f \) and \( D_2 f \) are continuous. Assume that \( f \) is 1-1; then both \( D_1 f \) and \( D_2 f \) cannot not be constant and equal to 0. So suppose that there is \((x_0, y_0) \in \mathbb{R}^2\) such that \( D_1 f (x_0, y_0) \neq 0 \). The continuity of \( D_1 f \) implies that there is an open set \( A \subset \mathbb{R}^2 \) containing \((x_0, y_0)\) such that \( D_1 f(x) \neq 0 \) for all \( x \in A \).

Define a function \( g : A \to \mathbb{R}^2 \) with

\[ g(x, y) = (f(x, y), y). \]

Then for all \((x, y) \in A\),

\[ g'(x, y) = \begin{pmatrix} D_1 f(x, y) & D_2 f(x, y) \\ 0 & 1 \end{pmatrix}, \]

and so \( \det(g'(x, y)) = D_1 f(x, y) \neq 0 \); furthermore, \( g \in C'(A) \) and \( g \) is 1-1. Then by Exercise 66, we know that \( g(A) \) is open. We now show that \( g(A) \) cannot be open actually.

Take a point \( (f(x_0, y_0), \tilde{y}) \in g(A) \) with \( y \neq y_0 \). Then for any \((x, y) \in A\), we must have

\[ g(x, y) = (f(x, y), y) = (f(x_0, y_0), \tilde{y}) \implies (x, y) = (x_0, y_0); \]

that is, there is no \((x, y) \in A\) such that \( g(x, y) = (f(x_0, y_0), \tilde{y}) \). This proves that \( f \) cannot be 1-1.

(b) We can write \( f : \mathbb{R}^n \to \mathbb{R}^m \) as \( f = (f^1, \ldots, f^m) \), where \( f^i : \mathbb{R}^n \to \mathbb{R} \) for every \( i = 1, \ldots, m \). As in (a), there is a mapping, say, \( f^1 \), a point \( a \in \mathbb{R}^n \), and an open set \( A \) containing \( a \) such that \( D_1 f^1(x) \neq 0 \) for all \( x \in A \). Define \( g : A \to \mathbb{R}^m \) as

\[ g \left( x^1, x^{-1} \right) = (f(x), x^{-1}), \]

where \( x^{-1} := (x^2, \ldots, x^n) \). Then as in (a), it follows that \( f \) cannot be 1-1. \( \square \)

EXERCISE 68 (2-38).

a. If \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( f'(a) \neq 0 \) for all \( a \in \mathbb{R} \), show that \( f \) is 1-1 (on all of \( \mathbb{R} \)).

b. Define \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( f(x, y) = (e^x \cos y, e^x \sin y) \). Show that \( \det(f'(x, y)) \neq 0 \) for all \((x, y) \) but \( f \) is not 1-1.

PROOF.
(a) Suppose that \( f \) is not 1-1. Then there exist \( a, b \in \mathbb{R} \) with \( a < b \) such that \( f(a) = f(b) \). It follows from the mean-value theorem that there exists \( c \in (a, b) \) such that
\[
0 = f(b) - f(a) = f'(c) (b - a),
\]
which implies that \( f'(c) = 0 \). A contradiction.

(b) We have
\[
f'(x, y) = \begin{pmatrix}
\mathbb{D}_x e^x \cos y & \mathbb{D}_y e^x \cos y \\
\mathbb{D}_x e^x \sin y & \mathbb{D}_y e^x \sin y
\end{pmatrix} = \begin{pmatrix}
e^x \cos y & -e^x \sin y \\
e^x \sin y & e^x \cos y
\end{pmatrix}.
\]
Then
\[
\det (f'(x, y)) = e^{2x} \left( \cos^2 y + \sin^2 y \right) = e^{2x} \neq 0.
\]
However, \( f(x, y) \) is not 1-1 since \( f(x, y) = f(x, y + 2k\pi) \) for all \( (x, y) \in \mathbb{R}^2 \) and \( k \in \mathbb{N} \).

