The Theory of Measures and Integration

A Solution Manual for Vestrup (2003)

Jianfei Shen

School of Economics, The University of New South Wales

Sydney, Australia

I hear, I forget;
I see, I remember;
I do, I understand.
Old Proverb

Contents

List of Figures

Preface

Sydney, *Jianfei Shen*

Acknowledgements

1

SET SYSTEMS

Remarks

Remark 1.1. Klenke (2008, Fig. 1.1, p.7) provides a chart to indicate the relationships among the set systems. Here I replicate his chart; see Figure 1.1.

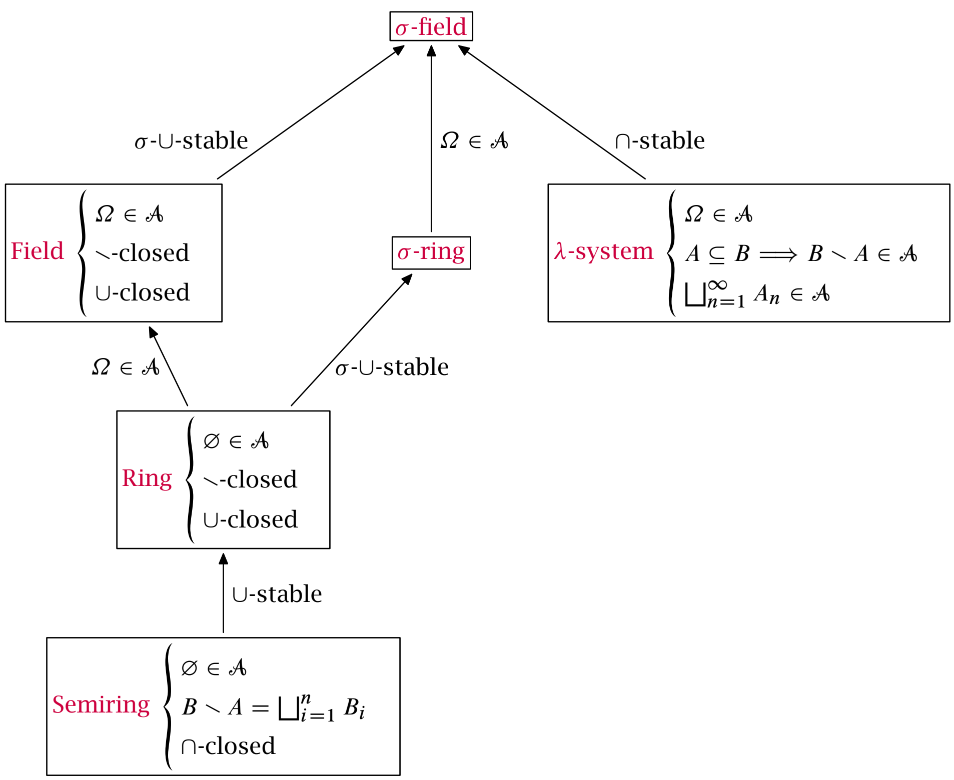

Figure 1.1. Inclusion between classes of sets $\mathcal{A} \subseteq 2^{\Omega}$

Semiring $\xrightarrow{\bigcup\text{-stable}}$ **Ring** See part (g) of Exercise 1.22;

σ-ring $\xrightarrow{\Omega \in \mathcal{A}}$ σ-field See part (b) of Exercise 1.43;

Ring $\xrightarrow{\Omega \in \mathcal{A}}$ Field $\Omega \in \mathcal{A}$ and $\mathcal{A}$ is closed under difference implies that $A \in \mathcal{A} \Longrightarrow A^c = \Omega - A \in \mathcal{A}$;

λ-systme $\xrightarrow{\bigcap\text{-stable}}$ σ-field See Exercise 1.10.

Remark 1.2. This notes is for Exercise 1.34 (p.17). See Klenke (2008, Example 1.40, p.18-19). We construct a measure for an infinitely often repeated random experiment with finitely many possible outcomes (Product measure, Bernoulli measure). Let S be the set of possible outcomes. For $s \in S$, let $p_s \geqslant 0$ be the probability that s occurs. Hence $\sum_{s \in S} p_s = 1$. For a fixed realization of the repeated experiment, let $z_1(\omega), z_2(\omega), \ldots \in S$ be the observed outcomes. Hence the space of *all* possible outcomes of the repeated experiment is $\Omega = S^{\mathbb{N}}$. We define the set of all sequences whose first n values are $z_1(\omega), \ldots, z_n(\omega)$:

$$[z_1(\omega), \ldots, z_n(\omega)] = \left\{\omega' \in \Omega : z_i(\omega') = z_i(\omega) \text{ for any } i = 1, \ldots, n\right\}. \tag{1.1}$$

Let $C_0 = \{\varnothing\}$. For $n \in \mathbb{N}$, define the class of *cylinder* sets that depend only on the first n coordinates

$$C_n = \{[z_1(\omega), \ldots, z_n(\omega)] : z_1(\omega), \ldots, z_n(\omega) \in S\}, \tag{1.2}$$

and let $C := \bigcup_{n=0}^{\infty} C_n$.

We interpret $[z_1(\omega), \ldots, z_n(\omega)]$ as the event where the outcome of the first experiment is $z_1(\omega)$, the outcome of the second experiment is $z_2(\omega)$ and finally the outcome of the n-th experiment is $z_n(\omega)$. The outcomes of the other experiments do not play a role for the occurrence of this event. As the individual experiments ought to be independent, we should have for any choice $z_1(\omega), \ldots, z_n(\omega) \in E$ that the probability of the event $[z_1(\omega), \ldots, z_n(\omega)]$ is the product of the probabilities of the individual events.

1.1 π-Systems, λ-Systems, and Semirings

1.1.1 π-Systems

► Exercise 1.3 (1.1.1). *Let $\Omega = (\alpha, \beta]$. Let $\mathcal{P}$ consists of $\varnothing$ along with the rsc subintervals of Ω. $\mathcal{P}$ is a π-system of subsets of $(\alpha, \beta]$.*

Proof. Let $A = (a, b]$ and $B = (c, d]$ be $\mathcal{P}$-sets. Then either $A \cap B = \varnothing \in \mathcal{P}$, or $A \cap B = (a \vee c, b \wedge d] \in \mathcal{P}$. □

► Exercise 1.4 (1.1.2). *Must $\varnothing$ be in every π-system?*

Solution. Not necessary. For example, let

$$\Omega = (0,1], \quad A = (0,1/2], \quad B = (1/4,1], \quad C = (1/4,1/2],$$

and let $\mathcal{P} = \{A, B, C\}$. Then $\mathcal{P}$ is a π-system on Ω, and $\varnothing \notin \mathcal{P}$. Generally, if $A \cap B \neq \varnothing$ for any A, B in a π-system, then $\varnothing$ does not in this π-system. □

► EXERCISE 1.5 (1.1.3). *List all π-systems consisting of at least two subsets of $\{\omega_1, \omega_2, \omega_3\}$.*

SOLUTION. These π-systems are:

- $\{\{\omega_i\}, \{\omega_i, \omega_j\}\}$, $(i,j) \in \{1,2,3\}^2$ and $j \neq i$;
- $\{\{\omega_i\}, \{\omega_1, \omega_2, \omega_3\}\}$;
- $\{\{\omega_i, \omega_j\}, \{\omega_1, \omega_2, \omega_3\}\}$;
- $\{\{\omega_i\}, \{\omega_i, \omega_j\}, \{\omega_1, \omega_2, \omega_3\}\}$;
- $\{\varnothing, \{\omega_i\}, \{\omega_i, \omega_j\}\}$, $i = 1,2,3$, and $j \neq i$;
- $\{\varnothing, \{\omega_i, \omega_j\}, \{\omega_1, \omega_2, \omega_3\}\}$;
- $\{\varnothing, \{\omega_i\}, \{\omega_i, \omega_j\}, \{\omega_1, \omega_2, \omega_3\}\}$. □

► EXERCISE 1.6 (1.1.4). *If $\mathcal{P}_k$ consists of the empty set and the k-dimensional rectangles of any one form, then $\mathcal{P}_k$ is a π-system of subsets of $\mathbb{R}^k$.*

PROOF. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{P}_k$ be two k-dimensional rectangles of any form. We also write $\boldsymbol{A} = A_1 \times A_2 \times \cdots \times A_k$ and $\boldsymbol{B} = B_1 \times \cdots \times B_k$, where A_i and B_i are rsc intervals for every $i \in \{1, \ldots, n\}$. We also assume that $\boldsymbol{A} \neq \varnothing$ and $\boldsymbol{B} \neq \varnothing$; for otherwise $\boldsymbol{A} \cap \boldsymbol{B} = \varnothing \in \mathcal{P}_k$ is trivial. Then

$$\boldsymbol{A} \cap \boldsymbol{B} = (A_1 \times \cdots \times A_k) \cap (B_1 \times \cdots \times B_k) = \bigtimes_{i=1}^{k} (A_i \cap B_i) \in \mathcal{P}_k$$

since $A_i \cap B_i$ is a rsc interval in $\mathbb{R}$. □

► EXERCISE 1.7 (1.1.5). *Let $\mathcal{P}$ consist of $\varnothing$ and all subsets of $\mathbb{R}^k$ that are neither open nor closed. Then $\mathcal{P}$ is not a π-system of subsets of $\mathbb{R}^k$.*

PROOF. To get some intuition, let $k = 1$. Consider two $\mathcal{P}$-sets: $A = (0, 1/2]$ and $B = [1/4, 1)$. Note that neither A nor B are open or closed on $\mathbb{R}$, but their intersection $A \cap B = [1/4, 1/2]$ is closed on $\mathbb{R}$, and is not in $\mathcal{P}$.

Now consider the k-dimensional case. Let $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{P}$; let $\boldsymbol{A} = \bigtimes_{i=1}^k A_i$ and $\boldsymbol{B} = \bigtimes_{i=1}^k B_i$; particularly, we let $A_i = (a_i, b_i]$ and $B_i = [c_i, d_i)$, where $a_i < c_i < b_i < d_i$. Then $(a_i, b_i] \cap [c_i, d_i) = [c_i, b_i] \neq \varnothing$, and $\boldsymbol{A} \cap \boldsymbol{B} = \bigtimes_{i=1}^k (A_i \cap B_i) = \bigtimes_{i=1}^k [c_i, b_i]$ is closed on $\mathbb{R}^k$. □

► EXERCISE 1.8 (1.1.6). *For each α in a nonempty index set A, let $\mathcal{P}_\alpha$ be a π-system over Ω.*

a. *The collection* $\bigcap_{\alpha\in A}\mathcal{P}_\alpha$ *is a* π*-system on* Ω*.*

b. *Let* $\mathcal{A}\subseteq 2^\Omega$*. Suppose that* $\{\mathcal{P}_\alpha : \alpha\in A\}$ *is the "exhaustive list" of all the* π*-system that contain* $\mathcal{A}$*. In other words, each* $\mathcal{P}_\alpha\supseteq\mathcal{A}$*, and any* π*-system that contains* $\mathcal{A}$ *coincides with some* $\mathcal{P}_\alpha$*. Then* $\bigcap_{\alpha\in A}\mathcal{P}_\alpha$ *is a* π*-system that contains* $\mathcal{A}$*. If* $\mathcal{Q}$ *is a* π*-system containing* $\mathcal{A}$*, then* $\bigcap_{\alpha\in A}\mathcal{P}_\alpha\subseteq\mathcal{Q}$*. The minimal* π*-system generated by* $\mathcal{A}$ *always exists.*

c. *Suppose that* $\mathcal{P}$ *is a* π*-system with* $\mathcal{P}\supseteq\mathcal{A}$*, and suppose that* $\mathcal{P}$ *is contained in any other* π*-system that contains* $\mathcal{A}$*. Then* $\mathcal{P}=\bigcap_{\alpha\in A}\mathcal{P}_\alpha$*, with notation as in (b). The minimal* π*-system containing* $\mathcal{A}$ *[which always exists] is also unique.*

PROOF. **(a)** Suppose $B, C\in\bigcap_{\alpha\in A}\mathcal{P}_\alpha$. Then $B, C\in\mathcal{P}_\alpha$ for every $\alpha\in A$. Since $\mathcal{P}_\alpha$ is a π-system, we have $B\cap C\in\mathcal{P}_\alpha$ for all $\alpha\in A$. Consequently, $B\cap C\in\bigcap_{\alpha\in A}\mathcal{P}_\alpha$, i.e., $\bigcap_{\alpha\in A}\mathcal{P}_\alpha$ is a π-system on Ω.

The analogous statement holds for rings, σ-rings, algebras and σ-algebras. However, it fails for semirings. A counterexample: let $\Omega=\{1,2,3,4\}$, $\mathcal{A}_1=\{\varnothing,\Omega,\{1\},\{2,3\},\{4\}\}$, and $\mathcal{A}_2=\{\varnothing,\Omega,\{1\},\{2\},\{3,4\}\}$. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are semirings but $\mathcal{A}_1\cap\mathcal{A}_2=\{\varnothing,\Omega,\{1\}\}$ is not.

(b) Since $2^\Omega\in\{\mathcal{P}_\alpha : \alpha\in A\}=:\Pi(\mathcal{A})$, the family $\Pi(\mathcal{A})$ is nonempty. It follows from (a) that $\bigcap_{\alpha\in A}\mathcal{P}_\alpha$ is a π-system containing $\mathcal{A}$. Finally, if $\mathcal{Q}$ is a π-system containing $\mathcal{A}$, then $\mathcal{Q}\in\Pi(\mathcal{A})$, hence $\bigcap_{\alpha\in A}\mathcal{P}_\alpha\subseteq\mathcal{Q}$.

(c) Since $\bigcap_{\alpha\in A}\mathcal{P}_\alpha$ is the π-system generated by $\mathcal{A}$, we have $\bigcap_{\alpha\in A}\mathcal{P}_\alpha\subseteq\mathcal{P}$; since $\mathcal{P}$ is contained in any other π-system that contains $\mathcal{A}$, we have $\mathcal{P}\subseteq\bigcap_{\alpha\in A}\mathcal{P}_\alpha$. □

1.1.2 λ-System

► EXERCISE 1.9 (1.1.7). *This exercise explores some equivalent definitions of a* λ*-system.*[1]

a. $\mathcal{L}$ *is a* λ*-system iff* $\mathcal{L}$ *satisfies* (λ_1)*,* (λ_2')*, and* (λ_3)*.*

b. *Every* λ*-system additionally satisfies* (λ_4)*,* (λ_5)*, and* (λ_6)*.*

c. $\mathcal{L}$ *is a* λ*-system iff* $\mathcal{L}$ *satisfies* (λ_1)*,* (λ_2')*, and* (λ_5)*.*

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[1] The conditions are:

$(\boldsymbol{\lambda_1})$ $\Omega\in\mathcal{L}$;
$(\boldsymbol{\lambda_2})$ $A\in\mathcal{L}\Longrightarrow A^c\in\mathcal{L}$;
$(\boldsymbol{\lambda_2'})$ $A, B\in\mathcal{L}\ \&\ A\subseteq B\Longrightarrow B\setminus A\in\mathcal{L}$;
$(\boldsymbol{\lambda_3})$ For any disjoint $\{A_n\}_{n=1}^\infty\subseteq\mathcal{L}$, $\bigcup_{n=1}^\infty A_n\in\mathcal{L}$;
$(\boldsymbol{\lambda_4})$ $A, B\in\mathcal{L}\ \&\ A\cap B=\varnothing\Longrightarrow A\cup B\in\mathcal{L}$;
$(\boldsymbol{\lambda_5})$ $\forall\ \{A_n\}_{n=1}^\infty\subseteq\mathcal{L}$, $A_n\uparrow\Longrightarrow\bigcup_{n=1}^\infty A_n\in\mathcal{L}$;
$(\boldsymbol{\lambda_6})$ $\forall\ \{A_n\}_{n=1}^\infty\subseteq\mathcal{L}$, $A_n\downarrow\Longrightarrow\bigcap_{n=1}^\infty A_n\in\mathcal{L}$.

d. *If a collection $\mathscr{L}$ is nonempty and satisfies (λ_2) and (λ_3), then $\mathscr{L}$ is a λ-system.*

Proof. **(a)** Let $\mathscr{L}$ be a λ-system. Then $\varnothing \in \mathscr{L}$ by (λ_1) and (λ_2). Suppose that $A, B \in \mathscr{L}$ and $A \subseteq B$. Then $B^c \in \mathscr{L}$ by $(\lambda 2)$ and $A \cap B^c = \varnothing$. By (λ_3), $B^c \cup A = B^c \cup A \cup \varnothing \cup \varnothing \cup \cdots \in \mathscr{L}$. By (λ_2) again, $B \smallsetminus A = (B^c \cup A)^c \in \mathscr{L}$.

To show the inverse direction, we need only to show that (λ_1) and (λ_2') imply (λ_2): if $A \in \mathscr{L}$, then $A^c = \Omega \smallsetminus A \in \mathscr{L}$.

(b) Let $\mathscr{L}$ be a λ-system, so it satisfies (λ_1)—(λ_3) and (λ_2'). To verify that (λ_4) holds, first notice that $\varnothing = \Omega^c \in \mathscr{L}$. If $A, B \in \mathscr{L}$ and $A \cap B = \varnothing$, then $A \cup B = A \cup B \cup \varnothing \cup \varnothing \cup \cdots \in \mathscr{L}$.

To see that (λ_5), let $\{A_n\} \subseteq \mathscr{L}$ be increasing. Let $B_1 = A_1$ and $B_n = A_n \smallsetminus A_{n-1}$ for $n \geqslant 2$. Then $\{B_n\} \subseteq \mathscr{L}$ by (λ_2') and is disjoint. Hence, $\bigcup A_n = \bigsqcup B_n \in \mathscr{L}$.

Finally, if $\{A_n\} \subseteq \mathscr{L}$ is decreasing, then $\{A_n^c\} \subseteq \mathscr{L}$ is increasing. Hence $\bigcup A_n^c \in \mathscr{L}$ by (λ_5). Then $\bigcap A_n = (\bigcup A_n^c)^c \in \mathscr{L}$.

(c) If $\mathscr{L}$ is a λ-system, it follows from (a) and (b) that (λ_2') and (λ_5) hold. Now suppose that (λ_1), (λ_2'), and (λ_5) hold. It follows from the *only if* part of (a) that (λ_1) and (λ_2') imply (λ_2). To see (λ_3) also hold, let $\{A_n\}_{n=1}^{\infty} \subseteq \mathscr{L}$ be a disjoint sequence. We can construct a nondecreasing sequence $\{B_n\}_{n=1}^{\infty}$ by letting $B_n = \bigcup_{i=1}^{n} A_i$. Notice that $B_n \in \mathscr{L}$ for all n. Hence, $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$, and by (λ_5), we have (λ_3).