This exercise shows that the non-singularity of \( \mathcal{D} f \) on \( A \) implies that \( f \) is locally 1-1 at each point of \( A \), but it does not imply that \( f \) is 1-1 on all of \( A \). See Munkres (1991, p. 69).

\[\Box\]

- Exercise 69 (2-39). Use the function \( f : \mathbb{R} \to \mathbb{R} \) defined by
\[
f(x) = \begin{cases}
\frac{\pi}{2} + x^2 \sin \frac{1}{x} & x \neq 0 \\
0 & x = 0
\end{cases}
\]
to show that continuity of the derivative cannot be eliminated from the hypothesis of Theorem 2-11.

Proof. If \( x \neq 0 \), then
\[
f'(x) = \frac{1}{2} + 2x \sin \frac{1}{x} - \cos \frac{1}{x}.
\]
If \( x = 0 \), then
\[
f'(0) = \lim_{h \to 0} \frac{h/2 + h^2 \sin (1/h)}{h} = \frac{1}{2}.
\]
Hence, \( f'(x) \) is not continuous at 0. It is easy to see that \( f \) is not injective for any neighborhood of 0 (see Figure 2.6).

2.6 Implicit Functions

- Exercise 70 (2-40). Use the implicit function theorem to re-do Exercise 45 (c).

Proof. Define \( f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \) by
SECTION 2.6 IMPLICIT FUNCTIONS

Figure 2.6.

\[ f^i(t, s) = \sum_{j=1}^{n} a_{ji}(t)s^j - b_i(t), \]

for \( i = 1, \ldots, n \). Then

\[ M := \begin{pmatrix} 
\mathbb{D}_2 f^1(t, s) & \cdots & \mathbb{D}_{1+n} f^1(t, s) \\
\vdots & \ddots & \vdots \\
\mathbb{D}_2 f^n(t, s) & \cdots & \mathbb{D}_{1+n} f^n(t, s)
\end{pmatrix} \begin{pmatrix} 
a_{11}(t) & \cdots & a_{n1}(t) \\
\vdots & \ddots & \vdots \\
a_{1n}(t) & \cdots & a_{nn}(t)
\end{pmatrix}, \]

and so \( \det(M) \neq 0 \).

It follows from the Implicit Function Theorem that for each \( t \in \mathbb{R} \), there is a unique \( s(t) \in \mathbb{R}^n \) such that \( f(t, s(t)) = 0 \), and \( s \) is differentiable.

\[ \square \]

\[ \text{Exercise 71 (2-41).} \ Let f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \ be differentiable. For each x \in \mathbb{R} define g_x: \mathbb{R} \to \mathbb{R} by g_x(y) = f(x, y). Suppose that for each x there is a unique y with g'_x(y) = 0; let c(x) be this y. \]

a. If \( \mathbb{D}_{2,2} f(x, y) \neq 0 \) for all \( x, y \), show that \( c \) is differentiable and

\[ c'(x) = -\frac{\mathbb{D}_{2,1} f(x, c(x))}{\mathbb{D}_{2,2} f(x, c(x))}. \]

b. Show that if \( c'(x) = 0 \), then for some \( y \) we have

\[ \mathbb{D}_{2,1} f(x, y) = 0, \quad \mathbb{D}_{2} f(x, y) = 0. \]

c. Let \( f(x, y) = x (y \log y - y) - y \log x. \) Find
\[
\max_{1/2 \leq x \leq 2} \left[ \min_{1/3 \leq y \leq 1} f(x, y) \right].
\]

**Proof.**

(a) For every \( x \), we have \( g'_x (y) = D_2 f(x, y) \). Since for every \( x \) there is a unique \( y = c(x) \) such that \( D_2 f(x, c(x)) = 0 \), the solution \( c(x) \) is the same as obtained from the Implicit Function Theorem; hence, \( c(x) \) is differentiable, and by differentiating \( D_2 f(x, c(x)) = 0 \) with respect to \( x \), we have

\[
D_{2,1} f(x, c(x)) + D_{2,2} f(x, c(x)) \cdot c'(x) = 0;
\]

that is,

\[
c'(x) = -\frac{D_{2,1} f(x, c(x))}{D_{2,2} f(x, c(x))}.
\]

(b) It follows from (a) that if \( c'(x) = 0 \), then \( D_{2,1} f(x, c(x)) = 0 \). Hence, there exists some \( y = c(x) \) such that \( D_{2,1} f(x, y) = 0 \). Furthermore, by definition, \( D_2 (x, c(x)) = D_2 f(x, y) = 0 \).

(c) We have

\[
D_2 f(x, y) = x \ln y - \ln x.
\]

Let \( D_2 f(x, y) = 0 \) we have \( y = c(x) = x^{1/x} \). Also, \( D_{2,2} f(x, y) = x/y > 0 \) since \( x, y > 0 \). Hence, for every fixed \( x \in [1/2, 2] \),

\[
\min_y f(x, y) = f(x, c(x)).
\]

![Figure 2.7](image_url)

*Figure 2.7.*

It is easy to see that \( c'(x) > 0 \) on \([1/2, 2]\), \( c(1) = 1 \), and \( c(a) = 1/3 \) for some \( a > 1/2 \) (see Figure 2.7). Therefore,

\[
\min_{1/3 \leq y \leq 1} f(x, y) = f(x, y^*(x)),
\]

where (see Figure 2.8)
\[ y^* (x) = \begin{cases} 
1/3 & \text{if } 1/2 \leq x \leq a \\
c(x) = x^{1/x} & \text{if } a < x \leq 1 \\
1 & \text{if } 1 < x \leq 2.
\end{cases} \]

**Figure 2.8.**

\[ \frac{1}{2} \leq x \leq a \quad \text{In this case, our problem is} \]
\[
\max_{1/2 \leq x \leq a} f(x, 1/3) = -\left(\frac{1 + \ln 3}{3}\right)x - \frac{1}{3}\ln x.
\]

It is easy to see that \( x^* = 1/2 \), and so \( f(x^*, 1/3) = \ln (4/3e) /6 \).

\[ a < x \leq 1 \quad \text{In this case, our problem is} \]
\[
\max_{a < x \leq 1} f(x, x^{1/x}) = -x^{1+1/x}.
\]

It is easy to see that the maximum of \( f \) occurs at \( x^* = a \) and \( y^*(x^*) = 1/3 \).

\[ 1 < x \leq 2 \quad \text{In this case, our problem is} \]
\[
\max_{1 < x \leq 2} f(x, 1) = -x - \ln x.
\]

The maximum of \( f \) occurs at \( x^* = 1 \).

Now, as depicted in Figure 2.9, we have \( x^* = 1/2 \), \( y^* = 1/3 \), and \( f(x^*, y^*) = \ln (4/3e) /6 \).

\( \Box \)
\[ f(x, y^*(x)) \]
3

INTEGRATION

3.1 Basic Definitions

- **Exercise 72 (3-1).** Let \( f : [0, 1] \times [0, 1] \rightarrow \mathbb{R} \) be defined by
  \[
  f(x, y) = \begin{cases} 
  0 & \text{if } 0 \leq x < 1/2 \\
  1 & \text{if } 1/2 \leq x \leq 1.
  \end{cases}
  \]

  Show that \( f \) is integrable and \( \int_{[0,1] \times [0,1]} f = 1/2 \).

**Proof.** Consider a partition \( P = (P_1, P_2) \) with \( P_1 = P_2 = (0, 1/2, 1) \). Then \( L(f, P) = U(f, P) = 1/2 \). It follows from Theorem 3-3 (the Riemann condition) that \( f \) is integrable and \( \int_{[0,1] \times [0,1]} f = 1/2 \).

- **Exercise 73 (3-2).** Let \( f : A \rightarrow \mathbb{R} \) be integrable and let \( g = f \) except at finitely many points. Show that \( g \) is integrable and \( \int_A f = \int_A g \).

**Proof.** Fix an \( \varepsilon > 0 \). It follows from the Riemann condition that there is a partition \( P \) of \( A \) such that
  \[
  U(f, P) - L(f, P) < \frac{\varepsilon}{2}.
  \]

Let \( P' \) be a refinement of \( P \) such that:

- for every \( x \in A \) with \( g(x) \neq f(x) \), it belongs to \( 2^n \) subrectangles of \( P' \), i.e., \( x \) is a corner of each subrectangle.
- for every subrectangle \( S \) of \( P' \),
  \[
  v(S) < \frac{\varepsilon}{2^n d(u-\ell)},
  \]

where
\[ d = \left| \{ x : f (x) \neq g (x) \} \right|. \]

\[ u = \sup_{x \in A} \{ g (x) \} - \inf_{x \in A} \{ f (x) \}. \]

\[ \ell = \inf_{x \in A} \{ g (x) \} - \sup_{x \in A} \{ f (x) \}. \]

\[ \begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{figure3.1.png}
\caption{Figure 3.1.}
\end{figure} \]