(d) If $\mathscr{L} \neq \varnothing$ and satisfies (λ_2) and (λ_3), then there exists some $A \in \mathscr{L}$ and so $\Omega = A \cup A^c \in \mathscr{L}$ by (λ_4). □

► Exercise 1.10 (1.1.8). *If $\mathscr{L}$ is a λ-system and a π-system, then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{L}$ whenever $A_n \in \mathscr{L}$ for all $n \in \mathbb{N}$. That is, $\mathscr{L}$ is closed under countable unions.*

Proof. This exercise proves that a λ-system which is $\bigcup$-stable is a σ-field (see Figure 1.1). Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathscr{L}$. Let $B_1 = A_1$ and $B_n = A_1^c \cap A_2^c \cap \cdots \cap A_{n-1}^c \cap A_n$ for all $n \geqslant 2$. Since $\mathscr{L}$ is a λ-system, $\{A_1^c, \ldots, A_{k-1}^c\} \subseteq \mathscr{L}$; since $\mathscr{L}$ is a π-system, $B_n \in \mathscr{L}$. It follows from (λ_3) that $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \in \mathscr{L}$. □

► Exercise 1.11 (1.1.9). *A λ-system is not necessarily a π-system.*

Proof. For example, let $\Omega = (0, 1]$. The following collection is a λ-system:

$$\mathscr{L} = \{\varnothing, \Omega, (0, 1/2], (1/4, 1], (1/2, 1], (0, 1/4]\}.$$

However, $\mathscr{L}$ is not a π-system because $(0, 1/2] \cap (1/4, 1] = (1/4, 1/2] \notin \mathscr{L}$. □

► Exercise 1.12 (1.1.10). *Find all λ-systems over $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ with at least three elements.*

Solution.

$$\begin{cases} \{\varnothing, \Omega, \{\omega_i\}, \{\omega_j, \omega_k, \omega_\ell\}\} & i \neq j \neq k \neq \ell \\ \{\varnothing, \Omega, \{\omega_i, \omega_j\}, \{\omega_k, \omega_\ell\}\} & i \neq j \neq k \neq \ell. \end{cases}$$

□

► EXERCISE 1.13 (1.1.11). *The collection consisting of $\varnothing$ and the rsc intervals is not a λ-system on $\mathbb{R}$.*

PROOF. This is not a λ-system, but is a semiring. Consider a nontrival rsc interval $(a, b]$. Note that $(a, b]^c = (-\infty, a] \cup (b, +\infty)$ is not a rsc interval, and so is not in this collection. □

► EXERCISE 1.14 (1.1.12). *Suppose that for each α in a nonempty index set A, $\mathscr{L}_\alpha$ is a λ-system over Ω.*

a. *The collection $\bigcap_{\alpha \in A} \mathscr{L}_\alpha$ is a λ-system on Ω.*

b. *Suppose that $\mathscr{A} \subseteq 2^\Omega$ is such that $\mathscr{A}$ is contained in each $\mathscr{L}_\alpha$, and suppose that $\{\mathscr{L}_\alpha : \alpha \in A\}$ is the "exhaustive list" of all the λ-system that contain $\mathscr{A}$. Then $\bigcap_{\alpha \in A} \mathscr{L}_\alpha$ is a λ-system that contains $\mathscr{A}$. If $\mathscr{Q}$ is a λ-system on Ω that contains $\mathscr{A}$, then $\bigcap_{\alpha \in A} \mathscr{L}_\alpha \subseteq \mathscr{Q}$. The minimal λ-system generated by $\mathscr{A}$ always exists.*

c. *Let $\mathscr{L}$ denote a λ-system over Ω with $\mathscr{L} \supseteq \mathscr{A}$ and where $\mathscr{L}$ is contained in any other λ-system also containing $\mathscr{A}$. Then $\mathscr{L} = \bigcap_{\alpha \in A} \mathscr{L}_\alpha$, with notation as in (b). Therefore, the λ-system generated by $\mathscr{A}$ always exists and is unique.*

PROOF. **(a)** It is clear that $\Omega \in \bigcap_{\alpha \in A} \mathscr{L}_\alpha$. Suppose $A \in \bigcap_{\alpha \in A} \mathscr{L}_\alpha$, then $A \in \mathscr{L}_\alpha$ for any $\alpha \in A$. Hence, $A^c \in \mathscr{L}_\alpha$ for any $\alpha \in A$. So $A^c \in \bigcap_{\alpha \in A} \mathscr{L}_\alpha$, i.e., $\bigcap_{\alpha \in A} \mathscr{L}_\alpha$ is closed under complementation. To see $\bigcap_{\alpha \in A} \mathscr{L}_\alpha$ is closed under disjoint unions, let $\{A_n\}_{n=1}^\infty \subseteq \bigcap_{\alpha \in A} \mathscr{L}_\alpha$ be a disjoint sequence. Then $\{A_n\}_{n=1}^\infty \subseteq \mathscr{L}_\alpha$ for any α implies $\bigcup_{n=1}^\infty A_n \in \mathscr{L}_\alpha$ for any α, which implies that $\bigcup_{n=1}^\infty A_n \in \bigcap_{\alpha \in A} \mathscr{L}_\alpha$.

(b) From (a) we know $\bigcap_{\alpha \in A} \mathscr{L}_\alpha$ is a λ-system, and since $\mathscr{A} \subseteq \mathscr{L}_\alpha$, $\forall\ \alpha \in A$, we know that $\mathscr{A} \subseteq \bigcap_{\alpha \in A} \mathscr{L}_\alpha$; hence, $\bigcap_{\alpha \in A} \mathscr{L}_\alpha$ is a λ-system that contains $\mathscr{A}$. $\bigcap_{\alpha \in A} \mathscr{L}_\alpha \subseteq \mathscr{Q}$ because $\mathscr{Q} \in \{\mathscr{L}_\alpha : \alpha \in A\}$.

(c) Since $\mathscr{L}$ is contained in any other λ-system containing $\mathscr{A}$, and $\bigcap_{\alpha \in A} \mathscr{L}_\alpha$ is such a λ-system, so $\mathscr{L} \subseteq \bigcap_{\alpha \in A} \mathscr{L}_\alpha$. Since $\mathscr{L} \in \{\mathscr{L}_\alpha : \alpha \in A\}$, so $\bigcap_{\alpha \in A} \mathscr{L}_\alpha \subseteq \mathscr{L}$. □

1.1.3 Semiring

► EXERCISE 1.15 (1.1.13). *Is $\mathscr{A} = \{\varnothing\} \cup \{(0, x] \colon 0 < x \leqslant 1\}$ a semiring over $(0, 1]$?*

SOLUTION. $\mathscr{A}$ is not a semiring on $(0, 1]$. Take $(0, x]$ and $(0, y]$ with $x < y$. Then $(0, y] \smallsetminus (0, x] = (x, y] \notin \mathscr{A}$ since $x > 0$ by definition. □

► EXERCISE 1.16 (1.1.14). *This exercise explores some alternative definitions of a semiring.*

a. *Some define $\mathscr{A}$ to be a semiring iff $\mathscr{A}$ is a nonempty π-system such that both $E, F \in \mathscr{A}$ and $E \subseteq F$ imply the existence of a finite collection $C_0, C_1, \dots, C_n \in \mathscr{A}$ with $E = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n \subseteq F$ and $C_i \smallsetminus C_{i-1} \in \mathscr{A}$ for $i = 1, \dots, n$. This definition of a semiring is equivalent to our definition of a semiring.*

b. *Some define $\mathcal{A}$ to be a semiring by stipulating (SR1), (SR2), and the following property: $A, B \in \mathcal{A}$ implies the existence of disjoint $\mathcal{A}$-sets $C_0, C_1, \dots, C_n$ with $B \setminus A = \bigcup_{i=0}^{n} C_i$. Note that here $B \setminus A$ is not necessarily a* proper *difference. If $\mathcal{A}$ is a semiring by this definition, then $\mathcal{A}$ is a semiring by our definition, but the converse is not necessarily true.*

PROOF. **(a)** We first show that (SR1), (SR2), and (SR3) imply the above definition. (SR1) and (SR2) imply that $\mathcal{A}$ is a nonempty π-system (since $\varnothing \in \mathcal{A}$). Let $E, F \in \mathcal{A}$ and $E \subseteq F$. By (SR3) there exists disjoint $D_1, \dots, D_n \in \mathcal{A}$ such that $F \setminus E = \bigcup_{i=1}^{n} D_i$. Let $C_0 = E$ and $C_i = E \cup D_1 \cup \cdots \cup D_i$ for $i = 1, \dots, n$. Then $E = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n = F$, and $C_i \setminus C_{i-1} = D_i \in \mathcal{A}$.

Now suppose (a) holds. (SR1): Since $\mathcal{A}$ is nonempty, there exists $E \in \mathcal{A}$; since $E \subseteq E$, there exists a finite collection $E = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n \subseteq E$, which implies that $C_0 = C_1 = \cdots = C_n$, and so $C_i \setminus C_{i-1} = \varnothing \in \mathcal{A}$. (SR2) holds trivially. (SR3): Let $A, B \in \mathcal{A}$ and $A \subseteq B$. Then by the assumption, there exists a finite collection $C_0, C_1, \dots, C_n \in \mathcal{A}$ with $A = C_0 \subseteq C_1 \subseteq \cdots \subseteq C_n \subseteq B$, and $B_n = C_n \setminus C_{n-1} \in \mathcal{A}$. Then $\{B_i\}_{i=1}^{n} \subseteq \mathcal{A}$ is disjoint, and

$$A \cup \left(\bigcup_{i=1}^{n} B_i \right) = A \cup \left[\bigcup_{i=1}^{n} (C_i \setminus C_{i-1}) \right] = A \cup (B \setminus A) = B.$$

(b) Some authors do apply this definition, for example, see Aliprantis and Border (2006); Dudley (2002). The proof is obvious. □

► EXERCISE 1.17 (1.1.15). *Let $\mathcal{A}$ consist of $\varnothing$ as well as all rsc rectangles $(\boldsymbol{a}, \boldsymbol{b}]$. The collection of all finite disjoint unions of $\mathcal{A}$-sets is a semiring over $\mathbb{R}^k$.*

PROOF. We prove a more general theorem. See Bogachev (2007, Lemma 1.2.14, p.8).

For any semiring $\mathcal{S}$, the collection of all finite unions of sets in $\mathcal{S}$ forms a ring $\mathcal{R}$.

Proof. It is clear that the class $\mathcal{R}$ admits finite unions. Suppose that $A = \bigcup_{i=1}^{n} A_n$ and $B = \bigcup_{j=1}^{k} B_k$, where $A_i, B_i \in \mathcal{S}$. Then we have $A \cap B = \bigcup_{i \leqslant n, j \leqslant k} A_i \cap B_j \in \mathcal{R}$. Note that $A_i \cap B_j \in \mathcal{A}$, $\forall\ i \in \{1, \dots, n\}$ and $j \in \{1, \dots, k\}$, since a semiring is $\bigcap$-stable. Hence $\mathcal{R}$ admits finite intersections. In addition,

$$A \setminus B = \bigcup_{i=1}^{n} \left(A_i \setminus \bigcup_{j=1}^{k} B_j \right) = \bigcup_{i=1}^{n} \bigcap_{j=1}^{k} \left(A_i \setminus B_j \right).$$

Since the set $A_i \setminus B_j = A_i \setminus \left(A_i \cap B_j \right)$ is a finite union of sets in $\mathcal{S}$, one has $A_i \setminus B_j \in \mathcal{R}$. Furthermore, $\bigcap_{j=1}^{k} \left(A_i \setminus B_j \right) \in \mathcal{S}$ because $\mathcal{S}$ is $\bigcap$-stable. Finally, the finite list $\left\{ A_i \setminus B_j \right\}_{i \in \{1,\dots,n\}, j \in \{1,\dots,k\}}$ is disjoint; hence, $A \setminus B$ is a finite disjoint union of sets in $\mathcal{S}$.

Now, since $\mathcal{A}$ is a semiring [which is a well known fact], we conclude that the collection of all finite disjoint unions of $\mathcal{A}$-sets is a ring over $\mathbb{R}^k$ [a ring is a semiring, see Exercise 1.22 (p.10)]. $\square$

► EXERCISE 1.18 (1.1.16). *An arbitrary intersection of semirings on Ω is not necessarily a semiring on Ω.*

SOLUTION. Unlike the other kinds of classes of families of sets (e.g., Exercise 1.8 and Exercise 11.2), the intersection of a collection of semirings need not be a semiring. For example, let $\Omega = \{0, 1, 2\}$, $\mathcal{A}_1 = \{\varnothing, \Omega, \{0\}, \{1\}, \{2\}\}$, and $\mathcal{A}_2 = \{\varnothing, \Omega, \{0\}, \{1, 2\}\}$. Then $\mathcal{A}_1$ and $\mathcal{A}_2$ are semirings (in fact, $\mathcal{A}_2$ is a field), but their intersection $\mathcal{A} = \mathcal{A}_1 \cap \mathcal{A}_2 = \{\varnothing, \Omega, \{0\}\}$ is not a semiring as $\Omega \smallsetminus \{0\} = \{1, 2\}$ is not a disjoint union of sets in $\mathcal{A}$.

Generally, let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two semirings, and $\Omega \in \mathcal{A}_1$ and $\Omega \in \mathcal{A}_2$. Then $\Omega \in \mathcal{A}_1 \cap \mathcal{A}_2$, and which means that the complement of every element in $\mathcal{A}_1 \cap \mathcal{A}_2$ should be expressed as finite union of disjoint sets in $\mathcal{A}_1 \cap \mathcal{A}_2$. As we have seen in the example, this is a demanding requirement.

Of course, there is no pre-requirement that Ω should be in a semiring. See the next Exercise 1.19. $\square$

► EXERCISE 1.19 (1.1.17). *If $\mathcal{A}$ is a semiring over Ω, must $\Omega \in \mathcal{A}$?*

SOLUTION. Not necessarily. In face, the simplest example of a semiring (a ring, a σ-ring) is just $\{\varnothing\}$. $\square$

► EXERCISE 1.20 (1.1.18). *Let $\mathcal{A}$ denote a semiring. Pick $n \in \mathbb{N}$, and let $A, A_1, \ldots, A_n \in \mathcal{A}$. Then there exists a finite collection $\{C_1, \ldots, C_m\}$ of disjoint $\mathcal{A}$-sets with $A \smallsetminus \bigcup_{i=1}^n A_i = \bigcup_{j=1}^m C_j$.*

PROOF. When $n = 1$, write $A \smallsetminus A_1 = A \smallsetminus (A \cap A_1)$ and invoke (SR3). Now assume that the result is true for $n \in \mathbb{N}$. Consider $n + 1$.

$$A \smallsetminus \bigcup_{i=1}^{n+1} A_i = \left(A \smallsetminus \bigcup_{i=1}^{n} A_i \right) \smallsetminus A_{n+1} = \left(\bigcup_{j=1}^{m} C_j \right) \smallsetminus A_{n+1} = \bigcup_{j=1}^{m} \left(C_j \smallsetminus A_{n+1} \right).$$

Now for each j, there exists disjoint sets $\{D_1^j, \ldots, D_{k_j}^j\} \subseteq \mathcal{A}$ such that

$$C_j \smallsetminus A_{n+1} = \bigcup_{r=1}^{k_j} D_r^j.$$

Then $\{D_r^j : j = 1, \ldots, m, r = 1, \ldots, m_j\}$ is a finite pairwise disjoint subset of $\mathcal{A}$, and

$$A \smallsetminus \bigcup_{i=1}^{n+1} A_i = \bigcup_{j=1}^{m} \bigcup_{r=1}^{m_j} D_r^j. \qquad \square$$

► EXERCISE 1.21 (1.1.19). *Other books deal with a system called a* ring. *We will not deal with rings of sets in this text, but since the reader might refer to other books that deal with rings, it is worthy to discuss the concept. A collection $\mathcal{R}$ of subsets of a nonempty set Ω is called a* ring *of subsets of Ω iff*

(R1) $\mathcal{R} \neq \varnothing$,

(R2) $A, B \in \mathcal{R}$ *implies* $A \cup B \in \mathcal{R}$, *and*

(R3) $A, B \in \mathcal{R}$ *implies* $A \setminus B \in \mathcal{R}$.

That is, a ring is a nonempty collection of subsets closed under unions and differences.

a. $\varnothing$ *is in every ring.*

b. $\mathcal{R}$ *is a ring iff* $\mathcal{R}$ *satisfies (R1), (R2), and*

 (R4) $A, B \in \mathcal{R}$ *with* $A \subseteq B$ *implies* $B \setminus A \in \mathcal{R}$.

c. *Every ring satisfies*

 (R5) $A, B \in \mathcal{R}$ *implies* $A \Delta B \in \mathcal{R}$.

d. *Every ring is a π-system.*

e. *Every ring is closed under finite unions and finite intersections.*

f. $\mathcal{R}$ *is a ring iff* $\mathcal{R}$ *a nonempty π-system that satisfies (R4) along with*

 (R6) $A, B \in \mathcal{R}$ *and* $A \cap B = \varnothing$ *imply* $A \cup B \in \mathcal{R}$.

g. $\mathcal{R}$ *is a ring iff* $\mathcal{R}$ *is a nonempty π-system that satisfies (R5).*

h. *Suppose that* $\{\mathcal{R}_\alpha : \alpha \in A\}$ *is the "exhaustive list" of all rings that contain* $\mathcal{A}$. *Then* $\bigcap_{\alpha \in A} \mathcal{R}_\alpha$ *is a ring that contains* $\mathcal{A}$, *and* $\bigcap_{\alpha \in A} \mathcal{R}_\alpha$ *is contained in any ring that contains* $\mathcal{A}$. *The minimal ring containing* $\mathcal{A}$ *is always exists and is unique.*

i. *The collection of finite unions of rsc intervals is a ring on* $\mathbb{R}$.

j. *Let Ω be uncountable. The collection of all amc subsets of Ω is a ring on Ω.*

PROOF. **(a)** By (R1), there exists some set $A \in \mathcal{R}$, it follows from (R3) that $\varnothing = A \setminus A \in \mathcal{R}$.

(b) We need only to prove that (R3) $\Longleftrightarrow$ (R4) under (R1) and (R2).

- (R3) $\Longrightarrow$ (R4) is obvious.
- (R4) $\Longrightarrow$ (R3): Let $A, B \in \mathcal{R}$, and note that $B \setminus A = (B \cup A) \setminus A \in \mathcal{R}$ since $A \subseteq A \cup B$, and $B \cup A \in \mathcal{R}$ by (R2).

(c) Let $A, B \in \mathcal{R}$. By (R3), $A \smallsetminus B \in \mathcal{R}$, and $B \smallsetminus A \in \mathcal{R}$; by (R2), $(A \smallsetminus B) \cup (B \smallsetminus A) \in \mathcal{R}$. Observe that $A \Delta B = (A \smallsetminus B) \cup (B \smallsetminus A)$, and we complete the proof.

(d) Let $A, B \in \mathcal{R}$. It is clear that $A \cap B = (A \cup B) \smallsetminus (A \Delta B)$. Note that $A \cup B \in \mathcal{R}$ [by (R2)], $A \Delta B \in \mathcal{R}$ [by (c)], and $(A \cup B) \smallsetminus (A \Delta B) \in \mathcal{R}$ [by (R3)]. Therefore, $A \cap B \in \mathcal{R}$ and $\mathcal{R}$ is a π-system.

(e) Just follows (R2) and (d).

(f) To see the *only if* part, suppose $\mathcal{R}$ is a ring. Then (d) means that $\mathcal{R}$ is a nonempty π-system, (R3) $\Longrightarrow$ (R4) [by part (b)], and (R2) $\Longrightarrow$ (R6) [by definition].

Now we prove the *if* part. Note that

(R1) By assumption;

(R2) Let $A, B \in \mathcal{R}$. We can write $A \cup B$ as

$$\begin{aligned} A \cup B &= (A \smallsetminus B) \cup (B \smallsetminus A) \cup (A \cap B) \\ &= \left[A \smallsetminus (A \cap B)\right] \cup \left[B \smallsetminus (A \cap B)\right] \cup (A \cap B). \end{aligned}$$

Now (R4) implies that $\left[A \smallsetminus (A \cap B)\right] \in \mathcal{R}$, and $\left[B \smallsetminus (A \cap B)\right] \in \mathcal{R}$; (R6) implies that $A \cup B \in \mathcal{R}$.[2]

(R3) Let $A, B \in \mathcal{R}$. Then $A \smallsetminus B = \varnothing \cup (A \smallsetminus B) = (A \cap A^c) \cup (A \cap B^c) = A \cap (A^c \cup B^c) = A \cap (A \cap B)^c = A \smallsetminus (A \cap B)$. Clearly, $A \cap B \subseteq A$, so (R4) implies that $A \smallsetminus B \in \mathcal{R}$.

(g) To see the *only if* part, suppose that $\mathcal{R}$ is a ring. Then (R1) and (d) implies $\mathcal{R}$ is a nonempty π-system, and we have (R5) by (c).

For the inverse direction, suppose that $\mathcal{R}$ satisfies the given assumptions.