With such a choice of partition of \( A \), we have

\[
U (g, P') - U (f, P') = \sum_{i=1}^{d} \left[ \sum_{j=1}^{2^n} \left[ M_{S_{ij}} (g) - M_{S_{ij}} (f) \right] v (S_{ij}) \right]
\]

\[
\leq d 2^n u v. \]

where \( v := \sup_{S \in P} \{ v (S) \} \) is the least upper bound of the volumes of the subrectangles of \( P' \). Similarly,

\[
L (g, P') - L (f, P') = \sum_{i=1}^{d} \left[ \sum_{j=1}^{2^n} \left[ m_{S_{ij}} (g) - m_{S_{ij}} (f) \right] v (S_{ij}) \right]
\]

\[
\geq d 2^n \ell v. \]

Therefore,

\[
U (g, P') - L (g, P') \leq \left[ U (f, P') + d 2^n u v \right] - \left[ L (f, P') + d 2^n \ell v \right]
\]

\[
\leq \frac{\varepsilon}{2} + d 2^n (u - \ell) v
\]

\[
= \frac{\varepsilon}{2} + d 2^n (u - \ell) \frac{\varepsilon}{2^{n+1} d (u - \ell)}
\]

\[ = \varepsilon; \]

that is, \( g \) is integrable. It is easy to see now that \( \int_A g = \int_A f. \)

\[ \square \]

**Exercise 74 (3-3).** Let \( f, g : A \to \mathbb{R} \) be integrable.

a. For any partition \( P \) of \( A \) and subrectangle \( S \), show that \( m_S (f) + m_S (g) \leq m_S (f + g) \) and \( M_S (f + g) \leq M_S (f) + M_S (g) \) and therefore \( L (f, P) + L (g, P) \leq L (f + g, P) \) and \( U (f + g, P) \leq U (f, P) + U (g, P). \)
b. Show that \( f + g \) is integrable and \( \int_A (f + g) = \int_A f + \int_A g \).

c. For any constant \( c \), show that \( \int_A cf = c \int_A f \).

**Proof.**

(a) We show that \( m_S (f) + m_S (g) \) is a lower bound of \( \{ (f + g) (x) : x \in S \} \). It is clear that \( m_S (f) \leq f (x) \) and \( m_S (g) \leq g (x) \) for any \( x \in S \). Then for every \( x \in S \) we have

\[
m_S (f) + m_S (g) \leq f (x) + g (x) = (f + g) (x).
\]

Hence, \( m_S (f) + m_S (g) \leq m_S (f + g) \).

Similarly, for every \( x \in S \) we have \( M_S (f) \geq f (x) \) and \( M_S (g) \geq g (x) \); hence, \( (f + g) (x) = f (x) + g (x) \leq M_S (f) + M_S (g) \) and so \( M_S (f + g) \leq M_S (f) + M_S (g) \).

Now for any partition \( P \) of \( A \) we have

\[
L (f, P) + L (g, P) = \sum_{S \in P} m_S (f) v (S) + \sum_{S \in P} m_S (g)
= \sum_{S \in P} [m_S (f) + m_S (g)] v (S) \tag{3.1}
\leq \sum_{S \in P} m_S (f + g) v (S)
= L (f + g, P).
\]

and

\[
U (f, P) + U (g, P) = \sum_{S \in P} M_S (f) v (S) + \sum_{S \in P} M_S (g) v (S)
= \sum_{S \in P} [M_S (f) + M_S (g)] v (S) \tag{3.2}
\geq \sum_{S \in P} M_S (f + g) v (S)
= U (f + g, P).
\]

(b) It follows from (3.1) and (3.2) that for any partition \( P \),

\[
U (f + g, P) - L (f + g, P) \leq \left[ U (f, P) + U (g, P) \right] - \left[ L (f, P) + L (g, P) \right] \]
\[
= \left[ U (f, P) - L (f, P) \right] + \left[ U (g, P) - L (g, P) \right].
\]

Since \( f \) and \( g \) are integrable, there exist \( P' \) and \( P'' \) such that for any \( \varepsilon > 0 \), we have \( U (f, P') - L (f, P') < \varepsilon /2 \) and \( U (g, P'') - L (g, P'') < \varepsilon /2 \). Let \( \bar{P} \) refine both \( P' \) and \( P'' \). Then

\[
U \left( f, \bar{P} \right) - L \left( f, \bar{P} \right) < \frac{\varepsilon}{2} \quad \text{and} \quad U \left( g, \bar{P} \right) - L \left( g, \bar{P} \right) < \frac{\varepsilon}{2}.
\]

Hence,
U \left( f + g, \bar{P} \right) - L \left( f + g, \bar{P} \right) < \varepsilon,

and so \( f + g \) is integrable.

Now, by definition, for any \( \varepsilon > 0 \), there exists a partition \( P \) (by using a common refinement partition if necessary) such that \( \int_A f < L \left( f, P \right) + \varepsilon/2 \), \( \int_A g < L \left( g, P \right) + \varepsilon/2 \), \( U \left( f, P \right) < \int_A f + \varepsilon/2 \), and \( U \left( g, P \right) < \int_A g + \varepsilon/2 \). Therefore,

\[
\int_A f + \int_A g - \varepsilon < L \left( f, P \right) + L \left( g, P \right) \leq \int_A \left( f + g \right) \\
\leq U \left( f + g, P \right) \\
\leq U \left( f, P \right) + U \left( g, P \right) \\
< \int_A f + \int_A g + \varepsilon.
\]

Hence, \( \int_A (f + g) = \int_A f + \int_A g \).

(c) First, suppose that \( c > 0 \). Then for any partition \( P \) and any subrectangle \( S \), we have \( m_S (cf) = cm_S (f) \) and \( M_S (cf) = cM_S (f) \). But then \( L \left( cf, P \right) = cl \left( f, P \right) \) and \( U \left( cf, P \right) = cU \left( f, P \right) \). Since \( f \) is integrable, for any \( \varepsilon > 0 \) there exists a partition \( P \) such that \( U \left( f, P \right) - L \left( f, P \right) < \varepsilon/c \). Therefore,

\[
U \left( cf, P \right) - L \left( cf, P \right) = c \left[ U \left( f, P \right) - L \left( f, P \right) \right] < \varepsilon;
\]

that is, \( cf \) is integrable. Further,\n
\[
c \int_A f - \frac{\varepsilon}{c} < cL \left( f, P \right) = L \left( cf, P \right) \leq \int_A cf \leq U \left( cf, P \right) = cU \left( f, P \right) \\
< c \int_A f + \frac{\varepsilon}{c};
\]

i.e., \( \int_A cf = c \int_A f \).

Now let \( c < 0 \). Then for any partition \( P \) of \( A \), we have \( m_S (cf) = cM_S (f) \) and \( M_S (cf) = cm_S (f) \). Hence \( L \left( cf, P \right) = cU \left( f, P \right) \) and \( U \left( cf, P \right) = cL \left( f, P \right) \). Since \( f \) is integrable, for every \( \varepsilon > 0 \), choose \( P \) such that \( U \left( f, P \right) - L \left( f, P \right) < -\varepsilon/c \). Then

\[
U \left( cf, P \right) - L \left( cf, P \right) = -c \left[ U \left( f, P \right) - L \left( f, P \right) \right] < \varepsilon;
\]

that is, \( cf \) is integrable. Furthermore,\n
\[
-c \int_A f + \frac{\varepsilon}{c} < -cL \left( f, P \right) = -U \left( cf, P \right) \leq -\int_A cf \leq -L \left( cf, P \right) = -cL \left( f, P \right) \\
< -c \int_A f - \frac{\varepsilon}{c};
\]

i.e., \( \int_A cf = c \int_A f \). \( \square \)
Exercise 75 (3-4). Let \( f : A \rightarrow \mathbb{R} \) and let \( P \) be a partition of \( A \). Show that \( f \) is integrable if and only if for each subrectangle \( S \) the function \( f \upharpoonright S \) is integrable, and that in this case \( \int_A f = \sum_S \int_S f \upharpoonright S \).