(R1) $\mathcal{R} \neq \varnothing$ by assumption;

(R2) Let $A, B \in \mathcal{R}$. Then$A \cup B = (A \Delta B) \cup (A \cap B) = (A \Delta B) \, \Delta \, (A \cap B)$. Since $\mathcal{R}$ is a π-system, $A \cap B \in \mathcal{R}$. Thus, (R5) implies (R2).

(R3) Let $A, B \in \mathcal{R}$. Note that $A \smallsetminus B = (A \Delta B) \cap A$. Then (R5) implies that $A \Delta B \in \mathcal{R}$, and $(A \Delta B) \cap A \in \mathcal{R}$ since $\mathcal{R}$ is a π-system.[3]

(h) Similar to Exercise 1.8 and Exercise 11.2.

(i) See Exercise 11.5 (p.147).

(j) (R1) is trivial. (R2) holds because every finite (in fact, countable) union of amc sets is amc (see, e.g., Rudin 1976). To see (R3), let A, B be amc. Since $A \smallsetminus B = A \smallsetminus (A \cap B) \subseteq A$, and $A \cap B \subseteq A$, we know that $A \smallsetminus B$ is amc. □

► Exercise 1.22 (1.1.20). *This problem explores the relationship between semirings and rings.*

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[2] For $A \smallsetminus B = A \smallsetminus (A \cap B)$, see part (g) of this exercise

[3] Vestrup (2003, p.6) hints that $A \smallsetminus B = A \Delta (A \cap B)$.

a. *Every ring is a semiring. However, not every semiring is a ring.*

b. *Let $\mathcal{A}$ denote a semiring on Ω, and let $\mathcal{R}$ consist of the finite disjoint unions of $\mathcal{A}$-sets. Then $\mathcal{R}$ is closed under finite intersections and disjoint unions.*

c. *If $A, B \in \mathcal{A}$ and $A \subseteq B$, then $B - A \in \mathcal{R}$.*

d. *$A \in \mathcal{A}$, $B \in \mathcal{R}$, and $A \subseteq B$ imply $B - A \in \mathcal{R}$.*

e. *$A, B \in \mathcal{R}$ and $A \subseteq B$ imply $B - A \in \mathcal{R}$.*

f. *$\mathcal{R}$ is the minimal ring generated by $\mathcal{A}$.*

g. *A semiring that satisfies (R2) is a ring.*

Proof. **(a)** Let $\mathcal{R}$ be a ring. Then (R1) [$\mathcal{R} \neq \varnothing$] and (R3) [$\mathcal{R}$ is closed under differences] imply that there exists $A \in \mathcal{R}$ such that $\varnothing = A \smallsetminus A \in \mathcal{R}$. Thus, (SR1) is satisfied. To see that $\mathcal{R}$ satisfies (SR2) [$\mathcal{R}$ is a π-system], refer Exercise 1.21 (d). Finally, (R4) [Exercise 1.21 (b)] implies (SR3).

To see a semiring is not necessary a ring, note that the collection $\mathcal{S} := \{\varnothing, (a, b] \mid a, b \in \mathbb{R}\}$ is a semiring, but is not a ring: let $-\infty < a < b < c < d < +\infty$, then $(a, b] \cup (c, d] \notin \mathcal{S}$.

Note that a semiring $\mathcal{S}$ is a ring *if* for any $A, B \in \mathcal{S}$ we have $A \cup B \in \mathcal{S}$ [Figure 1.1 (p.1), and part (g) of this exercise]. Any semiring generates a ring as in the Claim in Exercise 11.5 (p.147).

(b) Let $\mathcal{A}$ be a semiring on Ω, and let

$$\mathcal{R} := \left\{ \bigcup_{i=1}^{n} A_i : A_i \in \mathcal{A} \text{ and } n \in \mathbb{N} \right\}.$$

To prove $\mathcal{R}$ is closed under finite intersections, let $A = \bigcup_{j=1}^{m} A_j$, and $B = \bigcup_{k=1}^{n} B_k$, where the A_j's are disjoint and in $\mathcal{A}$, as are the B_k's. Then

$$A \cap B = \left(\bigcup_{j=1}^{m} A_j \right) \cap \left(\bigcup_{k=1}^{n} B_k \right) = \bigcup_{j=1}^{m} \bigcup_{k=1}^{n} \left(A_j \cap B_k\right) = \bigcup_{\substack{1 \leq j \leq m \\ 1 \leq k \leq n}} \left(A_j \cap B_k\right) \overset{\langle 1 \rangle}{\in} \mathcal{R},$$

where $\langle 1 \rangle$ holds because the $\left(A_j \cap B_k\right)$'s are disjoint and in $\mathcal{A}$ [by (SR2)]. Since the intersection of any two sets in $\mathcal{R}$ is in $\mathcal{R}$, it follows by induction that so is the intersection of finitely many sets in $\mathcal{R}$.

A disjoint union of finitely many sets in $\mathcal{R}$ is clearly in $\mathcal{R}$.

(c) Let $A, B \in \mathcal{A}$ and $A \subseteq B$. Then by (SR3), there exists disjoint $C_1, \ldots, C_k \in \mathcal{A}$ with $B \smallsetminus A = \bigcup_{i=1}^{k} C_i$. Thus, $B \smallsetminus A \in \mathcal{R}$ by definition.

(d) Let $A \in \mathcal{A}$, $B \in \mathcal{R}$, and $A \subseteq B$. Then,

$$B - A \overset{\langle 2 \rangle}{=} \left(\bigcup_{i=1}^{n} A_i \right) - A = \bigcup_{i=1}^{n} (A_i \setminus A) = \bigcup_{i=1}^{n} \left[A_i - (A_i \cap A) \right] \overset{\langle 3 \rangle}{\in} \mathcal{R},$$

where $\langle 2 \rangle$ follows the fact that $B \in \mathcal{R}$ [the A_i's are in $\mathcal{A}$ and disjoint], and $\langle 3 \rangle$ follows part (c) in this problem [note that $A_i \in \mathcal{A}, A \in \mathcal{A}$, and by (SR2), $A_i \cap A \in \mathcal{A}$].

(e) Let $A, B \in \mathcal{R}$ and $A \subseteq B$. Then

$$B \setminus A = \left(\bigcup_{k=1}^{n} B_k \right) \setminus \left(\bigcup_{j=1}^{m} A_j \right) = \bigcup_{k=1}^{n} \left[B_k \setminus \left(\bigcup_{j=1}^{m} A_j \right) \right] = \bigcup_{k=1}^{n} \left[\bigcap_{j=1}^{m} \left(B_k \setminus A_j \right) \right].$$

Note that $B_k, A_j \in \mathcal{A}$, then $B_k \cap A_j \in \mathcal{A}$ [(SR2)], and by part (c),

$$B_k \setminus A_j = B_k - \left(B_k \cap A_j \right) \in \mathcal{R}.$$

Furthermore, by part (b), $\bigcap_{j=1}^{m} \left(B_k \setminus A_j \right) \in \mathcal{R}$, and so $B - A \in \mathcal{R}$.

(f) Let $\Re(\mathcal{A})$ be the class of rings containing $\mathcal{A}$, and let $\mathcal{C} \in \Re(\mathcal{A})$. By definition, if $A \in \mathcal{R}$, then $A = \bigcup_{i=1}^{n} A_i$, where $\{A_i\}_{i=1}^{n} \subseteq \mathcal{A}$ are disjoint. Then $A \in \mathcal{C}$ since $\mathcal{C}$ is a ring containing $\mathcal{A}$. Hence, $\mathcal{R}$ is the minimal ring containing $\mathcal{A}$.

(g) Let $\mathcal{A}$ be a semiring satisfying (R2) [$A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$]. Then $\mathcal{A}$ is nonempty since $\varnothing \in \mathcal{A}$ by definition of a semiring. By (R2), $\mathcal{A}$ is $\bigcup$-stable; hence, to prove $\mathcal{A}$ is a ring, we need only to prove that $\mathcal{A}$ is closed under difference. Let $A, B \in \mathcal{A}$. Then

$$A \setminus B = A - (A \cap B) \overset{1}{=} \bigcup_{i=1}^{k} C_i \in \mathcal{A},$$

where $\{C_i\}_{i=1}^{k} \subseteq \mathcal{A}$ are disjoint, and equality (1) follows (SR3). □

► EXERCISE 1.23 (1.1.21). *Let Ω be infinite, and let $\mathcal{A} \subseteq 2^{\Omega}$ have cardinality $\aleph_0$. We will show that the ring generated by $\mathcal{A}$ has cardinality $\aleph_0$.*

a. *Given $\mathcal{C} \subseteq 2^{\Omega}$, let $\mathcal{C}^*$ denote the collection of all finite unions of differences of $\mathcal{C}$-sets. If* $\operatorname{card}(\mathcal{C}) = \aleph_0$, *then* $\operatorname{card}(\mathcal{C}^*) = \aleph_0$. *Also, $\varnothing \in \mathcal{C}$ implies $\mathcal{C} \subseteq \mathcal{C}^*$.*

b. *Let $\mathcal{A}_0 = \mathcal{A}$, and define $\mathcal{A}_n = \mathcal{A}_{n-1}^*$ for $n \geqslant 1$. Then $\mathcal{A} \subseteq \bigcup_{n=0}^{\infty} \mathcal{A}_n \subseteq \Re(\mathcal{A})$, where $\Re(\mathcal{A})$ is the minimal ring generated by $\mathcal{A}$ and where [without loss of generality] $\varnothing \in \mathcal{A}$. Also,* $\operatorname{card}(\bigcup_{n=0}^{\infty} \mathcal{A}_n) = \aleph_0$.

c. *$\bigcup_{n=0}^{\infty} \mathcal{A}_n$ is a ring on Ω, and from the fact that $\Re(\mathcal{A})$ is the minimal ring containing $\mathcal{A}$, we have $\bigcup_{n=0}^{\infty} \mathcal{A}_n = \Re(\mathcal{A})$, and thus* $\operatorname{card}(\Re(\mathcal{A})) = \aleph_0$.

d. *We may generalize: if $\mathcal{A}$ is infinite, then* $\operatorname{card}(\mathcal{A}) = \operatorname{card}(\Re(\mathcal{A}))$.

PROOF. **(a)** Let $\mathcal{C}' := \{C_i \setminus C_j : C_i, C_j \in \mathcal{C}\}$. Since $\operatorname{card}(\mathcal{C}) = \aleph_0$ [$\mathcal{C}$ is countable], we can write $\mathcal{C}$ as

$$\mathcal{C} = \{C_n\}_{n=1}^{\infty}.$$

We now show that $\operatorname{card}(\mathcal{C}') = \aleph_0$. Notice that for any $C_n \in \mathcal{C}$, we can construct a bijection on $\mathbb{N}$ onto $C_n \setminus \mathcal{C} := \{C_n \setminus C_i : C_i \in \mathcal{C}\}$ as follows

$$f_{C_n}(i) = C_n \setminus C_i,$$

but which means that $C_n \setminus \mathcal{C}$ is countable. Then,

$$\mathcal{C}' = \bigcup_{C_n \in \mathcal{C}} [C_n \setminus \mathcal{C}]$$

is a countable union of countable sets, so it is countable [under the Axiom of Choice, see (Hrbacek and Jech, 1999, Corollary 3.6, p. 75)].

Now we show that for any $n \in \mathbb{N}$, the set $\mathcal{C}_n^*$ defined by

$$\mathcal{C}_n^* = \left\{ \bigcup_{i=1}^{n} C_i' : C_i' \in \mathcal{C}' \text{ and } C_i' \neq C_j' \text{ whenever } i \neq j \right\}$$

is countable. We prove this claim with the Induction Principle on $n \in \mathbb{N}$. Clearly, this claim holds with $n = 1$ since in this case, $\mathcal{C}_1^* = \mathcal{C}'$. Assume that it is true for some $n \in \mathbb{N}$. We need to prove the case of $n + 1$. However,

$$\mathcal{C}_{n+1}^* = \mathcal{C}_n^* \cup \bar{C}',$$

where

$$\bar{\mathcal{C}}' := \left\{ C' \in \mathcal{C}' : C' \neq C_i' \ \forall\ i \leqslant n \right\}.$$

Because $\mathcal{C}'$ is countable, we conclude that $\bar{\mathcal{C}}' \subseteq \mathcal{C}'$ is amc. Therefore, $\mathcal{C}_{n+1}^*$ is countable. Hence, by the Induction Principle, $\operatorname{card}(\mathcal{C}_n^*) = \aleph_0$ for any $n \in \mathbb{N}$, and

$$\mathcal{C}^* = \bigcup_{n \in \mathbb{N}} \mathcal{C}_n^* \tag{1.3}$$

is countable.

We now show that if $\varnothing \in \mathcal{C}$, then $\mathcal{C} \subseteq \mathcal{C}^*$. Let $C \in \mathcal{C}$, then $C \in \mathcal{C}'$ because $C = C \setminus \varnothing$; therefore,

$$\mathcal{C} \subseteq \mathcal{C}' \subseteq \mathcal{C}^*.$$

[Remember that $\mathcal{C}' = \mathcal{C}_1^*$ and (11.1).]

(b) By the definition of $\mathcal{A}_n$, we know $\mathcal{A}_1 = \mathcal{A}^*$, the collection of all finite unions of differences of $\mathcal{A}$ sets. Since $\varnothing \in \mathcal{A}$, we know from part (a) that $\mathcal{A} \subseteq \mathcal{A}^* = \mathcal{A}_1$; therefore,

$$\mathcal{A} \subseteq \bigcup_{n=0}^{\infty} \mathcal{A}_n. \tag{1.4}$$

We are now ready to prove that $\bigcup_{n=0}^{\infty} \subseteq \Re(\mathscr{A})$. We use the Induction Principle to prove that

$$\mathscr{A}_i \subseteq \Re(\mathscr{A}), \quad \forall\, i \in \mathbb{N}. \tag{Pi}$$

Clearly, **P**0 holds as $\mathscr{A}_0 = \mathscr{A} \subseteq \Re(\mathscr{A})$. Now assume **P**$n$ holds. We need to prove **P**$n+1$. Notice that $\mathscr{A}_{n+1} = \mathscr{A}^*$, the collection of all finite unions of differences of $\mathscr{A}_n$-sets, we can write a generic element of $\mathscr{A}_{n+1}$ as

$$A_{n+1} = \bigcup_{j=1}^{m} A'_j,$$

where $A'_j = A'_n \smallsetminus A''_n$, and $A'_n, A''_n \in \mathscr{A}_n$. Since $\mathscr{A}_n \subseteq \Re(\mathscr{A})$ by **P**n, we know that $A_j = A'_n \smallsetminus A''_n \in \Re(\mathscr{A})$ by (R3); therefore, $A_{n+1} = \bigcup_{j=1}^{m} A'_j \in \Re(\mathscr{A})$ by (R2). This proves **P**$n+1$. Then, by the Induction Principle, we know that $\mathscr{A}_n \subseteq \Re(\mathscr{A})$, $\forall\, n \in \mathbb{N}$; therefore,

$$\bigcup_{n=0}^{\infty} \mathscr{A}_n \subseteq \Re(\mathscr{A}). \tag{1.5}$$

Combine (11.2) and (1.5) we have

$$\mathscr{A} \subseteq \bigcup_{n=0}^{\infty} \mathscr{A}_n \subseteq \Re(\mathscr{A}). \tag{1.6}$$

To prove $\mathrm{card}(\bigcup_{n=0}^{\infty} \mathscr{A}_n) = \aleph_0$, we first use the Induction Principle again to prove that $\mathscr{A}_n$ is countable, $\forall\, n \in \mathbb{N}$. Clearly, $\mathscr{A}_1 = \mathscr{A}^*$ is countable by part (a). Assume $\mathscr{A}_n$ is countable, then $\mathscr{A}_{n+1} = \mathscr{A}^*$ is countable by part (a) once again. Therefore, $\bigcup_{n=0}^{\infty} \mathscr{A}_n$ is countable [under the Axiom of Choice].

(c) Clearly, $\bigcup_{n=0}^{\infty} \mathscr{A}_n := \widetilde{\mathscr{A}} \neq \varnothing$, so (R1) is satisfied. To see (R2) and (R3), let $A, B \in \widetilde{\mathscr{A}}$. Then there exist $m, n \in \mathbb{N}$ such that $A \in \mathscr{A}_m$ and $B \in \mathscr{A}_n$. We have shown in part (a) that

$$\mathscr{A}_{n+1} = \mathscr{A}_n^* \supseteq \mathscr{A}_n$$

[along with the Induction Principle]. Therefore, either $\mathscr{A}_m \subseteq \mathscr{A}_n$ [if $m \leqslant n$] or $\mathscr{A}_n \subseteq \mathscr{A}_m$ [if $n \leqslant m$]. Without loss of generality, we assume that $m \leqslant n$, i.e., $\mathscr{A}_m \subseteq \mathscr{A}_n$; therefore, $A \in \mathscr{A}_m \Longrightarrow A \in \mathscr{A}_n$. Therefore, $A, B \in \mathscr{A}_n$ implies that

$$A \cup B = (A \smallsetminus \varnothing) \cup (B \smallsetminus \varnothing) \in \mathscr{A}_n^* = \mathscr{A}_{n+1} \subseteq \widetilde{\mathscr{A}},$$

[this proves (R2)], and

$$A \smallsetminus B = \left(\bigcup_{i=1}^{n_A} A_i\right) \smallsetminus \left(\bigcup_{j=1}^{n_B} B_j\right) = \bigcup_{i=1}^{n_A+n_B} \left(A_i \smallsetminus B_j\right) \in \mathscr{A}_n^* \subseteq \bigcup_{n=1}^{\infty} \mathscr{A}_n,$$

[this proves (R3)]. Hence, $\widetilde{\mathscr{A}}$ is a ring, and $\widetilde{\mathscr{A}} = \Re(\mathscr{A})$; furthermore, we have $\mathrm{card}(\Re(\mathscr{A})) = \aleph_0$.

(d) Straightforward. □

1.2 Fields

► Exercise 1.24 (1.2.1). *The collection $\mathcal{F} = \{A \subseteq \Omega : A$ is finite or A^c is finite$\}$ is a field on Ω.*

Proof. $\Omega \in \mathcal{F}$ because $\Omega^c = \varnothing$ is finite; let $A \in \mathcal{F}$. If A is finite, $A^c \in \mathcal{F}$ as $(A^c)^c = A$ is finite; if A^c is finite $A^c \in \mathcal{F}$. Thus, $\mathcal{F}$ is closed under complements. Finally, let $A, B \in \mathcal{F}$. There are two cases: (i) both A and B are finite, then $A \cup B$ is finite, whence $A \cup B \in \mathcal{F}$; (ii) at least one of A^c or B^c is finite. Assume that B^c is. We have $(A \cup B)^c = A^c \cap B^c \subseteq B^c$, and thus $(A \cup B)^c$ is finite, so that gain $A \cup B \in \mathcal{F}$. □

► Exercise 1.25 (1.2.2). *Let $\mathcal{F} \subseteq 2^{\Omega}$ be such that $\Omega \in \mathcal{F}$ and $A \smallsetminus B \in \mathcal{F}$ whenever $A, B \in \mathcal{F}$. Then $\mathcal{F}$ is a field on Ω.*

Proof. We need to check $\mathcal{F}$ satisfies (F1)–(F3). $\Omega \in \mathcal{F}$ by assumption. Let $A = \Omega$ and $B \in \mathcal{F}$. Then $B^c = \Omega \smallsetminus B \in \mathcal{F}$. Let $A, B \in \mathcal{F}$. Then $A^c, B^c \in \mathcal{F}$. Since $(A \cup B)^c = A^c \cap B^c = A^c \smallsetminus B \in \mathcal{F}$, we must have $A \cup B = \left[(A \cup B)^c\right]^c \in \mathcal{F}$. □

► Exercise 1.26 (1.2.3). *Every λ-system that is closed under* arbitrary *differences is a field.*

Proof. We only need to show that it is closed under finite unions, and it comes from the previous exercise. □

► Exercise 1.27 (1.2.4). *Let $\mathcal{F} \subseteq 2^{\Omega}$ satisfy (F1) and (F2), and suppose that $\mathcal{F}$ is closed under finite* disjoint *unions. Then $\mathcal{F}$ is not necessarily a field.*

Solution. For example, let $\Omega = \{1, 2, 3, 4\}$, and

$$\mathcal{F} = \left\{\varnothing, \Omega, \{1, 2\}, \{3, 4\}, \{2, 3\}, \{1, 4\}\right\}.$$

$\mathcal{F}$ satisfies all the requirements, but which is not a field since, for example,

$$\{1, 2\} \cup \{2, 3\} = \{1, 2, 3\} \notin \mathcal{F}.$$ □

► Exercise 1.28 (1.2.5). *Suppose that $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}_3 \subseteq \cdots$, where $\mathcal{F}_n$ is a field on Ω for each $n \in \mathbb{N}$. Then $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is a field on Ω.*

Proof. (F1) $\Omega \in \mathcal{F}_n$, for each $n \in \mathbb{N}$, so $\Omega \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$; [Of course, it is enough to check that $\Omega \in \mathcal{F}_n$ for some $\mathcal{F}_n$.] (F2) Suppose $A \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$. Then there exist $n \in \mathbb{N}$ such that $A \in \mathcal{F}_n$. So $A^c \in \mathcal{F}_n \implies A^c \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$; (F3) Let $A, B \in \bigcup_{n=1}^{\infty} \mathcal{F}_n$. Then $\exists\ m \in \mathbb{N}$ such that $A \in \mathcal{F}_m$, and $\exists\ n \in \mathbb{N}$ such that $B \in \mathcal{F}_n$. Hence, $A \cup B \in \mathcal{F}_m \cup \mathcal{F}_n \subseteq \bigcup_{n=1}^{\infty} \mathcal{F}_n$. □

► Exercise 1.29 (1.2.6). *The collection consisting of $\mathbb{R}^k$, $\varnothing$, and all k-dimensional rectangles of all forms fails to be a field on $\mathbb{R}^k$.*

SOLUTION. Consider $k = 1$ and $[a, b]$, where $a, b \in \mathbb{R}$. Then $[a, b]^c = (-\infty, a) \cup (b, +\infty)$ is not a interval.