Proof. Let \( P \) be a partition of \( A \), and \( S \) be a subrectangle with respect to \( P \).

**Only if:** Suppose that \( f \) is integrable. Then there exists a partition \( P_1 \) of \( A \) such that \( U(f, P_1) - L(f, P_1) < \varepsilon \) for any given \( \varepsilon > 0 \). Let \( P_2 \) be a common refinement of \( P \) and \( P_1 \). Then
\[
U(f, P_2) - L(f, P_2) \leq U(f, P_1) - L(f, P_1) < \varepsilon,
\]
and there are rectangles \( \{S_2^1, \ldots, S_2^n\} = S_2(S) \) with respect to \( P_2 \), such that \( S = \bigcup_{i=1}^n S_2^i \). Therefore,
\[
U(f, P_2) - L(f, P_2) = \sum_{S_2^i} \left[ M_{S_2^i}(f) - m_{S_2^i}(f) \right] v(S_2^i) \\
\geq \sum_{S_2 \in S_2(S)} \left[ M_{S_2}(f) - m_{S_2}(f) \right] v(S_2) \\
= U(f \upharpoonright S, P_2) - L(f \upharpoonright S, P_2);
\]
that is, \( f \upharpoonright S \) is integrable.

If: Now suppose that \( f \upharpoonright S \) is integrable for each \( S \). For each partition \( P' \), let \( |P'| \) be the number of subrectangles induced by \( P' \). Let \( P_S \) be a partition such that
\[
U(f \upharpoonright S, P_S) - L(f \upharpoonright S, P_S) < \frac{\varepsilon}{2|P'|}.
\]
Let \( P' \) be the partition of \( A \) obtained by taking the union of all the sub-sequences defining the partitions of the \( P_S \); see Figure 3.2. Then there are

![Figure 3.2](image-url)
refinements $P'_S$ of $P_S$ whose rectangles are the set of all subrectangles of $P'$ which are contained in $S$. Hence,

$$\sum_S \int_S f \upharpoonright S - \varepsilon < \sum_S L(f \upharpoonright S, P_S) \leq \sum_S L(f \upharpoonright S, P'_S) = L(f, P') \leq U(f, P') = \sum_S U(f \upharpoonright S, P'_S) \leq \sum_S U(f \upharpoonright S, P_S) < \sum_S \int_S f \upharpoonright S + \varepsilon.$$ 

Therefore, $f$ is integrable, and $\int_A f = \sum_S \int_S f \upharpoonright S$. □

**Exercise 76 (3-5).** Let $f, g: A \to \mathbb{R}$ be integrable and suppose $f \leq g$. Show that $\int_A f \leq \int_A g$.

**Proof.** Since $f$ is integrable, the function $-f$ is integrable by Exercise 74 (c); then $g - f$ is integrable by Exercise 74 (b). It is easy to see $\int_A (g - f) \geq 0$ since $g \geq f$. It follows from Exercise 74 that $\int_A (g - f) = \int_A (g + (-f)) = \int_A g + \int_A (-f) = \int_A g - \int_A f$; hence, $\int_A f \leq \int_A g$. □

**Exercise 77 (3-6).** If $f: A \to \mathbb{R}$ is integrable, show that $|f|$ is integrable and $|\int_A f| \leq \int_A |f|$.

**Proof.** Let $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$. Then

$$f = f^+ - f^- \quad \text{and} \quad |f| = f^+ + f^-.$$

It is evident that for any partition $P$ of $A$, both $U(f^+, P) - L(f^+, P) \leq U(f, P) - L(f, P)$ and $U(f^-, P) - L(f^-, P) \leq U(f, P) - L(f, P)$; hence, both $f^+$ and $f^-$ are integrable if $f$ is. Further,

$$|\int_A f| = |\int_A (f^+ - f^-)| = |\int_A f^+ - \int_A f^-| \leq \int_A f^+ + \int_A f^- = \int_A (f^+ + f^-) = \int_A |f|.$$ 

**Exercise 78 (3-7).** Let $f: [0, 1] \times [0, 1] \to \mathbb{R}$ be defined by
SECTION 3.3  FUBINI'S THEOREM

51

\[ f(x, y) = \begin{cases} 
0 & \text{if } x \text{ irrational} \\
0 & \text{if } x \text{ rational, } y \text{ irrational} \\
1/q & \text{if } x \text{ rational, } y = p/q \text{ is lowest terms.}
\end{cases} \]

Show that \( f \) is integrable and \( \int_{[0,1] \times [0,1]} f = 0. \)

PROOF. \( \square \)

### 3.2 Measure Zero and Content Zero

**Exercise 79 (3-8).** Prove that \([a_1, b_1] \times \cdots \times [a_n, b_n]\) does not have content 0 if \(a_i < b_i\) for each \(i\).