The $k \geqslant 2$ case can be generalized easily. For example, let

$$\boldsymbol{A} = \bigtimes_{i=1}^{k} [a_i, b_i].$$

Then $\boldsymbol{A}^c$ is not a rectangle. □

► EXERCISE 1.30 (1.2.7). *The collection consisting of ∅ and the finite disjoint unions of k-dimensional rsc subrectangles of the given k-dimensional rsc rectangle $(\boldsymbol{a}, \boldsymbol{b}]$ is a field on Ω.*

PROOF. A more general proposition can be found in Folland (1999, Proposition 1.7). Denote the set system given in the problem as $\mathcal{S}$, a semiring, and the collection of ∅ and the finite disjoint unions of k-dimensional rsc subrectangles as $\mathcal{A}$ First $\Omega = \bigcup_{i \in \varnothing} I_i$ by definition, where $I_i \in \mathcal{S}$. If $A, B \in \mathcal{S}$ and $B^c = \bigcup_{i=1}^{n} C_i$, where $C_i \in \mathcal{S}$. Then $A \smallsetminus B = \bigcup_{i=1}^{n} (A \cap C_i)$ and $A \cup B = (A \smallsetminus B) \cup B$. Hence $A \smallsetminus B \in \mathcal{A}$ and $A \cup B \in \mathcal{A}$. It now follows by induction that if $A_1, \dots, A_n \in \mathcal{S}$, then $\bigcup_{i=1}^{n} A_i \in \mathcal{A}$. It is easy to see that $\mathcal{A}$ is closed under complements. □

► EXERCISE 1.31 (1.2.8). *An arbitrary intersection of fields on Ω is a field on Ω.*

PROOF. Let $\{\mathcal{F}_\alpha : \alpha \in A\}$ be a set of fields on Ω, where A is some arbitrary set of indexes. Then

(F1) $\Omega \in \bigcap_{\alpha \in A} \mathcal{F}_\alpha$ since $\Omega \in \mathcal{F}_\alpha$ for any $\alpha \in A$.

(F2) Let $B \in \bigcap_{\alpha \in A} \mathcal{F}_\alpha$, then $A^c \in \mathcal{F}_\alpha$, for any $\alpha \in A$; hence $A^c \in \bigcap_{\alpha \in A} \mathcal{F}_\alpha$.

(F3) Let $B, C \in \bigcap_{\alpha \in A} \mathcal{F}_\alpha$. Then $B, C \in \mathcal{F}_\alpha$, $\forall\, \alpha \in A$. Hence, $B \cup C \in \mathcal{F}_\alpha$, $\forall\, \alpha \in A$, and $B \cap C \in \bigcap_{\alpha \in A} \mathcal{F}_\alpha$.

□

► EXERCISE 1.32 (1.2.9). *Let Ω be arbitrary, and let $\mathcal{A} \subseteq 2^\Omega$. There exists a unique field $\mathcal{F}$ on Ω with the properties that (i) $\mathcal{A} \subseteq \mathcal{F}$, and (ii) if $\mathcal{G}$ is a field with $\mathcal{A} \subseteq \mathcal{G}$, then $\mathcal{F} \subseteq \mathcal{G}$. This field $\mathcal{F}$ is called the [minimal] field [on Ω] generated by $\mathcal{A}$.*

PROOF. Let $\{\mathcal{F}_\alpha : \alpha \in A\}$ be the exhaustive set of fields on Ω containing $\mathcal{A}$. Then $\bigcap_{\alpha \in A} \mathcal{F}_\alpha$ is the desired field. □

► EXERCISE 1.33 (1.2.10). *Let $A_1, \dots, A_n \subsetneq \Omega$ be disjoint. What does a typical element in the minimal field generated by $\{A_1, \dots, A_n\}$ look like?*

SOLUTION. Refer to Ash and Doléans-Dade (2000, Exercise 1.2.8). To save notation, let $\mathbf{F}$ denote the minimal field generated by $\mathcal{A} := \{A_1, \dots, A_n\}$. We consider an element of $\mathbf{F} \smallsetminus \{\Omega, \varnothing\}$. We can write a typical element $B \in \mathbf{F}$ as follows,

$$B = B_1 * B_2 * \cdots * B_m,$$

where $*$ is an set operation either $\cup$ or $\cap$, and $B_i \in \left\{A_1, \ldots, A_n, A_1^c, \ldots, A_n^c\right\}$ for each $i \in \{1, \ldots, m\}$. ◻

► EXERCISE 1.34 (1.2.11). *Let S be finite, and Ω denote the set of sequences of elements of S. For each $\omega \in \Omega$, write*

$$\omega = \left(z_1(\omega), z_2(\omega), \ldots\right),$$

so that $z_k(\omega)$ denotes the k-th term of ω for all $k \in \mathbb{N}$. For $n \in \mathbb{N}$ and $H \subseteq S^n$, let

$$C_n(H) := \{\omega \in \Omega \mid z_1(\omega), \ldots, z_n(\omega) \in H\}.$$

Let

$$\mathcal{F} := \left\{C_n(H) \,\middle|\, n \in \mathbb{N}, H \subseteq S^n\right\}.$$

Then $\mathcal{F}$ is a field of subsets of S^∞. [The sets $C_n(H)$ are called cylinders of rank *n, and $\mathcal{F}$ is collection of all cylinders of all ranks.]*

PROOF. See Remark 1.2 (p.2) for more details about Cylinders. To prove $\mathcal{F}$ is a field, note that

(F1) $\Omega \in \mathcal{F}$. Consider $C_\infty(S^\infty)$; then $\omega \in C_\infty(S^\infty)$, $\forall\ \omega \in \Omega$, which means $\Omega \subseteq C_\infty(S^\infty)$. Hence,

$$\Omega = C_\infty\left(S^\infty\right) \in \mathcal{F}.$$

(F2) To prove that $\mathcal{F}$ is closed under complements, consider any $C_n(H) \in \mathcal{F}$. By definition,

$$C_n(H) := \{\omega \in \Omega \mid z_1(\omega), \ldots, z_n(\omega) \in H\}.$$

Then,

$$\begin{aligned}
\left[C_n(H)\right]^c &= \left\{\omega \in \Omega : [z_1(\omega), \ldots, z_n(\omega)] \notin H\right\} \\
&= \left\{\omega \in \Omega : [z_1(\omega), \ldots, z_n(\omega)] \in H^c\right\} \\
&= C_n\left(H^c\right) \\
&\in \mathcal{F}.
\end{aligned}$$

(π-system) Finally, we need to prove $\mathcal{F}$ is closed under finite intersections.[4]

[4] It is hard to prove that $\mathcal{F}$ is closed under finite unions. See below for my first but failed try.

(Wrong!) Let $C_m(G), C_n(H) \in \mathcal{F}$, where $m, n \in \mathbb{N}$ and $G \subseteq S^m$, $H \subseteq S^n$. By definition,

$$\begin{aligned}
C_m(G) \cup C_n(H) &= \left\{\omega \in \Omega \,\middle|\, [z_1(\omega), \ldots, z_m(\omega)] \in G\right\} \bigcup \{\omega \in \Omega \mid [z_1(\omega), \ldots, z_n(\omega)] \in H\} \\
&\overset{1}{=} \left\{\omega \in \Omega \,\middle|\, [z_1(\omega), \ldots, z_{m\wedge n}(\omega)] \in (H \cup G)\right\} \\
&\overset{2}{=} C_{m\wedge n}\left(G_{m\wedge n} \cup H_{m\wedge n}\right) \\
&\in \mathcal{F},
\end{aligned}$$

Consider two cylinders, $C_m(G)$ and $C_n(H)$, where $m, n \in \mathbb{N}$, $G \subseteq S^m$, and $H \subseteq S^n$. We need to prove that $C_m(G) \cap C_n(H) \in \mathcal{F}$. In fact,

$$C_m(G) \cap C_n(H) = C_{m \vee n} \left((G_{m \wedge n} \cap H_{m \wedge n}) \times \left(G_{m-(m \wedge n)} \cup H_{n-(m \wedge n)} \right) \right) \in \mathcal{F},$$

---

where, for example, $G_{m \wedge n}$ in equality (2), $G_{m \wedge n} \subseteq S^{m \wedge n}$, $G_{m-(m \wedge n)} \subseteq S^{m-(m \wedge n)}$, and $G_{m \wedge n} \times G_{m-(m \wedge n)} = G$.

To see why equality (1) holds, we need the following facts:

Claim 1. Suppose that $m \leqslant n$, $H = G \times H_{n-m}$, and $G \subseteq S^m$. Then $C_m(G) \supseteq C_n(H)$.

Proof. Pick any $\omega' \in C_n(H)$. By definition,

$$\left[z_1(\omega'), \ldots, z_n(\omega') \right] \in H = G \times H_{n-m},$$

which means that

$$\left[z_1(\omega'), \ldots, z_m(\omega') \right] \in G \Longrightarrow \omega' \in C_m(G).$$

Claim 2. If $G \subseteq H \subseteq S^n$, then $C_n(G) \subseteq C_n(H)$.

Proof. Straightforward.

Claim 3. For any $m, n \in \mathbb{N}$, and $G \subseteq S^m$, $H \subseteq S^n$, we have

$$C_m(G) \cup C_n(H) \subseteq C_{m \wedge n}(G_{m \wedge n} \cup H_{m \wedge n}).$$

Proof. Without loss of any generality, we assume that $m \wedge n = m$. Pick any $\omega' \in C_m(G) \cup C_n(H)$. Then,

$$\left[z_1(\omega'), \ldots, z_m(\omega') \right] \in G, \quad \text{or} \quad \left[z_1(\omega'), \ldots, z_n(\omega') \right] \in H.$$

From Claim 2, we have

$$\left[z_1(\omega'), \ldots, z_m(\omega') \right] \in G \cup H_m, \quad \text{or} \quad \left[z_1(\omega'), \ldots, z_n(\omega') \right] \in (G \cup H_m) \times H_{n-m},$$

where $H_m \subseteq S^m$, and $H = H_m \times H_{n-m}$. Then, by Claim 1, if $m \wedge n = m$, we have

$$\omega' \in C_m(G \cup H_m).$$

Claim 4. For any $m, n \in \mathbb{N}$, and $G \subseteq S^m$, $H \subseteq S^n$, we have

$$C_m(G) \cup C_n(H) \supseteq C_{m \wedge n}(G_{m \wedge n} \cup H_{m \wedge n}).$$

Proof. We still assume that $m \wedge n = m$. Pick any $\omega' \in C_{m \wedge n}(G_{m \wedge n} \cup H_{m \wedge n}) = C_m(G \cup H_m)$. By definition,

$$\left[z_1(\omega'), \ldots, z_m(\omega') \right] \in G \cup H_m;$$

that is,

$$\left[z_1(\omega'), \ldots, z_m(\omega') \right] \in G \quad \text{or} \quad \left[z_1(\omega'), \ldots, z_m(\omega') \right] \in H_m. \qquad \square$$

where $G_{m\wedge n}, H_{m\wedge n} \subseteq S^{m\wedge n}$, $G_{m-(m\wedge n)} \subseteq S^{m-(m\wedge n)}$, $H_{n-(m\wedge n)} \subseteq S^{n-(m\wedge n)}$, $G = G_{m\wedge n} \times G_{m-(m\wedge n)}$, $H = H_{m\wedge n} \times H_{n-(m\wedge m)}$, and we define $G_0 = H_0 = \varnothing$. □

► EXERCISE 1.35 (1.2.12). *Suppose that $\mathcal{A}$ is a semiring on Ω with $\Omega \in \mathcal{A}$. The collection of finite disjoint unions of $\mathcal{A}$-sets is a field on Ω. [Compare with Example 3 and Exercise 1.30.]*

PROOF. Let $\mathcal{A}$ be a semiring, and $\Omega \in \mathcal{A}$. Let $\mathcal{F}$ be the collection of finite disjoint unions of $\mathcal{A}$-sets, tha is, $A \in \mathcal{F}$ iff for some $n \in \mathbb{N}$ we have $A = \bigcup_{i=1}^n A_i$, where A_i's are disjoint $\mathcal{A}$-sets. $\mathcal{F}$ is a field: (i) $\Omega \in \mathcal{F}$ since $\Omega = \Omega \cup \varnothing \in \mathcal{F}$. (ii) Let $A \in \mathcal{F}$. Then $A = \bigcup_{i=1}^n A_i$, where $n \in \mathbb{N}$ and $\{A_i\}_{i=1}^n \subseteq \mathcal{A}$. To prove $A^c \in \mathcal{F}$, we need only to prove $A_i^c \in \mathcal{F}$ since $A^c = \bigcap_{i=1}^n A_i^c$, and $\mathcal{A}$ is a semiring [$\bigcap$-stable]. But $A_i^c \in \mathcal{F}$ is directly from (SR3) and the fact that $\Omega \in \mathcal{A}$ since $A_i^c = \Omega \smallsetminus A_i = \bigcup_{j=1}^{n^i} C_j^i$, where $\left\{C_j^i\right\}_{j=1}^{n^i} \subseteq \mathcal{A}$ is disjoint, and $n^i \in \mathbb{N}$, $\forall\ i \in \{1, \dots, n\}$, that is, each A_i^c is a finite disjoint union of $\mathcal{A}$-sets. Thus, $\mathcal{F}$ is closed under complements.

Instead of proving that $\mathcal{F}$ satisfies (F3) directly, we prove that $\mathcal{F}$ is a π-system. Let $B_1, B_2 \in \mathcal{F}$. Then

$$B_1 \cap B_2 = \left(\bigcup_{i=1}^n A_i\right) \cap \left(\bigcup_{j=1}^k A_j\right) = \bigcup_{i=1}^n \left[\bigcup_{j=1}^k \left(A_i \cap A_j\right)\right] = \bigcup_{i,j} \left(A_i \cap A_j\right).$$

Note that $A_i \cap A_j \in \mathcal{A}$ by (SR2). Hence $B_1 \cap B_2 \in \mathcal{F}$.

□

► EXERCISE 1.36 (1.2.13). *Let $f : \Omega \to \Omega'$. Given $\mathcal{A}' \subseteq 2^{\Omega'}$, let $f^{-1}(\mathcal{A}') = \{f^{-1}(A') : A' \in \mathcal{A}'\}$, where $f^{-1}(A')$ is the usual inverse image of A' under f.*

a. *If $\mathcal{A}'$ is a field on Ω', then $f^{-1}(\mathcal{A}')$ is a field on Ω.*

b. *$f(\mathcal{A})$ may not be a field over Ω' even if $\mathcal{A}$ is a field on Ω.*

PROOF. **(a)** Let $\mathcal{A}'$ be a field on Ω'. (i) Since $\Omega = f^{-1}(\Omega')$ and $\Omega' \in \mathcal{A}'$, we have that $\Omega \in f^{-1}(\mathcal{A}')$. (ii) If $A \in f^{-1}(\mathcal{A}')$, then $A = f^{-1}(A')$ for some $A' \in \mathcal{A}'$. Therefore, $A^c = [f^{-1}(A')]^c = f^{-1}((A')^c)$, and $(A')^c \in \mathcal{A}'$ since $\mathcal{A}'$ is a field. It follows that $A^c \in f^{-1}(\mathcal{A}')$, so that $f^{-1}(\mathcal{A}')$ is closed under complements. (iii) To see that $f^{-1}(\mathcal{A}')$ is closed under finite unions, let $\{A_i\}_{i=1}^n \subseteq \mathcal{A}$, where $n \in \mathbb{N}$. Therefore, for each $i \in \{1, \dots, n\}$, there is $A_i' \in \mathcal{A}'$ with $A_i = f^{-1}(A_i)$. Therefore,

$$\bigcup_{i=1}^n A_i = \bigcup_{i=1}^n f^{-1}\left(A_i'\right) = f^{-1}\left(\bigcup_{i=1}^n A_i'\right) \in f^{-1}(\mathcal{A}'),$$

since $\bigcup_{i=1}^n A_i' \in \mathcal{A}'$.

(b) The simplest case is that f is not onto [surjective]. In this case, $f(\Omega) \subsetneq \Omega'$; that is, $\Omega' \notin \mathscr{A}'$, and so $\mathscr{A}'$ is not a field on Ω'. $\square$

► EXERCISE 1.37 (1.2.14). *Let Ω be infinite, and let $\mathcal{A} \subseteq 2^{\Omega}$ have cardinality $\aleph_0$. Let $f(\mathcal{A})$ denote the minimal field generated by $\mathcal{A}$ [Exercise 1.32]. We will show that* $\operatorname{card}(f(\mathcal{A})) = \aleph_0$.

a. *Given a collection $\mathcal{C}$, let $\mathcal{C}^*$ denote the collection of*

  i. *finite unions of $\mathcal{C}$-sets,*

  ii. *finite unions of differences of $\mathcal{C}$-sets, and*

  iii. *finite unions of complements of $\mathcal{C}$-sets.*

  If $\varnothing \in \mathcal{C}$, then $\mathcal{C} \subseteq \mathcal{C}^*$. If $\operatorname{card}(\mathcal{C}) = \aleph_0$, *then* $\operatorname{card}(\mathcal{C}^*) = \aleph_0$.

PROOF. $\square$

► EXERCISE 1.38 (1.2.15). *Some books work with a system of sets called an* algebra. *An algebra on Ω is a nonempty collection of subsets of Ω that satisfies (F2) and (F3).*

a. *$\mathcal{F}$ is an algebra on Ω iff $\mathcal{F}$ is a ring on Ω with $\Omega \in \mathcal{F}$.*

b. *$\mathcal{F}$ is an algebra iff $\mathcal{F}$ is a field. Thus algebra and field are synonymous.*

PROOF.

(a: $\Longrightarrow$) Suppose $\mathcal{F}$ is an algebra. Then,

(R1) $\mathcal{F} \neq \varnothing$ by assumption.

(R2) $\mathcal{F}$ is $\bigcup$-stable follows (F3).