PROOF. Similar to the \([a, b]\) case. \( \square \)

**Exercise 80 (3-9).**

a. Show that an unbounded set cannot have content 0.

b. Give an example of a closed set of measure 0 which does not have content 0.

PROOF.

(a) Finite union of bounded sets is bounded.

(b) \(\mathbb{Z}\) or \(\mathbb{N}\). \( \square \)

**Exercise 81 (3-10).**

a. If \(C\) is a set of content 0, show that the boundary of \(C\) has content 0.

b. Give an example of a bounded set \(C\) of measure 0 such that the boundary of \(C\) does not have measure 0.

PROOF. \( \square \)

### 3.3 Fubini's Theorem

**Exercise 82 (3-27).** If \( f : [a, b] \times [a, b] \to \mathbb{R} \) is continuous, show that

\[ \int_a^b \int_a^y f(x, y) \, dx \, dy = \int_a^b \int_x^b f(x, y) \, dy \, dx. \]

PROOF. As illustrated in Figure 3.3,
CHAPTER 3 INTEGRATION

$$C = \{(x,y) \in [a,b]^2 : a \leq x \leq y \text{ and } a \leq y \leq b\}$$
$$= \{(x,y) \in [a,b]^2 : a \leq x \leq b \text{ and } x \leq y \leq b\}.$$ 

**Figure 3.3. Fubini’s Theorem**

**Exercise 83 (3-30).** Let $C$ be the set in Exercise 17. Show that

$$\int_{[0,1]} \left( \int_{[0,1]} 1_C(x,y) \, dx \right) \, dy = \int_{[0,1]} \left( \int_{[0,1]} 1_C(x,y) \, dy \right) \, dx = 0.$$

**Proof.** There must be typos. □

**Exercise 84 (3-31).** If $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ and $f : A \to \mathbb{R}$ is continuous, define $F : A \to \mathbb{R}$ by

$$F(x) = \int_{[a_1,x_1] \times \cdots \times [a_n,x_n]} f.$$ 

What is $\partial_i F(x)$, for $x \in \text{int}(A)$?

**Solution.** Let $c \in \text{int}(A)$. Then
\[ \mathbb{D}_1 f (c) = \lim_{h \to 0} \frac{F (c^i, c^i + h) - F (c)}{h} \]

\[ = \lim_{h \to 0} \frac{\int_{a_1 c^i}^{c^i + h} \left( f \left( \left[ a_1 c^i \right] \times \left[ a_i, c^i + h \right] \times \cdots \times [a_n, c^n] \right) \right) \, dx_i - F (c)}{h} \]

\[ = \lim_{h \to 0} \frac{\int_{a_1 c^i}^{c^i + h} \left( f \left( \left[ a_1 c^i \right] \times \cdots \times [a_i, c^i + h] \times \cdots \times [a_n, c^n] \right) \right) \, dx_i}{h} \]

\[ = \int_{a_1 c^i}^{c^i + h} \left( f \left( \left[ a_1 c^i \right] \times \cdots \times [a_i, c^i + h] \times \cdots \times [a_n, c^n] \right) \right) \, dx_i. \]

\[ \therefore \]

**Exercise 85 (3-32**). Let \( f : [a, b] \times [c, d] \to \mathbb{R} \) be continuous and suppose \( \mathbb{D}_2 f \) is continuous. Define \( F (y) = \int_a^b f (x, y) \, dx \). Prove Leibnitz’s rule: \( F' (y) = \int_a^b \mathbb{D}_2 f (x, y) \, dx \).