(R3) The assumption of $\Omega \in \mathcal{F}$ and (F2) imply that if $A, B \in \mathcal{F}$, then $A^c = \Omega - A \in \mathcal{F}$ and $B^c = \Omega - B \in \mathcal{F}$. Then

$$\left[\left(A^c \cup B\right) \overset{\text{(F3)}}{\in} \mathcal{F}\right] \overset{\text{(F2)}}{\Longrightarrow} \left[\left(A^c \cup B\right)^c \in \mathcal{F}\right] \Longrightarrow [A \smallsetminus B \in \mathcal{F}].$$

This proves that $\mathcal{F}$ is closed under difference.

(a: $\Longleftarrow$) Suppose $\mathcal{F}$ is a ring and $\Omega \in \mathcal{F}$. To prove $\mathcal{F}$ is an algebra on Ω, note that

(A1) $\mathcal{F} \neq \varnothing$ since $\mathcal{F}$ is a ring.

(F2) Let $A \in \mathcal{F}$. Because $\Omega \in \mathcal{F}$ and (R3), we have $A^c = \Omega - A \in \mathcal{F}$. This proves that $\mathcal{F}$ is closed under difference.

(F3) $\bigcup$-stability follows (R2).

(b) We need only to prove that $\mathcal{F}$ is an field if $\mathcal{F}$ is an algebra since the reverse direction is trivial.

Suppose $\mathcal{F}$ is an algebra. We want to show $\Omega \in \mathcal{F}$. Since $\mathcal{F} \neq \varnothing$ by definition of an algebra, there must exist $A \in \mathcal{F}$. Then $A^c \in \mathcal{F}$ by (F2), and so $\Omega = A \cup A^c \in \mathcal{F}$ by (F3). □

1.3 σ-Fields

► Exercise 1.39 (1.3.1). *A collection $\mathcal{F}$ of sets is called a* monotone class *iff (MC1) for every nondecreasing sequence $\{A_n\}_{n=1}^{\infty}$ of $\mathcal{F}$-sets we have $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$, and (MC2) for every nonincreasing sequence $\{A_n\}_{n=1}^{\infty}$ of $\mathcal{F}$-sets we have $\bigcap_{n=1}^{\infty} A_n \in \mathcal{F}$.*

a. *If $\mathcal{F}$ is both a field and a monotone class, then $\mathcal{F}$ is a σ-field.*

b. *A field is a monotone class if and only if it is a σ-field.*

Proof. See Chung (2001, Theorem 2.1.1).

a. Let $\mathcal{F}$ is both a field and a MC. Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$, then $B_n = \bigcup_{i=1}^{n} A_n \in \mathcal{F}$ since $\mathcal{F}$ is a *field*, $B_n \subseteq B_{n+1}$, and $\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n \in \mathcal{F}$.

b. We only need to show the "IF" part. But it is trivial: A σ-filed is a field and a MC.

□

► Exercise 1.40 (1.3.2). *This problem discusses some equivalent formulations of a σ-field.*

a. *$\mathcal{F}$ satisfies (S1), (S2), and closure under amc intersections iff $\mathcal{F}$ is a σ-field.*

b. *Every field that is closed under countable disjoint unions is a σ-field.*

c. *If $\mathcal{F}$ satisfies (S1), closure under differences, and closure under countable unions or closure under countable intersections, then $\mathcal{F}$ is a σ-field.*

Proof. **(a)** For the "ONLY IF" part, let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ and $\mathcal{F}$ satisfy (S1) and (S2). Then $A_n^c \in \mathcal{F}$ for any $n \in \mathbb{N}$; hence, $\bigcup_{n=1}^{\infty} A_n = \left(\bigcap_{n=1}^{\infty} A_n^c\right)^c \in \mathcal{F}$. The "IF" part is proved by the same logic.

(b) We need only to prove $\mathcal{F}$ is closed under countable unitions. Let $\mathcal{F}$ be a field, and $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$. Let

$$B_k = A_k \cap \left(\bigcup_{i=1}^{n-1} A_i\right)^c.$$

It is clear that $\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{F}$ is disjoint, and $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} A_n$. This completes the proof.

(c) We only need to prove (S2), that is, $\mathscr{F}$ is closed under complementation. Let $A \in \mathscr{F}$. By (S1), $\Omega \in \mathscr{F}$, then $A^c = \Omega \setminus A \in \mathscr{F}$ since by assumption, $\mathscr{F}$ is closed under difference. □

▶ EXERCISE 1.41 (1.3.3). *Prove the following claims.*

a. *A finite union of σ-fields on Ω is not necessarily a field on Ω.*

b. *If a finite union of σ-fields on Ω is a field, then it is a σ-field as well.*

c. *Given σ-fields $\mathscr{F}_1 \subsetneq \mathscr{F}_2 \subsetneq \cdots$ on Ω, it is not necessarily the case that $\bigcup_{n=1}^{\infty} \mathscr{F}_n$ is a σ-field.*

PROOF. **(a)** Let $\{\mathscr{F}_i\}_{i=1}^n$ be a class of σ-fields, and consider $\bigcup_{i=1}^n A_i$, where $A_i \in \mathscr{F}_i$. Note that it is possible that $\bigcup_{i=1}^n A_i \notin \mathscr{F}_j$ for any j, so $\bigcup_{i=1}^n A_i \notin \bigcup_{i=1}^n \mathscr{F}_i$. For example (Athreya and Lahiri, 2006, Exercise 1.5, p.32), let

$$\Omega = \{1,2,3\}, \quad \mathscr{F}_1 = \big\{\{1\},\{2,3\},\Omega,\varnothing\big\}, \quad \mathscr{F}_2 \big\{\{1,2\},\{3\},\Omega,\varnothing\big\}.$$

It is easy to verify that $\mathscr{F}_1$ and $\mathscr{F}_2$ are both σ-fields, but $\mathscr{F}_1 \cup \mathscr{F}_2$ is not a field since $\{1\} \cup \{3\} = \{1,3\} \notin \mathscr{F}_1 \cup \mathscr{F}_2$.

(b) Without loss of any generality, we here just consider two σ-fields, $\mathscr{F}_1$ and $\mathscr{F}_2$, on Ω. Consider a sequence $\{A_n\}_{n=1}^{\infty} \subseteq \mathscr{F}_1 \cup \mathscr{F}_2$. Then we can construct two sequences, one in $\mathscr{F}_1$ and one in $\mathscr{F}_2$. Particularly, the sequence of sets $\{A_n^1\} \subseteq \mathscr{F}_1$ is constructed as follows:

$$A_n^1 = \begin{cases} A_n, & \text{if } A_n \in \mathscr{F}_1 \\ \varnothing, & \text{otherwise.} \end{cases}$$

The sequence of sets $\{A_n^2\} \subseteq \mathscr{F}_2$ is constructed similarly. Then $\bigcup_{k=1}^{\infty} A_k^1 \in \mathscr{F}_1$ and $\bigcup_{m=1}^{\infty} A_m^2 \in \mathscr{F}_2$ since both $\mathscr{F}_1$ and $\mathscr{F}_2$ are σ-fields, and

$$\bigcup_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^1\right) \cup \left(\bigcup_{n=1}^{\infty} A_n^2\right).$$

If $\mathscr{F}_1 \cup \mathscr{F}_2$ is a field, we have

$$\bigcup_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^1\right) \cup \left(\bigcup_{n=1}^{\infty} A_n^2\right) \in \mathscr{F}_1 \cup \mathscr{F}_2.$$

(c) See Broughton and Huff (1977) for a more general result. Let $\Omega = \mathbb{N}$ and for all $n \in \mathbb{N}$, let

$$\mathscr{F}_n = \sigma\left(\big\{\{1\},\ldots,\{n\}\big\}\right).$$

Since $\big\{\{1\},\ldots,\{m\}\big\} \subsetneq \big\{\{1\},\ldots,\{n\}\big\}$ when $m < n$, we have $\mathscr{F}_1 \subsetneq \mathscr{F}_2 \subsetneq \cdots$. It it clear that $\{1\},\{2\},\ldots \in \bigcup_{n=1}^{\infty} \mathscr{F}_n$, but

$$\bigcup_{n=1}^{\infty} \{n\} = \{1, 2, \ldots\} \notin \bigcup_{n=1}^{\infty} \mathscr{F}_n,$$

since there does not exist a $\mathscr{F}_n$ such that $\{1, 2, \ldots\} \in \mathscr{F}_n$, for any $n \in \mathbb{N}$. □

► EXERCISE 1.42 (1.3.5). *A subset $A \subseteq \mathbb{R}$ is called* nowhere dense *iff every open interval I contains an open interval J such that $J \cap A = \varnothing$. Clearly $\varnothing$ and all subsets of a nowhere dense set are nowhere dense. A subset $A \subseteq \mathbb{R}$ is called a* set of the first category *iff A is a countable union of nowhere dense sets.*

a. *An amc union of sets of the first category is of the first category.*

b. *Let $\mathscr{F} = \{A \subseteq \mathbb{R} : A$ or A^c is a set of the first category$\}$. Then $\mathscr{F}$ is a σ-field of subsets of $\mathbb{R}$.*

PROOF. Refer Gamelin and Greene (1999, Section 1.2) for the more detailed definitions and discussion of *nowhere dense* and the *first category set*.

(a) Consider a countable sequence of sets of the first category, $\{A_n\}_{n=1}^{\infty}$. Then $A_n = \bigcup_{i=1}^{\infty} A_i^n$ for any $n \in \mathbb{N}$, where $\{A_i^n\}_{i=1}^{\infty}$ are nowhere dense. Clearly, the amc unions of amc unions is still amc, which proves the claim.

(b) Let $\mathscr{F} = \{A \subseteq \mathbb{R} : A$ or A^c is a set of the first category$\}$. Then $\Omega \in \mathscr{F}$ since $\varnothing$ is of the first category and $\Omega = \varnothing^c$. To see $\mathscr{F}$ is closed under complementation, let $A \in \mathscr{F}$. (i) If A is of the first category, then $A^c \in \mathscr{F}$ since $(A^c)^c = A$ is of the first category; (ii) If A^c is of the first category, then $A^c \in \mathscr{F}$ by the definition of $\mathscr{F}$. In any case, $A \in \mathscr{F}$ implies that $A^c \in \mathscr{F}$.

Finally, to see $\mathscr{F}$ is σ-$\bigcup$-stable, let $\{A_n\}_{n=1}^{\infty}$ be a sequence of $\mathscr{F}$-sets. There are two cases: (i) Each A_n is of the first category. Then part (a) of this exercise implies that $\bigcup_{n=1}^{\infty} A_n \in \mathscr{F}$. (ii) Some A_n^c is of the first category. In this case, we assume without loss of generality that A_1^c is of the first category, and we have that $\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c \subseteq A_1^c$. It is trivial that $\left(\bigcup_{n=1}^{\infty} A_n\right)^c$ is of the first category since A_1^c is, and every subset of the first category is of the first category. Particularly, let $A_1^c = \bigcup_{n=1}^{\infty} B_n$, where the B_n's are nowhere dense sets. Since $\left(\bigcup_{n=1}^{\infty} A_n\right)^c \subseteq A_1^c$, we must can rewrite $\left(\bigcup_{n=1}^{\infty} A_n\right)^c$ as

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} C^n,$$

where every C_n is a subset of B_n and some C_n's maybe be empty. Note that then every C_n is nowhere dense no matter $C_n = \varnothing$ or not. Consequently, $\left(\bigcup_{n=1}^{\infty} A_n\right)^c$ is of the first category by definition. □

► EXERCISE 1.43 (1.3.6). *A σ-ring of subsets of Ω is a nonempty collection of subsets of Ω that is closed under differences as well as countable unions.*

a. *Every σ-ring is closed under finite unions and amc intersections.*

b. *$\mathcal{F}$ is a σ-field iff $\mathcal{F}$ is a σ-ring with $\Omega \in \mathcal{F}$.*

c. *State and prove an existence and uniqueness result regarding the [minimal] σ-ring generated by a collection $\mathcal{A}$ of subsets of Ω.*

PROOF. **(a)** Let $\mathcal{R}$ be a σ-ring. We first prove that $\varnothing \in \mathcal{R}$. Since $\mathcal{R} \neq \varnothing$, there exists $A \in \mathcal{R}$; moreover, since $\mathcal{R}$ is closed under difference, we have $\varnothing = A \smallsetminus A \in \mathcal{R}$. Now consider an arbitrary sequence of $\mathcal{R}$-sets $A_1, \dots, A_n, \varnothing, \varnothing, \dots$. Because $\mathcal{R}$ is σ-$\bigcup$-stable, we know that

$$\bigcup_{i=1}^{n} A_i = (A_1 \cup A_2 \cup \cdots \cup A_n) \cup (\varnothing \cup \varnothing \cup \cdots) \in \mathcal{R},$$

which proves that $\mathcal{R}$ is $\bigcup$-stable.

To see $\mathcal{R}$ is closed under amc intersections, let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{R}$. Then $A = \bigcup_{n=1}^{\infty} A_n \in \mathcal{R}$. Let

$$A'_n = A \smallsetminus A_n, \quad \forall\, n \in \mathbb{N}.$$

Then $\{A'_n\}_{n=1}^{\infty} \subseteq \mathcal{R}$, $\bigcup_{n=1}^{\infty} A'_n \in \mathcal{R}$, and

$$A \smallsetminus \left(\bigcup_{n=1}^{\infty} A'_n \right) = \bigcap_{n=1}^{\infty} A_n \in \mathcal{R}$$

since $A \smallsetminus \left(\bigcup_{n=1}^{\infty} A'_n\right) \in \mathcal{R}$. [Basically, I let A be the universal space, and A'_n be the complements of A_n in A.]

(b) Suppose that $\mathcal{F}$ is a σ-field. Then $\Omega \in \mathcal{F}$ be (S1). To see $\mathcal{F}$ is closed under difference, let $A, B \in \mathcal{F}$. Then (S2) implies that $B^c \in \mathcal{F}$. Since $\mathcal{F}$ is $\bigcap$-stable, we have $A \smallsetminus B = A \cap B^c \in \mathcal{F}$. The fact that $\mathcal{F}$ is σ-$\bigcup$-stable follows (S3).

Now suppose that $\mathcal{F}$ is a σ-ring with $\Omega \in \mathcal{F}$. We need only to prove that $\mathcal{F}$ satisfies (S2). Let $A \in \mathcal{F}$. Since $\Omega \in \mathcal{F}$ and $\mathcal{F}$ is closed under difference, we have

$$A^c = \Omega \smallsetminus A \in \mathcal{F}.$$

(c) Standard. Omitted. □

► EXERCISE 1.44 (1.3.9).

a. *If $\mathcal{A} \subseteq \mathcal{A}' \subseteq \sigma(\mathcal{A})$, then $\sigma(\mathcal{A}') = \sigma(\mathcal{A})$.*

b. *For any collection $\varnothing \neq \mathcal{A} \subseteq 2^{\Omega}$, $\pi(\mathcal{A}) \subseteq \lambda(\mathcal{A}) \subseteq \sigma(\mathcal{A})$.*

c. *If the nonempty collection $\mathcal{A}$ is finite, then $\sigma(\mathcal{A}) = f(\mathcal{A})$.*

d. *For arbitrary collection $\mathcal{A}$, we have $\sigma(\mathcal{A}) = \sigma\left(f(\mathcal{A})\right)$.*

e. *For arbitrary collection $\mathcal{A}$, we have $f\left(\sigma(\mathcal{A})\right) = \sigma\left(f(\mathcal{A})\right)$.*

PROOF. **(a)** On the first hand, $\mathcal{A} \subseteq \mathcal{A}'$ implies that $\sigma(\mathcal{A}) \subseteq \sigma(\mathcal{A}')$; on the second hand, $\mathcal{A}' \subseteq \sigma(\mathcal{A})$ implies that $\sigma(\mathcal{A}') \subseteq \sigma\left(\sigma(\mathcal{A})\right) = \sigma(\mathcal{A})$. We thus get the equality.

(b) Let $\Pi(\mathcal{A})$, $\Lambda(\mathcal{A})$, and $\Sigma(\mathcal{A})$ denote the collection of all π-systems, λ-systems, and σ-fields of subsets of Ω that contain $\mathcal{A}$, respectively. With this, we may define

$$\pi(\mathcal{A}) = \bigcap_{\mathcal{P}\in\Pi(\mathcal{A})} \mathcal{P}, \quad \lambda(\mathcal{A}) = \bigcap_{\mathcal{L}\in\Lambda(\mathcal{A})} \mathcal{L}, \quad \text{and} \quad \sigma(\mathcal{A}) = \bigcap_{\mathcal{F}\in\Sigma(\mathcal{A})} \mathcal{F}.$$

It is easy to see that any σ-field containing $\mathcal{A}$ is a λ-system containing $\mathcal{A}$; hence $\Sigma(\mathcal{A}) \subseteq \Lambda(\mathcal{A})$, and so $\lambda(\mathcal{A}) \subseteq \sigma(\mathcal{A})$.

(c) It is clear that $f(\mathcal{A}) \subseteq \sigma(\mathcal{A})$; since $0 < |\mathcal{A}| < \infty$, the field $f(\mathcal{A})$ is finite and so it is a σ-field. Then $\sigma(\mathcal{A}) \subseteq f(\mathcal{A})$.

(d) On the first hand, $f(\mathcal{A}) \subseteq \sigma(\mathcal{A})$ implies that $\sigma\left(f(\mathcal{A})\right) \subseteq \sigma\left(\sigma(\mathcal{A})\right) = \sigma(\mathcal{A})$. On the second hand, $\mathcal{A} \subseteq f(\mathcal{A})$ and so $\sigma(\mathcal{A}) \subseteq \sigma\left(f(\mathcal{A})\right)$.

(e) It follows from (d) that $\sigma\left(f(\mathcal{A})\right) = \sigma(\mathcal{A})$. By definition, $f\left(\sigma(\mathcal{A})\right)$ is the minimal field containing $\sigma(\mathcal{A})$. But $\sigma(\mathcal{A})$ itself is a field; hence $f\left(\sigma(\mathcal{A})\right) = \sigma(\mathcal{A})$. □

► EXERCISE 1.45 (1.3.16). *Let $\mathcal{F} = \sigma(\mathcal{A})$, where $\varnothing \neq \mathcal{A} \subseteq 2^{\Omega}$. For each $B \in \mathcal{F}$ there exists a countable subcollection $\mathcal{A}_B \subseteq \mathcal{A}$ with $B \in \sigma(\mathcal{A}_B)$.*

PROOF. Let

$$\mathcal{B} = \left\{B \in \mathcal{F} : \exists \mathcal{A}_B \subseteq \mathcal{A} \text{ such that } \mathcal{A}_B \text{ is countable and } B \in \sigma(\mathcal{A}_B)\right\}. \tag{1.7}$$

It is clear that $\mathcal{B} \subseteq \mathcal{F}$. For any $B \in \mathcal{A}$, take $\mathcal{A}_B = \{B\}$; then $\mathcal{A}_B = \{B\}$ is countable and $B \in \sigma\left(\{B\}\right) = \{\varnothing, \Omega, B, B^c\}$; hence $\mathcal{A} \subseteq \mathcal{B}$. We now show that $\mathcal{B}$ is a σ-field. Obviously, $\Omega \in \mathcal{B}$ since $\Omega \in \mathcal{F}$ and $\Omega \in \sigma\left(\{\Omega\}\right) = \{\varnothing, \Omega\}$. If $B \in \mathcal{B}$, then $B \in \mathcal{F}$ and there exists a countable $\mathcal{A}_B \subseteq \mathcal{A}$ such that $B \in \sigma(\mathcal{A}_B)$; but which mean that $B^c \in \mathcal{F}$ and $B^c \in \sigma(\mathcal{A}_B)$, i.e., $B^c \in \mathcal{B}$. Similarly, it is easy to see that $\mathcal{B}$ is closed under countable unions. Thus, $\mathcal{B}$ is a σ-field containing $\mathcal{A}$, and so $\mathcal{F} \subseteq \mathcal{B}$. We thus proved that $\mathcal{B} = \mathcal{F}$ and the get the result. □

► EXERCISE 1.46 (1.3.18). *Given $\varnothing \neq \mathcal{A} \subseteq 2^{\Omega}$ and $\varnothing \neq B \subseteq \Omega$, let $\mathcal{A} \cap B = \{A \cap B : A \in \mathcal{A}\}$ and let $\sigma(\mathcal{A}) \cap B = \{A \cap B : A \in \sigma(\mathcal{A})\}$.*

a. *$\sigma(\mathcal{A}) \cap B$ is a σ-field on B.*

b. *Next, define $\sigma_B(\mathcal{A} \cap B)$ to be the minimal σ-field over B generated by the class $\mathcal{A} \cap B$. Then $\sigma_B(\mathcal{A} \cap B) = \sigma(\mathcal{A}) \cap B$.*

PROOF. This claim can be found in Ash and Doléans-Dade (2000, p. 5).

(a) $B \in \sigma(\mathcal{A}) \cap B$ as $\Omega \in \sigma(\mathcal{A})$. If $C \in \sigma(\mathcal{A}) \cap B$, then $C = A \cap B$ with $A \in \sigma(\mathcal{A})$; hence $B \smallsetminus C = A^c \cap B \in \sigma(\mathcal{A}) \cap B$. To see that $\sigma(\mathcal{A}) \cap B$ is closed under countable unions, let $\{C_n\}_{n=1}^{\infty} \subseteq \sigma(\mathcal{A}) \cap B$. Then each $C_n = A_n \cap B$ with $A_n \in \sigma(\mathcal{A})$. Hence,

$$\bigcup_{n=1}^{\infty} C_n = \bigcup_{n=1}^{\infty} (A_n \cap B) = \left(\bigcup_{n=1}^{\infty} A_n\right) \cap B \in \sigma(\mathscr{A}) \cap B.$$

(b) First, $\mathscr{A} \subseteq \sigma(\mathscr{A})$, hence $\mathscr{A} \cap B \subseteq \sigma(\mathscr{A}) \cap B$. Since $\sigma(\mathscr{A}) \cap B$ is a σ-field on B by (a), we have $\sigma_B(\mathscr{A} \cap B) \subseteq \sigma(\mathscr{A}) \cap B$. To establish the reverse inclusion we must show that $A \cap B \in \sigma_B(\mathscr{A} \cap B)$ for all $A \in \sigma(\mathscr{A})$. We use the *good sets principle*. Let

$$\mathscr{G} = \{A \in \sigma(\mathscr{A}) : A \cap B \in \sigma_B(\mathscr{A} \cap B)\}.$$

We now show that $\mathscr{G}$ is a σ-field containing $\mathscr{A}$. It is evident that $\Omega \in \mathscr{G}$. If $A \in \mathscr{G}$, then $A \cap B \in \sigma_B(\mathscr{A} \cap B)$ and $A \in \sigma(\mathscr{A})$; hence, $A^c \cap B = B \smallsetminus (A \cap B) \in \sigma_B(\mathscr{A} \cap B)$ implies that $A^c \in \mathscr{G}$. To see $\mathscr{G}$ is closed under countable unions, let $\{A_n\}_{n=1}^{\infty} \subseteq \mathscr{G}$ with $A_n \cap B \in \sigma_B(\mathscr{A} \cap B)$ for all $n \in \mathbb{N}$. Then

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \cap B = \bigcup_{n=1}^{\infty} (A_n \cap B) \in \sigma_B(\mathscr{A} \cap B).$$

Since $\mathscr{A} \subseteq \mathscr{G}$, we have $\sigma(\mathscr{A}) \subseteq \mathscr{G}$; hence $\sigma(\mathscr{A}) = \mathscr{G}$: every set in $\sigma(\mathscr{A})$ is good. □

► Exercise 1.47 (1.3.19). *Suppose that $\mathscr{A} = \{A_1, A_2, \ldots\}$ is a disjoint sequence of subsets of Ω with $\bigcup_{n=1}^{\infty} A_n = \Omega$. Then each $\sigma(\mathscr{A})$-set is the union of an at most countable subcollection of $A_1, A_2, \ldots$.*

Proof. Let

$$\mathscr{C} = \left\{A \in \sigma(\mathscr{A}) : A \text{ is an at most countable union of } \mathscr{A}\text{-sets}\right\}.$$

It is easy to see that $\Omega \in \mathscr{C}$ since $\Omega = \bigcup_{n=1}^{\infty} A_n$. If $A \in \mathscr{C}$, then $A = \bigcup_{i \in J} A_i$, where J is at most countable. Hence $A^c = \left(\bigcup_{n=1}^{\infty} A_n\right) \smallsetminus \left(\bigcup_{i \in J} A_i\right)$ is an at most countable union of $\mathscr{A}$-sets, that is, $\mathscr{C}$ is closed under complements. It is also easy to see that $\mathscr{C}$ is closed under countable unions and $\mathscr{A} \subseteq \mathscr{C}$. Hence, $\mathscr{C}$ is a σ-field and $\sigma(\mathscr{A}) = \mathscr{C}$. □

► Exercise 1.48 (1.3.20). *Let $\mathscr{P}$ denote a π-system on Ω, and let $\mathscr{L}$ denote a λ-system on Ω with $\mathscr{P} \subseteq \mathscr{L}$. We will show that $\sigma(\mathscr{P}) \subseteq \mathscr{L}$. Let $\lambda(\mathscr{P})$ denote the λ-system generated by $\mathscr{P}$, and for each subset $A \subseteq \Omega$ we define $\mathscr{G}_A = \{C \subseteq \Omega : A \cap C \in \lambda(\mathscr{P})\}$.*

Proof. See Vestrup (2003, Claim 1, p. 82). □

► Exercise 1.49 (1.3.21). *Let $\mathscr{F}$ denote a field on Ω, and let $\mathscr{M}$ denote a monotone class on Ω [See Exercise 1.39]. We will show that $\mathscr{F} \subseteq \mathscr{M}$ implies that $\sigma(\mathscr{F}) \subseteq \mathscr{M}$. Let $m(\mathscr{F})$ denote the* minimal monotone class on Ω generated by *$\mathscr{F}$. That is, $m(\mathscr{F})$ is the intersection of all monotone classes on Ω containing the collection $\mathscr{F}$.*

a. *To prove the claim, it is sufficient to show that $\sigma(\mathscr{F}) \subseteq m(\mathscr{F})$.*

b. *If $m(\mathcal{F})$ is a field, then $\sigma(\mathcal{F}) \subseteq m(\mathcal{F})$.*

c. $\Omega \in m(\mathcal{F})$.

d. *Let $\mathcal{G} = \{A \subseteq \Omega : A^c \in m(\mathcal{F})\}$. $\mathcal{G}$ is a monotone class on Ω and $m(\mathcal{F}) \subseteq \mathcal{G}$.*

e. *$m(\mathcal{F})$ is indeed closed under complements.*

f. *Let $\mathcal{G}_1 = \left\{A \subseteq \Omega : A \cup B \in m(\mathcal{F}) \text{ for all } B \in \mathcal{F}\right\}$. Then $\mathcal{G}_1$ is a monotone class such that $\mathcal{F} \subseteq \mathcal{G}_1$ and $m(\mathcal{F}) \subseteq \mathcal{G}_1$.*

g. *Let $\mathcal{G}_2 = \left\{B \subseteq \Omega : A \cup B \in m(\mathcal{F}) \text{ for all } A \in m(\mathcal{F})\right\}$. Then $\mathcal{G}_2$ is a monotone class such that $\mathcal{F} \subseteq \mathcal{G}_2$, and $m(\mathcal{F}) \subseteq \mathcal{G}_2$.*

h. *$m(\mathcal{F})$ is closed under finie unions, and hence is a field.*

Proof. *Halmos' Monotone Class Theorem* is proved in every textbook. See Billingsley (1995, Theorem 3.4), Ash and Doléans-Dade (2000, Theorem 1.3.9), or Chung (2001, Theorem 2.1.2), among others. The above outline is similar to Chung (2001).

(a) By definition. In fact, $\sigma(\mathcal{F}) = m(\mathcal{F})$.

(b) By Exercise 1.39: A field is a σ-field iff it is also an M.C. If $m(\mathcal{F})$ is a field, then it is a σ-field containing $\mathcal{F}$; hence, $\sigma(\mathcal{F}) \subseteq m(\mathcal{F})$.

(c) $\Omega \in \mathcal{F} \subseteq m(\mathcal{F})$.

(d) Let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{G}$ be monotone; then $\left\{A_n^c\right\}_{n=1}^{\infty}$ is also monotone. The DeMorgan identities

$$\left(\bigcup_{n=1}^{\infty} A_n\right)^c = \bigcap_{n=1}^{\infty} A_n^c, \quad \text{and} \quad \left(\bigcap_{n=1}^{\infty} A_n\right)^c = \bigcup_{n=1}^{\infty} A_n^c$$

show that $\mathcal{G}$ is a M.C. Since $\mathcal{F}$ is closed under complements and $\mathcal{F} \subseteq m(\mathcal{F})$, it is clear that $\mathcal{F} \subseteq \mathcal{G}$. Hence $m(\mathcal{F}) \subseteq \mathcal{G}$ by the minimality of $m(\mathcal{F})$.

(e) By (d), $m(\mathcal{F}) \subseteq \mathcal{G}$, which means that for any $A \in m(\mathcal{F})$, we have $A^c \in m(\mathcal{F})$. Hence, $m(\mathcal{F})$ is closed under implementation.

(f) Let $\mathcal{G}_1 = \left\{A \subseteq \Omega : A \cup B \in m(\mathcal{F}) \text{ for all } B \in \mathcal{F}\right\}$. If $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{G}_1$ is monotone, then $\{A_n \cup B\}_{n=1}^{\infty}$ is also monotone. The identities

$$\left(\bigcup_{n=1}^{\infty} A_n\right) \cup B = \bigcup_{n=1}^{n} (A_n \cup B), \quad \text{and} \quad \left(\bigcap_{n=1}^{\infty} A_n\right) \cup B = \bigcap_{n=1}^{n} (A_n \cup B)$$

show that $\mathcal{G}_1$ is a M.C. Since $\mathcal{F}$ is closed under finite unions and $\mathcal{F} \subseteq m(\mathcal{F})$, it follows that $\mathcal{F} \subseteq \mathcal{G}_1$, and so $m(\mathcal{F}) \subseteq \mathcal{G}_1$ by the minimality of $m(\mathcal{F})$.

(g) As in (f) we can show $\mathcal{G}_2$ is a M.C. By (f), $m(\mathcal{F}) \subseteq \mathcal{G}_1$, which means that for any $A \in m(\mathcal{F})$ and $B \in \mathcal{F}$ we have $A \cup B \in m(\mathcal{F})$. This in turn means that $\mathcal{F} \subseteq \mathcal{G}_2$. Hence, $m(\mathcal{F}) \subseteq \mathcal{G}_2$.

(h) Since $m(\mathcal{F}) \subseteq \mathcal{G}_2$, for any $B \in m(\mathcal{F})$ and $A \in m(\mathcal{F})$, we have $A \cup B \in m(\mathcal{F})$; that is, $m(\mathcal{F})$ is closed under finite unions. □

1.4 The Borel σ-Field

► Exercise 1.50 (1.4.1). *Show directly that*[5] $\sigma(\mathcal{A}_3) = \sigma(\mathcal{A}_3^*)$, $\sigma(\mathcal{A}_4) = \sigma(\mathcal{A}_7)$, *and* $\sigma(\mathcal{A}_4^*) = \sigma(\mathcal{A}_{10})$.

Proof. (i) It is clear that $\sigma\left(\mathcal{A}_3^*\right) \subseteq \sigma(\mathcal{A}_3)$. We only need to show that $\sigma(\mathcal{A}_3) \subseteq \sigma\left(\mathcal{A}_3^*\right)$. Since $[x, \infty) = \bigcup [r_n, \infty)$, where $\{r_n\}_{n=1}^{\infty} \subseteq \mathbb{Q}$, we complete the proof. □

► Exercise 1.51 (1.4.2). *All amc subsets of* $\mathbb{R}$ *are Borel sets. All subsets of* $\mathbb{R}$ *that differ from a Borel set by at most countably many points are Borel sets. That is, if the symmetric difference* $C \Delta B$ *is amc and* $B \in \mathcal{B}$, *then* $C \in \mathcal{B}$.

Proof. Let $A = \{x_n\}_{n=1}^{\infty} \subseteq \mathbb{R}$. Then $A = \bigcup_{n=1}^{\infty} \{x_n\}$. $\{x_n\}$ is a Borel set. □

► Exercise 1.52 (1.4.3). *The Borel σ-field on* $(0, 1]$ *is denoted by* $\mathcal{B}_{(0,1]}$ *and is defined as the σ-field on* $(0, 1]$ *generated by the rsc subintervals of* $(0, 1]$. $\mathcal{B}_{(0,1]}$ *may be equivalently defined by* $\{B \cap (0, 1] \mid B \in \mathcal{B}\}$.

Proof. It follows from Exercise 1.46 that $\sigma_B(\mathcal{A} \cap B) = \sigma(\mathcal{A}) \cap B$ for any $\varnothing \neq \mathcal{A} \subseteq 2^{\Omega}$ and $\varnothing \neq B \subseteq \Omega$. In particular, we have $\mathcal{B}_{(0,1]} = \mathcal{B} \cap B$. □

► Exercise 1.53 (1.4.4). $\mathcal{B}$ *is generated by the compact subsets of* $\mathbb{R}$.

Proof. Denote

$$\mathcal{A}_{11} = \{A \subseteq \mathbb{R}^n : A \text{ is compact}\}.$$

Let $A \in \mathcal{A}_{11}$. Every compact set is closed (Heine-Borel Theorem); hence $A \in \mathcal{A}_{10}$. It follows that $\sigma(\mathcal{A}_{11}) \subseteq \sigma(\mathcal{A}_{10})$. Now let $A \in \mathcal{A}_{10}$. The sets $A_K = A \cap [-K, K]^n$, $K \in \mathbb{N}$, are compact; hence the countable union $A = \bigcup_{K=1}^{\infty} A_K$ is in $\sigma(\mathcal{A}_{11})$. It follows that $\mathcal{A}_{10} \subseteq \sigma(\mathcal{A}_{11})$ and thus $\sigma(\mathcal{A}_{10}) \subseteq \sigma(\mathcal{A}_{11})$. □

[5] Notation: $\mathcal{A}_3$ = intervals of the form $[x, \infty)$, $\mathcal{A}_4$ = intervals of the form (x, ∞), $\mathcal{A}_7$ = intervals of the form $[a, b)$, and $\mathcal{A}_{10}$ = closed subsets of $\mathbb{R}$.

2

MEASURES

Remark 2.1 (The de Finetti Notation). I find the *de Finetti Notation* is very excellent. Here I cite Pollard (2001, Sec.4, Ch.1).

Ordinary algebra is easier than Boolean algebra. The correspondence $A \iff \mathbb{1}_A$ between $A \subseteq \Omega$ and their indicator functions,

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases}$$

transforms Boolean algebra into ordinary pointwise algebra with functions.

The operations of union and intersection correspond to pointwise maxima ($\vee$) and pointwise minima ($\wedge$), or pointwise products:

$$\mathbb{1}_{\bigcup_i A_i}(x) = \bigvee_i \mathbb{1}_{A_i}(x), \quad \text{and} \tag{2.1}$$

$$\mathbb{1}_{\bigcap_i A_i}(x) = \bigwedge_i \mathbb{1}_{A_i}(x) = \prod_i \mathbb{1}_{A_i}(x). \tag{2.2}$$

Complements corresponds to subtraction from one:

$$\mathbb{1}_{A^c}(x) = 1 - \mathbb{1}_A(x). \tag{2.3}$$

Derived operations, such as the set theoretic difference $A \setminus B := A \cap B^c$ and the symmetric difference, $A \Delta B := (A \setminus B) \cup (B \setminus A)$, also have simple algebraic counterparts:

$$\mathbb{1}_{A \setminus B}(x) = \left[\mathbb{1}_A(x) - \mathbb{1}_B(x)\right]^+ := \max\{0, \mathbb{1}_A(x) - \mathbb{1}_B(x)\}, \tag{2.4}$$

$$\mathbb{1}_{A \Delta B}(x) = |\mathbb{1}_A(x) - \mathbb{1}_B(x)|. \tag{2.5}$$

The algebra looks a little cleaner if we omit the argument x. For example, the horrendous set theoretic relationship

$$\left(\bigcap_{i=1}^n A_i\right) \Delta \left(\bigcap_{i=1}^n B_i\right) \subseteq \bigcup_{i=1}^n (A_i \Delta B_i)$$

corresponds to the pointwise inequality

$$\left|\prod_{i=1}^{n} \mathbb{1}_{A_i} - \prod_{i=1}^{n} \mathbb{1}_{B_i}\right| \leqslant \bigvee_{i=1}^{n} \left|\mathbb{1}_{A_i} - \mathbb{1}_{B_i}\right|,$$

whose verification is easy: when the right-hand side takes the value 1 the inequality is trivial, because the left-hand side can take only the values 0 or 1; and when right-hand side takes the value 0, we have $\mathbb{1}_{A_i} = \mathbb{1}_{B_i}$ for i, which makes the left-hand side zero.

2.1 Measures

► Exercise 2.2 (2.1.1). *This problem deals with some other variants of properties (M1)–(M3).*

a. *Some define a* probability measure P on a σ-field $\mathscr{A}$ of subsets of Ω *by stipulating that (i)* $0 \leqslant P(A) \leqslant 1$ *for all* $A \in \mathscr{A}$*, (ii)* $P(\Omega) = 1$*, and (iii)* P *is countably additive. This is a special case of our definition of a measure.*

b. *If (M1) and (M3) hold for a set function* μ *defined on a field* $\mathscr{A}$ *with* $\mu(A) < +\infty$ *for some* $A \in \mathscr{A}$*, then* μ *is a measure on* $\mathscr{A}$*.*

Proof. **(a)** If $\mu\colon \mathscr{A} \to [0, +\infty]$ is a measure, define a new set-valued function $P\colon \mathscr{A} \to \overline{\mathbb{R}}$ as

$$P(A) = \frac{\mu(A)}{\mu(\Omega)}, \quad \forall\, A \in \mathscr{A}.$$

(b) We only need to check (M2): $\mu(\varnothing) = 0$. Since $\mathscr{A}$ is a field, $\varnothing \in \mathscr{A}$. Consider the following sequence $\{A, \varnothing, \varnothing, \dots\}$. (M3) implies that

$$\mu(A) = \mu(A \cup \varnothing \cup \varnothing \cup \cdots) = \mu(A) + \sum \mu(\varnothing).$$

Since $\mu(A) < +\infty$, we have $\mu(\varnothing) = 0$. □

► Exercise 2.3 (2.1.2). *Let* $\Omega = \{\omega_1, \dots, \omega_n\}$*, and let* $p_1, \dots, p_n \in [0, +\infty]$*. Define* μ *on* 2^{Ω} *as in Example 2. Then* $(\Omega, 2^{\Omega}, \mu)$ *is a measure space, and* μ *is* σ*-finite iff* $p_n < +\infty$ *for each* $n \in \mathbb{N}$*.*

Proof. To prove $(\Omega, 2^{\Omega}, \mu)$ is a measure space, we only need to prove that μ is a measure on 2^{Ω} since 2^{Ω} is a σ-field. Clearly (M1) and (M2) hold. To see (M3) hold, let $A_1, \dots, A_m \in 2^{\Omega}$ be disjoint (Since Ω is finite, we need only to check the finite additivity). Then

$$
\begin{aligned}
\mu\left(\bigcup_{i=1}^{m} A_n\right) &= \sum\left\{p_k : k \text{ is such that } \omega_k \in A_i \text{ for some } i \in \{1,\dots,n\}\right\}\\
&= \sum_{i=1}^{m}\sum\left\{p_k : k \text{ is such that } \omega_k \in A_i\right\}\\
&= \sum_{i=1}^{m}\mu\left(A_i\right).
\end{aligned}
$$

If $p_i = +\infty$ for at least one i, then μ is not σ-finite. If each p_i is finite then μ is σ-finite: take $A_i = \{\omega_i\}$, where $i \in \{1,\dots,n\}$. □

► Exercise 2.4 (2.1.3). *Let $\mathcal{A} = \{\varnothing, \Omega\}$, $\mu(\varnothing) = 0$, and $\mu(\Omega) = +\infty$. Then $(\Omega, \mathcal{A}, \mu)$ is a measure space, but μ fails to be σ-finite.*

Proof. $\{\varnothing, \Omega\}$ is a (trivial) σ-field. (M1) and (M2) hold. Now check (M3):

$$
\mu\left(\varnothing \cup \Omega\right) = \mu\left(\Omega\right) = 0 + \mu\left(\Omega\right) = \mu\left(\varnothing\right) + \mu\left(\Omega\right).
$$

Notice that $\Omega = \Omega \cup \varnothing$ or $\Omega = \Omega$, but by the hypothesis, $\mu\left(\Omega\right) = +\infty$. □

► Exercise 2.5 (2.1.4). *Let Ω be uncountable. Let $\mathcal{A} = \{A \subseteq \Omega : A$ is amc or A^c is amc$\}$. Write $\mu(A) = 0$ if A is amc and $\mu(A) = +\infty$ if A^c is amc. Then $(\Omega, \mathcal{A}, \mu)$ is a measure space, and μ is not σ-finite.*

Proof. We show $\mathcal{A}$ is a σ-field first. $\Omega \in \mathcal{F}$ since $\Omega^c = \varnothing$ is amc. If $A \in \mathcal{A}$, then either A or A^c is amc. If A is amc, $A^c \in \mathcal{A}$ because $(A^c)^c = A$ is amc; if A^c is amc, $A^c \in \mathcal{A}$ by definition of $\mathcal{A}$. To see that $\mathcal{A}$ is closed under countably union, let $\{A_n\}_{n=1}^{\infty} \subset \mathcal{A}$. There are two cases: (i) Each A_n is amc. Then $\bigcup_{n=1}^{\infty} A_n$ is amc, whence is a $\mathcal{A}$-set, and (ii) At least one A_n is such that A_n^c is amc. Without loss of generality, we assume A_1^c is amc. Since $(\bigcup_{n=1}^{\infty} A_n)^c = \bigcap_{n=1}^{\infty} A_n^c \subseteq A_1^c$, it follows that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

We then show that μ is a measure on $\mathcal{A}$. It is clear that μ is nonnegative and $\mu\left(\varnothing\right) = 0$. Now let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be disjoint. If each A_n is amc, then $\bigcup_{n=1}^{\infty} A_n$ is amc, and so $\mu(\bigcup_{n=1}^{\infty} A_n) = 0 = \sum_{n=1}^{\infty} \mu\left(A_n\right)$; if there is at least one A_n, say A_1, so that A_1^c is amc, then $(\bigcup_{n=1}^{\infty} A_n)^c$ is amc. Hence, $+\infty = \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu\left(A_n\right) = \mu(A_1) + \sum_{n=2}^{\infty} \mu\left(A_n\right) = +\infty$.