**Proof.** We have

\[ F' (y) = \lim_{h \to 0} \frac{F (y + h) - F (y)}{h} \]

\[ = \lim_{h \to 0} \frac{\int_a^b f (x, y + h) \, dx - \int_a^b f (x, y) \, dx}{h} \]

\[ = \lim_{h \to 0} \frac{\int_a^b f (x, y + h) - f (x, y) \, dx}{h} \]

By DCT, we have

\[ F' (y) = \int_a^b \left[ \lim_{h \to 0} \frac{f (x, y + h) - f (x, y)}{h} \right] \, dx \]

\[ = \int_a^b \mathbb{D}_2 f (x, y) \, dx. \]

\[ \therefore \]

**Exercise 86 (3-33**). If \( f : [a, b] \times [c, d] \to \mathbb{R} \) is continuous and \( \mathbb{D}_2 f \) is continuous, define \( F (x, y) = \int_a^x f (t, y) \, dt \).

a. Find \( \mathbb{D}_1 F \) and \( \mathbb{D}_2 F \).

b. If \( G (x) = \int_a^x f (t, x) \, dt \), find \( G' (x) \).

**Solution.**

(a) \( \mathbb{D}_1 F (x, y) = f (x, y) \), and \( \mathbb{D}_2 F = \int_a^x \mathbb{D}_2 f (t, y) \, dt \).

(b) It follows that \( G (x) = F (g (x), x) \). Then

\[ G' (x) = g' (x) \mathbb{D}_1 F (g (x), x) + \mathbb{D}_2 F (g (x), x) \]

\[ = g' (x) f (g (x), x) + \int_a^x \mathbb{D}_2 f (t, x) \, dt. \]

\[ \therefore \]
4

INTEGRATION ON CHAINS

4.1 Algebraic Preliminaries

Exercise 87 (4-1*). Let \(e_1, \ldots, e_n\) be the usual basis of \(\mathbb{R}^n\) and let \(\varphi_1, \ldots, \varphi_n\) be the dual basis.

a. Show that \(\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(e_{i_1}, \ldots, e_{i_k}) = 1\). What would the right side be if the factor \((k + \ell)!/k!\ell!\) did not appear in the definition of \(\wedge\)?

b. Show that \(\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(v_1, \ldots, v_k)\) is the determinant of the \(k \times k\) minor of

\[
\begin{pmatrix}
v_1 \\
\vdots \\
v_k
\end{pmatrix}
\]

obtained by selecting columns \(i_1, \ldots, i_k\).

Proof.

(a) Since \(\varphi_{ij} \in \mathcal{T}(\mathbb{R}^n)\), for every \(j = 1, \ldots, k\), we have

\[
\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}(e_{i_1}, \ldots, e_{i_k}) = \frac{k!}{1! \cdots 1!} \text{Alt} \left( \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}(e_{i_1}, \ldots, e_{i_k}) \right) = \sum_{\sigma \in S_k} (\text{sgn}(\sigma)) \varphi_{i_1}(e_{\sigma(i_1)}) \cdots \varphi_{i_k}(e_{\sigma(i_k)})
\]

If the factor \((k + \ell)!/k!\ell!\) did not appear in the definition of \(\wedge\), then the solution would be \(1/k!\).

(b) □

Exercise 88 (4-9*). Deduce the following properties of the cross product in \(\mathbb{R}^3\).

\[
e_1 \times e_1 = 0 \quad e_2 \times e_1 = -e_3 \quad e_3 \times e_1 = e_2
\]
a. \(e_1 \times e_2 = e_3 \quad e_2 \times e_2 = 0 \quad e_3 \times e_2 = -e_1\)
\(e_1 \times e_3 = -e_2 \quad e_2 \times e_3 = e_1 \quad e_3 \times e_3 = 0\)

Proof.
(a) We just do the first line.

\[
\begin{align*}
(w, z) &= \begin{vmatrix} e_1 \\ e_1 \\ w \end{vmatrix} = 0 \implies z = e_1 \times e_1 = \mathbf{0}, \\
(w, z) &= \begin{vmatrix} e_2 \\ e_1 \\ w \end{vmatrix} = -w_3 \implies e_2 \times e_1 = -e_3, \\
(w, z) &= \begin{vmatrix} e_3 \\ e_1 \\ w \end{vmatrix} = w_2 \implies e_3 \times e_1 = e_2.
\end{align*}
\]
References


Index

Directional derivative, 24