Since Ω is uncountable, which cannot be covered by a sequence of countable $\mathcal{A}$-sets. Therefore, in any cover of Ω, there exists a set A so that A^c is amc. But which means that μ is not σ-finite since $\mu(A) = +\infty$. □

► Exercise 2.6 (2.1.5). *Let Ω be arbitrary, and let $\mathcal{A} = \{A \subseteq \Omega : A$ is amc or A^c is amc$\}$. Define μ over $\mathcal{A}$ by stating that $\mu(A) = 0$ if A is amc, and $\mu(A) = 1$ if A^c is amc.*

a. *μ is not well-defined if Ω is amc, but μ is well-defined if Ω is uncountable.*

b. *μ is σ-finite measure on the σ-field $\mathcal{A}$ when Ω is uncountable.*

Proof. **(a)** If Ω is amc, we can find a set A such that both A and A^c are amc. But then (i) $\mu(A) = 0$ since A is amc, and (ii) $\mu(A) = 1$ since A^c is amc. A contradiction.

However, if Ω is uncountable, then the previous issue will not occur because if both A and A^c are amc, then $A \cup A^c = \Omega$ is amc. A contradiction.

(b) We have proved in Exercise 2.5 that $\mathcal{A}$ is a σ-field. To prove that μ is σ-finite, consider $\{\Omega, \varnothing, \varnothing, \dots\}$. □

► Exercise 2.7 (2.1.6). *Suppose that $\mathcal{A}$ is a* finite *σ-field on Ω. Suppose that μ is defined on $\mathcal{A}$ such that (M1), (M2), and (M4) hold. Then $(\Omega, \mathcal{A}, \mu)$ is a measure space.*

Proof. Since $\mathcal{A}$ is a *finite* σ-field, any *countable* union of $\mathcal{A}$-sets must take the following form

$$A_1 \cup A_2 \cup \cdots \cup A_n \cup \varnothing \cup \varnothing \cup \cdots$$

Then the proof is straightforward. □

► Exercise 2.8 (2.1.7). *Let $\mathcal{A} = \{A \subseteq \Omega : A$ is finite or A^c is finite$\}$. Define μ on $\mathcal{A}$ by*

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is finite} \\ 1 & \text{if } A^c \text{ is finite.} \end{cases}$$

a. *μ fails to be well-defined when Ω is finite.*

b. *If Ω is infinite, then μ satisfies (M1), (M2), and (M4).*

c. *Let* $\operatorname{card}(\Omega) = \aleph_0$. *Then μ is finitely additive but not countably subadditive.*

d. *When Ω is uncountable, μ is a measure. Is μ σ-finite?*

Proof. **(a)** Let Ω be finite, and both A and A^c are finite. Then $\mu(A) = 0$ and $\mu(A) = 1$ occurs.

(b) The nonnegativity and $\mu(\varnothing) = 0$ are obvious. To see finite additivity, let $\{A_i\}_{i=1}^n \subseteq \mathcal{A}$ be disjoint, and $\bigcup_{i=1}^n A_i \in \mathcal{A}$. If each A_i is finite, then $\bigcup_{i=1}^n A_i$ is finite, whence $\mu(\bigcup_{i=1}^n A_i) = 0 = \sum_{i=1}^n \mu(A_i)$; if, say, A_1^c is finite, then $[\bigcup_{i=1}^n A_i]^c = \bigcap_{i=1}^n A_i^c \subseteq A_1^c$ is finite, and $\mu(\bigcup_{i=1}^n A_i) = 1$. Notice that $A_j \subseteq A_1^c$ for all $j = 2, 3, \dots, n$ since $\{A_i\}_{i=1}^n$ is disjoint. Hence $A_2, A_3, \dots, A_n$ are all finite if A_1^c is finite. Therefore, $\sum_{i=1}^n \mu(A_i) = 1 = \mu(\bigcup_{i=1}^n A_i)$.

(c) Since $\operatorname{card}(\Omega) = \aleph_0$, Ω is infinitely countable. Then μ is well-defined and finitely additive by part (b). To show μ fails to be countably subadditive, let $\Omega = \{\omega_1, \omega_2, \dots\}$, and $A_n = \{\omega_n\}$. Hence $\mu(A_n) = 0$ and so $\sum_{n=1}^\infty \mu(A_n) = 0$. But $\bigcup_{n=1}^\infty A_n = \Omega$ and $\mu(\bigcup_{n=1}^\infty A_n) = 1$ since $(\bigcup_{n=1}^\infty A_n)^c = \varnothing$ is finite.

(d) μ is σ-finite when Ω is uncountable. Just consider the following sequence of sets $\{\Omega, \varnothing, \varnothing, \ldots\}$. Note that $\mu(\Omega) = 1 < +\infty$ as $\Omega^c = \varnothing$ is finite, and $\mu(\varnothing) = 0 < +\infty$ as $\varnothing$ is finite. Finally, $\Omega = \Omega \cup \varnothing \cup \varnothing \cup \cdots$. □

► EXERCISE 2.9 (2.1.8). *Let* card$(\Omega) = \aleph_0$ *and* $\mathcal{A} = 2^{\Omega}$. *Let*

$$\mu(A) = \begin{cases} 0 & \text{if } A \text{ is finite} \\ +\infty & \text{if } A \text{ is infinite.} \end{cases}$$

Then μ is well-defined, μ satisfies (M1), (M2), and (M4), and that (M3) fails. Also, μ is σ-finite.

PROOF. It is straightforward to see that μ is well-defined, nonnegative, and $\mu(\varnothing) = 0$. Use the ways as in the previous exercise, we prove that μ is finite additive, but not countable additive. To prove μ is σ-finite, note that card$(\Omega) = \aleph_0$ (Ω is infinitely countable), Ω can be expressed as $\Omega = \{\omega_1, \omega_2, \ldots\}$; hence, we can just consider the following sequence $\{A_1 = \{\omega_1\}, A_2 = \{\omega_2\}, \ldots\}$. □

► EXERCISE 2.10 (2.1.9). *(M5) is not true if the hypothesis $\mu(A) < +\infty$ is omitted.*

PROOF. Suppose $A \subseteq B$ with $A, B, B \setminus A \in \mathcal{A}$. Then $\mu(B) = \mu(A) + \mu(B \setminus A)$. If $\mu(A) = +\infty$, then $\mu(B) = +\infty$ since $\mu(B \setminus A) \geqslant 0$. Then $\mu(B) - \mu(A) = (+\infty) - (+\infty)$ is undefined. □

► EXERCISE 2.11 (2.1.10). *Let μ denote a measure on a σ-field $\mathcal{A}$, and let $A, A_1, A_2, \ldots \in \mathcal{A}$.*

a. $\mu(A) = \sum_{k=1}^{\infty} \mu(A \cap A_k)$ *when the A_k's are disjoint with* $\bigcup_{k=1}^{\infty} A_k = \Omega$.

b. $\mu(A_1 \Delta A_2) = 0$ *iff* $\mu(A_1) = \mu(A_2) = \mu(A_1 \cap A_2)$.

c. $\mu(A_2) = 0$ *forces both* $\mu(A_1 \cup A_2) = \mu(A_1)$ *and* $\mu(A_1 \Delta A_2) = 0$.

d. $\mu(A_2) = 0$ *forces* $\mu(A_1 \setminus A_2) = \mu(A_1)$.

PROOF. **(a)** We have $A = A \cap \left(\bigcup_k A_k\right) = \bigcup_k (A \cap A_k)$, and $\{A \cap A_k\} \subset \mathcal{A}$ is disjoint. Hence,

$$\mu(A) = \mu\left(\bigcup_{k=1}^{\infty} (A \cap A_k)\right) = \sum_{k=1}^{\infty} \mu(A \cap A_k).$$

(b) If $\mu(A_1) = \mu(A_2) = \mu(A_1 \cap A_2)$, then

$$\mu(A_1) = \mu(A_1 \setminus A_2) + \mu(A_1 \cap A_2) \Longrightarrow \mu(A_1 \setminus A_2) = 0,$$

$$\mu(A_2) = \mu(A_1 \setminus A_2) + \mu(A_1 \cap A_2) \Longrightarrow \mu(A_1 \setminus A_2) = 0.$$

Therefore, $\mu(A_1 \Delta A_2) = \mu(A_1 \setminus A_2) + \mu(A_1 \setminus A_2) = 0$.

If $\mu(A_1 \Delta A_2) = 0$, then $\mu(A_1 \setminus A_2) = \mu(A_1 \setminus A_2) = 0$. But then $\mu(A_1) = \mu(A_1 \setminus A_2) + \mu(A_1 \cap A_2) = \mu(A_1 \cap A_2)$ and $\mu(A_2) = \mu(A_1 \cap A_2)$.

(c) $0 \leqslant \mu(A_1 \cap A_2) \leqslant \mu(A_2) = 0$ implies that $\mu(A_1 \cap A_2) = 0$. By the inclusion-exclusion principle,

$$\mu(A_1 \cup A_2) = \mu(A_1) + \mu(A_2) - \mu(A_1 \cap A_2) = \mu(A_1).$$

(d) Since $A_1 \cup A_2 = (A_1 \Delta A_2) \cup (A_1 \cap A_2)$, and $(A_1 \Delta A_2) \cap (A_1 \cap A_2) = \varnothing$, we have

$$\begin{aligned} \mu(A_1 \cup A_2) = \mu(A_1 \Delta A_2) + \mu(A_1 \cap A_2) &= \mu(A_1 \Delta A_2) \\ &= \mu(A_1 \smallsetminus A_2) + \mu(A_1 \smallsetminus A_2) \\ &= \mu(A_1 \smallsetminus A_2). \end{aligned}$$ □

► EXERCISE 2.12 (2.1.11). *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space such that there is $B \in \mathcal{A}$ with $0 < \mu(B) < +\infty$. Fix such a B, and define $\mu_B : \mathcal{A} \to \overline{\mathbb{R}}$ by the formula $\mu_B(A) = \mu(A \cap B) / \mu(B)$.*

a. *$(\Omega, \mathcal{A}, \mu_B)$ is a measure space.*

b. *Suppose in addition that Ω is the disjoint union of an amc collection of sets $B_n \in \mathcal{A}$ such that each B_n has finite measure, and suppose that μ is finite. Then for all $A \in \mathcal{A}$ we have $\mu(A) = \sum_n \mu_{B_n}(A) \cdot \mu(B_n)$. Also, for each $i \in \mathbb{N}$ we have*

$$\mu_{B_i}(A) = \frac{\mu_A(B_i) \cdot \mu(A)}{\sum_n \mu_A(B_n) \cdot \mu(A)}.$$

This formula is known as Bayes' Rule.

PROOF. **(a)** If suffices to show that μ_B is a measure on $\mathcal{A}$ since $\mathcal{A}$ is a σ-field. (M1) To see $\mu_B(A) \geqslant 0$, note that $\mu(B) > 0$ and $\mu(A \cap B) \geqslant 0$. (M2) To see $\mu_B(\varnothing) = 0$, note that $\mu_B(\varnothing) = \dfrac{\mu(\varnothing \cap B)}{\mu(B)} = 0$. (M3) For countable additivity, let $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ be disjoint. Then

$$\begin{aligned} \mu_B\Big(\bigcup_{n=1}^{\infty} A_n\Big) = \frac{\mu\left[B \cap \left(\bigcup_{n=1}^{\infty} A_n\right)\right]}{\mu(B)} &= \frac{\mu\left[\bigcup_{n=1}^{\infty} (A_n \cap B)\right]}{\mu(B)} \\ &= \frac{\sum_{n=1}^{\infty} \mu(A_n \cap B)}{\mu(B)} \\ &= \sum_{n=1}^{\infty} \frac{\mu(A_n \cap B)}{\mu(B)} \\ &= \sum_{n=1}^{\infty} \mu_B(A_n). \end{aligned}$$

(b) By the assumption, we can write Ω as $\Omega = \bigcup_{n=1}^{\infty} B_n$, where $\{B_n\}_{n=1}^{\infty} \subseteq \mathcal{A}$ is disjoint, and $\mu(B_n) < +\infty$. Since μ is finite, $\mu(\Omega) < +\infty$. For the first claim,

$$\begin{aligned}\sum_n \mu_{B_n}(A)\cdot\mu(B_n) = \sum_n \frac{\mu(A\cap B_n)}{\mu(B_n)}\cdot\mu(B_n) &= \sum_n \mu(A\cap B_n)\\ &= \mu\left[\bigcup_n (A\cap B_n)\right]\\ &= \mu(A\cap\Omega)\\ &= \mu(A).\end{aligned}$$

For the *Bayes' Rule*,

$$\mu_{B_i}(A) = \frac{\mu(B_i\cap A)}{\mu(A)} = \frac{\dfrac{\mu(B_i\cap A)}{\mu(A)}\cdot\mu(A)}{\sum_n \mu_{B_n}(A)\cdot\mu(B_n)} = \frac{\mu_A(B_i)\cdot\mu(A)}{\sum_n \mu_{B_n}(A)\cdot\mu(B_n)}. \qquad \square$$

► EXERCISE 2.13 (2.1.12). *Let* $S = \{s_1,\dots,s_m\}$, *and let* $\{p_u : u \in S\}$ *denote a collection of nonnegative numbers with* $\sum_{u\in S} p_u = 1$. *Let* Ω *denote the set of sequences of* S. *For each* $\omega\in\Omega$, *write* $\omega = (z_1(\omega), z_2(\omega),\dots)$. *Given* $n\in\mathbb{N}$ *and* $H\subseteq S^n$, *let*

$$C_n(H) = \left\{\omega\in\Omega\colon \big(z_1(\omega),\dots,z_n(\omega)\big)\in H\right\}.$$

Such a set is called a cylinder of rank n. *Let* $\mathcal{F} = \{C_n(H)\colon n\in\mathbb{N}, H\subseteq S^n\}$, *so that* $\mathcal{F}$ *consists of all cylinders of all ranks. Define* $\mu:\mathcal{F}\to\overline{\mathbb{R}}$ *as follows: if for some* $n\in\mathbb{N}$ *and* $H\subseteq S^n$ *we have* $A = \{\omega\in\Omega\colon (z_1(\omega),\dots,z_n(\omega))\in H\}$, *write* $\mu(A) = \sum\big\{p_{i_1},\dots,p_{i_n} : (i_1,\dots,i_n)\in H\big\}$.

a. μ *is well defined.*

2.2 CONTINUITY OF MEASURES

I complete the Claim 1 of Vestrup (2003, p. 43) before working out the exercises for this section. Note that we sometimes take the following notation (see Rosenthal, 2006, p. 34):

$$\begin{aligned}&\liminf A_n = [A_n \text{ ev.}]\text{: } A_n \text{ eventually},\\ &\limsup A_n = [A_n \text{ i.o.}]\text{: } A_n \text{ infinitely often}.\end{aligned}$$

Ash (2009) provides an excellent treatment of lim sup and lim inf for real number sequences.

CLAIM 5. Let $\{A_n\}_{n=1}^{\infty}$ denote a sequence of subsets of Ω. Then we have the following properties:

a. $\liminf A_n = \big\{\omega\in\Omega : \omega \text{ is in all but finitely many of } A_1, A_2,\dots\big\}$.

b. $\limsup A_n = \big\{\omega\in\Omega : \omega \text{ is in infinitely many of } A_1, A_2,\dots\big\}$.

c. $\liminf A_n \subseteq \limsup A_n$.

d. If $\{A_n\}_{n=1}^{\infty}$ is nondecreasing, then $\lim A_n$ exists and equals $\bigcup_{n=1}^{\infty} A_n$.

e. If $\{A_n\}_{n=1}^{\infty}$ is nonincreasing, then $\lim A_n$ exists and equals $\bigcap_{n=1}^{\infty} A_n$.

f. If $A_1, A_2, \dots$ are disjoint, then $\lim A_n$ exists and equals $\varnothing$.

Proof. **(b)** If $\omega \in \limsup A_n$, then for all $k \in \mathbb{N}$, there exist some $n \geqslant k$ such that $\omega \in A_n$. Hence, ω is in infinitely many of $A_1, A_2, \dots$. Conversely, if ω is in infinitely many of $A_1, A_2, \dots$, then for all $k \in \mathbb{N}$, there exists $n \geqslant k$ such that $\omega \in A_n$. Therefore, $\omega \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n = \limsup A_n$.

(e) Since $A_{i+1} \subseteq A_i$, we get $\bigcup_{k \geqslant n} A_k = A_n$. Therefore,

$$\limsup A_n = \bigcap_{n=1}^{\infty} \left(\bigcup_{k \geqslant n} A_k \right) = \bigcap_{n=1}^{\infty} A_n.$$

Likewise

$$\liminf A_n = \bigcup_{n=1}^{\infty} \left(\bigcap_{k \geqslant n} A_k \right) \supseteq \bigcap_{k=1}^{\infty} A_k = \limsup A_n \supseteq \liminf A_n.$$

Thus equality prevails and so $\lim A_n = \bigcap_{n=1}^{\infty} A_n$. □

► Exercise 2.14 (2.2.1). $\liminf \mathbb{1}_{A_n} = \mathbb{1}_{\liminf A_n}$ *and* $\limsup \mathbb{1}_{A_n} = \mathbb{1}_{\limsup A_n}$.

Proof. We first show

$$\mathbb{1}_{\bigcap_{n=k}^{\infty} A_n} = \inf_{n \geqslant k} \mathbb{1}_{A_n}, \tag{2.6}$$

$$\mathbb{1}_{\bigcup_{n=k}^{\infty} A_n} = \sup_{n \geqslant k} \mathbb{1}_{A_n}. \tag{2.7}$$

To prove (2.6), we must show that the two functions are equal. But

$$\begin{aligned}
\mathbb{1}_{\bigcap_{n=k}^{\infty} A_n}(\omega) = 1 &\iff \omega \in \bigcap_{n=k}^{\infty} A_n \\
&\iff \omega \in A_n \text{ for all } n \geqslant k \\
&\iff \mathbb{1}_{A_n}(\omega) = 1 \text{ for all } n \geqslant k \\
&\iff \inf_{n \geqslant k} \mathbb{1}_{A_n}(\omega) = 1.
\end{aligned}$$

Similarly, (2.7) holds since

$$\begin{aligned}\mathbb{1}_{\bigcup_{n=k}^{\infty} A_n}(\omega) = 1 &\iff \omega \in \bigcup_{n=k}^{\infty} A_n \\ &\iff \omega \in A_n \text{ for some } n \geqslant k \\ &\iff \mathbb{1}_{A_n}(\omega) = 1 \text{ for some } n \geqslant k \\ &\iff \sup_{n \geqslant k} \mathbb{1}_{A_n}(\omega) = 1.\end{aligned}$$

Hence,

$$\mathbb{1}_{\liminf A_n} = \mathbb{1}_{\bigcup_{n=1}^{\infty}(\bigcap_{k=n}^{\infty} A_k)} = \sup_{n \geqslant 1} \mathbb{1}_{\bigcap_{k=n}^{\infty} A_k} = \sup_{n \geqslant 1} \inf_{k \geqslant n} \mathbb{1}_{A_k} = \liminf \mathbb{1}_{A_n},$$

$$\mathbb{1}_{\limsup A_n} = \mathbb{1}_{\bigcap_{n=1}^{\infty}(\bigcup_{k=n}^{\infty} A_k)} = \inf_{n \geqslant 1} \mathbb{1}_{\bigcup_{k=n}^{\infty} A_k} = \inf_{n \geqslant 1} \sup_{k \geqslant n} \mathbb{1}_{A_k} = \limsup \mathbb{1}_{A_n}.$$

Alternatively, we have

$$\begin{aligned}\liminf \mathbb{1}_{A_n}(\omega) = 1 \iff \mathbb{1}_{A_n}(\omega) = 1 \text{ ev.} &\iff \omega \in A_n \text{ ev.} \\ &\iff \omega \in \liminf A_n \\ &\iff \mathbb{1}_{\liminf A_n}(\omega) = 1,\end{aligned}$$

$$\begin{aligned}\limsup \mathbb{1}_{A_n}(\omega) = 1 \iff \mathbb{1}_{A_n}(\omega) = 1 \text{ i.o.} &\iff \omega \in A_n \text{ i.o.} \\ &\iff \omega \in \limsup A_n \\ &\iff \mathbb{1}_{\limsup A_n}(\omega) = 1.\end{aligned}$$ □

► EXERCISE 2.15 (2.2.2). *Show that* $\liminf A_n \subseteq \limsup A_n$ *without using the representations of* $\liminf A_n$ *and* $\limsup A_n$ *given in parts (a) and (b) of Claim 1.*

PROOF. Notice that

$$\begin{aligned}\omega \in \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n &\iff \exists\, k^* \in \mathbb{N} \text{ such that } \omega \in A_n,\ \forall\, n \geqslant k^* \\ &\Longrightarrow \forall\, k \in \mathbb{N},\ \exists\, n \geqslant k \text{ such that } \omega \in A_n \\ &\Longrightarrow \omega \in \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n,\end{aligned}$$

where the first "$\Longrightarrow$" holds because if $k < k^*$, then $\omega \in A_n$ for all $n \geqslant k^*$; if $k \geqslant k^*$, then $\omega \in A_n$ for all $n \geqslant k$. □

► EXERCISE 2.16 (2.2.3). $(\liminf A_n)^c = \limsup A_n^c$ *and* $(\limsup A_n)^c = \liminf A_n^c$.

PROOF. These results are analog to $-\liminf x_n = \limsup(-x_n)$ and $-\limsup x_n = \liminf(-x_n)$. We have two methods to prove these claims. Here is the **Method 1**:

$$(\liminf A_n)^c = \left(\bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} A_n\right)^c = \bigcap_{k=1}^{\infty} \left(\bigcap_{n=k}^{\infty} A_n\right)^c = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} A_n^c = \limsup A_n^c.$$

$$(\limsup A_n)^c = \left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n\right)^c = \bigcup_{k=1}^{\infty}\left(\bigcup_{n=k}^{\infty} A_n\right)^c = \bigcup_{k=1}^{\infty}\bigcap_{n=k}^{\infty} A_n^c = \liminf A_n^c.$$

Here is **Methods 2**:

$$\begin{aligned} x \in (\liminf A_n)^c &\iff \neg\left[(\exists\, N \in \mathbb{N})\,(\forall\, n \geqslant N)\,(x \in A_n)\right] \\ &\iff (\forall\, N \in \mathbb{N})\,(\exists\, n \geqslant N)\,\left(x \in A_n^c\right) \\ &\iff x \in \limsup A_n^c; \end{aligned}$$

$$\begin{aligned} x \in (\limsup A_n)^c &\iff \neg\left[(\forall\, N \in \mathbb{N})\,(\exists n \geqslant N)\, x \in A_n\right] \\ &\iff (\exists\, N \in \mathbb{N})\,(\forall\, n \geqslant N)\,\left(x \in A_n^c\right) \\ &\iff x \in \liminf A_n^c. \end{aligned}$$ □

► Exercise 2.17 (2.2.4). *If $B_n \neq A_n$ for at most finitely many $n \in \mathbb{N}$, then* $\liminf A_n = \liminf B_n$ *and* $\limsup A_n = \limsup B_n$*. In other words,* $\liminf A_n$*,* $\limsup A_n$ *[and* $\lim A_n$*] are not changed if a finite number of A_k's are altered.*

Proof. Just follow Claim 1. □

► Exercise 2.18 (2.2.5). *We have the following relations:*

a. $\limsup(A_n \cup B_n) = \limsup A_n \cup \limsup B_n$ *and* $\liminf(A_n \cap B_n) = \liminf A_n \cap \liminf B_n$.

b. $\limsup(A_n \cap B_n) \subseteq \limsup A_n \cap \limsup B_n$ *and* $\liminf(A_n \cup B_n) \supseteq \liminf A_n \cup \liminf B_n$*. Both containment relations can be strict.*

c. $\mathbb{1}_{\limsup(A_n \cup B_n)} = \max\{\mathbb{1}_{\limsup A_n}, \mathbb{1}_{\limsup B_n}\}$ *and* $\mathbb{1}_{\liminf(A_n \cap B_n)} = \min\{\mathbb{1}_{\liminf A_n}, \mathbb{1}_{\liminf B_n}\}$.

Proof. **(a)** By definition, $\limsup(A_n \cup B_n) = [A_n \cup B_n \text{ i.o.}] = [A_n \text{ i.o.}] \cup [B_n \text{ i.o.}]$, and $\liminf(A_n \cap B_n) = [A_n \cap B_n \text{ ev.}] = [A_n \text{ ev.}] \cap [B_n \text{ ev.}]$.

(b) We have

$$\begin{aligned} \omega \in \limsup(A_n \cap B_n) &\iff \omega \text{ is in infinitely many of } A_1 \cap B_1, A_2 \cap B_2, \ldots \\ &\overset{*}{\Longrightarrow} \omega \text{ is in infinitely many of } A_1, A_2, \ldots \text{ and } B_1, B_2, \ldots \\ &\Longrightarrow \omega \in \limsup A_n \text{ and } \omega \in \limsup B_n \\ &\Longrightarrow \omega \in \limsup A_n \cap \limsup B_n, \end{aligned}$$

where $(*)$ holds with "$\Longrightarrow$" rather than "$\iff$" because, e.g., let

$$\omega \in \begin{cases} A_i & \text{if } i \text{ is odd} \\ B_j & \text{if } j \text{ is even;} \end{cases} \tag{2.8}$$

then ω is in infinitely many of $A_1, A_2, \ldots$, and ω is in infinitely many of $B_1, B_2, \ldots$. However, if $A_n \cap B_n = \varnothing$ for all $n \in \mathbb{N}$, then ω is not in any of $A_n \cap B_n$.

Likewise,

$$
\begin{aligned}
&\omega \in \liminf A_n \cup \liminf B_n \\
\iff &\omega \text{ is in all but finitely many of } A_1, A_2, \ldots \text{ or } B_1, B_2, \ldots \\
\stackrel{**}{\Longrightarrow} &\omega \text{ is in all but finitely many of } (A_1 \cup B_1), (A_2 \cup B_2), \ldots \\
\iff &\omega \in \liminf (A_n \cup B_n),
\end{aligned}
$$

where $(**)$ holds with "$\Longrightarrow$" rather than "$\iff$" because, e.g., consider (2.8); then ω is in all of $A_1 \cup B_1$, $A_2 \cup B_2$, ..., but there does not exist N such that ω is in all A_n for all $n \geqslant N$ or N' such that ω is in all B_n for all $n \geqslant N'$.

(c) We have

$$
\begin{aligned}
\mathbb{1}_{\limsup (A_n \cup B_n)}(\omega) = 1 &\iff \omega \in \limsup (A_n \cup B_n) \\
&\iff \omega \in \left(\limsup A_n \cup \limsup B_n\right) \\
&\iff \max\{\mathbb{1}_{\limsup A_n}(\omega), \mathbb{1}_{\limsup B_n}(\omega)\} = 1.
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{1}_{\liminf (A_n \cap B_n)} = 1 &\iff \omega \in \liminf (A_n \cap B_n) \\
&\iff \omega \in \left(\liminf A_n \cap \liminf B_n\right) \\
&\iff \min\left\{\mathbb{1}_{\liminf A_n}(\omega), \mathbb{1}_{\liminf B_n}(\omega)\right\} = 1. \qquad \square
\end{aligned}
$$

► EXERCISE 2.19 (2.2.6). *If $A_n \to A$ and $B_n \to B$, then $A_n \cup B_n \to A \cup B$, $A_n^c \to A^c$, $A_n \cap B_n \to A \cap B$,$A_n \smallsetminus B_n \to A \smallsetminus B$, and $A_n \Delta B_n \to A \Delta B$.*

PROOF. (i) We have $\limsup (A_n \cup B_n) = \limsup A_n \cup \limsup B_n = A \cup B$, $\liminf (A_n \cup B_n) \supseteq \liminf A_n \cup \liminf B_n = A \cup B$, and $\liminf (A_n \cup B_n) \subseteq \limsup (A_n \cup B_n) = A \cup B$. Therefor, $A_n \cup B_n \to A \cup B$.

(ii) Notice that $\limsup A_n^c = (\liminf A_n)^c = A^c$, and on the other hand $\liminf A_n^c = (\limsup A_n)^c = A^c$.

(iii) We have $\liminf (A_n \cap B_n) = \liminf A_n \cap \liminf B_n = A \cap B$, $\limsup (A_n \cap B_n) \subseteq \limsup A_n \cap \limsup B_n = A \cap B$, and $\limsup (A_n \cap B_n) \supseteq \liminf (A_n \cap B_n)$.

(iv) Note that $A_n \smallsetminus B_n = A_n \cap B_n^c$. We have known that $B_n^c \to B^c$, so $A_n \smallsetminus B_n \to A \cap B^c = A \smallsetminus B$ by (ii) and (iii).

(v) $A_n \Delta B_n = (A_n \smallsetminus B_n) \cup (B_n \smallsetminus A_n)$. Since $A_n \smallsetminus B_n \to A \smallsetminus B$ and $B_n \smallsetminus A_n \to B \smallsetminus A$ by (iv), we have $A_n \Delta B_n \to (A \smallsetminus B) \cup (B \smallsetminus A) = A \Delta B$ by (i). $\square$

► EXERCISE 2.20 (2.2.7). *If A_n is B or C as n is even or odd, then* $\liminf A_n = B \cap C$*, and* $\limsup A_n = B \cup C$.

PROOF. We have

$$
\begin{aligned}
\liminf A_n &= [A_n \text{ ev.}] = B \cap C, \\
\limsup A_n &= [A_n \text{ i.o.}] = B \cup C. \qquad \square
\end{aligned}
$$

► EXERCISE 2.21 (2.2.8). $\limsup A_n \smallsetminus \liminf A_n = \limsup (A_n \cap A_{n+1}^c) = \limsup (A_n^c \cap A_{n+1})$.

PROOF. We have

$$
\begin{aligned}
&x \in \limsup (A_n \cap A_{n+1}^c) \\
&\qquad \iff (\forall\, N \in \mathbb{N})(\exists\, n \geqslant N)(x \in A_n \text{ and } x \in A_{n+1}^c) \\
&\qquad \iff (x \in A_n \text{ i.o.}) \text{ and } (x \in A_n^c \text{ i.o.}) \\
&\qquad \iff x \in (\limsup A_n \cap \limsup A_n^c) = \limsup A_n \setminus \liminf A_n .
\end{aligned}
$$

The other equality can be proved similarly. ◻

► EXERCISE 2.22 (2.2.9). a. $\limsup_n \liminf_k (A_n \cap A_k^c) = \varnothing$.

b. $A \setminus \limsup_k A_k = \liminf_k (A \setminus A_k)$.

c. $\limsup_n \left(\liminf_k A_k \setminus A_n\right) = \varnothing$.

d. $\limsup_n (A \setminus A_n) = A \setminus \liminf_n A_n$ *and* $\limsup_n (A_n \setminus A) = \limsup_n A_n \setminus A$.

e. $\limsup_n (A \Delta A_n) = \left(A \setminus \liminf_n A_n\right) \cup \limsup_n (A_n \setminus A)$.

f. $A_n \to A$ *implies that* $\limsup_n (A \Delta A_n) = \limsup_n (A_n \setminus A)$.

g. *For arbitrary set E, F, G and H we have* $(E \Delta F)\, \Delta\, (G \Delta H) = (E \Delta G)\, \Delta\, (F \Delta H)$. *We also have for any set A that*

$$
\begin{aligned}
\limsup_k A_k \setminus \liminf_k A_k &= \liminf_k A_k \Delta \limsup_k A_k \\
&= \left(\liminf_k A_k \Delta \limsup_k A_k\right) \Delta\, (A \Delta A) \\
&= \left(\liminf_k A_k \Delta A\right) \Delta \left(\limsup_k A_k \Delta A\right).
\end{aligned}
$$

PROOF. **(a)**

$$
\begin{aligned}
\limsup_n \liminf_k (A_n \cap A_k^c)
&= \limsup_n \left[\liminf_k (A_n \cap A_k^c)\right] \\
&= \limsup_n \left[\liminf_k A_n \cap \liminf_k A_k^c\right] && \text{[by Exercise 2.18(a)]} \\
&= \limsup_n \left[A_n \cap \liminf_k A_k^c\right] \\
&= \limsup_n \left[A_n \cap \left(\limsup_k A_k\right)^c\right] \\
&\subseteq \left(\limsup_n A_n\right) \cap \left[\limsup_n \left(\limsup_k A_k\right)^c\right] && \text{[by Exercise 2.18(b)]} \\
&= \left(\limsup_n A_n\right) \cap \left(\limsup_k A_k\right)^c \\
&= \varnothing.
\end{aligned}
$$

(b) $\liminf_k (A \smallsetminus A_k) = \liminf_k (A \cap A_k^c) = \liminf_k A \cap \liminf_k A_k^c = A \cap (\limsup_k A_k)^c = A \smallsetminus \limsup_k A_k$.

(c) $\limsup_n (\liminf_k A_k \smallsetminus A_n) \subseteq [\limsup_n (\liminf_k A_k)] \cap (\limsup_n A_n^c) = \left(\liminf_k A_k\right) \cap \left(\liminf_n A_n\right)^c = \varnothing$.

(d) $\limsup_n (A \smallsetminus A_n) = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty \left(A \cap A_n^c\right) = \bigcap_{k=1}^\infty \left[A \cap \left(\bigcup_{n=k}^\infty A_n^c\right)\right] = A \cap \left[\bigcap_{k=1}^\infty \bigcup_{n=k}^\infty A_n^c\right] = A \cap \left(\limsup_n A_n^c\right) = A \cap \left(\liminf_n A_n\right)^c = A \smallsetminus \liminf_n A_n$, and $\limsup_n (A_n \smallsetminus A) = \bigcap_{k=1}^\infty \bigcup_{n=k}^\infty (A_n \cap A^c) = \bigcap_{k=1}^\infty \left[\left(\bigcup_{n=k}^\infty A_n\right) \cap A^c\right] = \limsup_n A_n \smallsetminus A$.

(e) $\limsup_n (A \Delta A_n) = \limsup_n [(A \smallsetminus A_n) \cup (A_n \smallsetminus A)] = \limsup_n (A \smallsetminus A_n) \cup \limsup_n (A_n \smallsetminus A) = (A \smallsetminus \liminf_n A_n) \cup \limsup_n (A_n \smallsetminus A)$.

(f) If $A_n \to A$, then $\limsup A_n = \liminf A_n = A$. Hence, $\limsup_n (A \Delta A_n) = \left(A \smallsetminus \liminf_n A_n\right) \cup \limsup_n (A_n \smallsetminus A) = (A \smallsetminus A) \cup \limsup_n (A_n \smallsetminus A) = \limsup_n (A_n \smallsetminus A)$.

(g) We first show for all $A, B, C \in 2^\Omega$,

$$
(A \Delta B) \Delta C = A \Delta (B \Delta C). \tag{2.9}
$$

This equation hold because[1]

[1] This proof is not elegant. See Resnick (1999, Exercise 1.43).

$$
\begin{aligned}
(A\Delta B)\Delta C &= \big[(A \smallsetminus B)\cup(B\smallsetminus A)\big]\,\Delta C \\
&= \Big\{\big[(A \smallsetminus B)\cup(B\smallsetminus A)\big]\smallsetminus C\Big\}\cup\Big\{C\smallsetminus\big[(A \smallsetminus B)\cup(B\smallsetminus A)\big]\Big\} \\
&= \Bigg\{\Big[\big(A\cap B^c\big)\cup(A^c\cap B)\Big]\cap C^c\Bigg\}\cup\Bigg\{\Big[\big(A\cap B^c\big)\cup(A^c\cap B)\Big]^c\cap C\Bigg\} \\
&= \big[(A\cap B^c\cap C^c)\cup(A^c\cap B\cap C^c)\big]\cup\Bigg\{\Big[\big(A\cap B^c\big)^c\sqcap(A^c\cap B)^c\Big]\cap C\Bigg\} \\
&\overset{*}{=} \big[(A\cap B^c\cap C^c)\cup(A^c\cap B\cap C^c)\big]\cup\Big\{\big[(A\cap B)\cup(A^c\cap B^c)\big]\cap C\Big\} \\
&= \big[(A\cap B^c\cap C^c)\cup(A^c\cap B\cap C^c)\big]\cup\big[(A\cap B\cap C)\cup(A^c\cap B^c\cap C)\big] \\
&= \Bigg\{A\cap\Big[\big(B^c\cap C^c\big)\cup(B\cap C)\Big]\Bigg\}\bigcup\Bigg\{A^c\cap\Big[\big(B\cap C^c\big)\cup\big(B^c\cap C\big)\Big]\Bigg\} \\
&= \Bigg\{A\cap\Big[(B\cup C)\sqcap\big(B^c\cup C^c\big)\Big]^c\Bigg\}\bigcup\big[A^c\cap(B\Delta C)\big] \\
&\overset{**}{=} \big[A\cap(B\Delta C)^c\big]\bigcup\big[A^c\cap(B\Delta C)\big] \\
&= A\Delta\,(B\Delta C)\,,
\end{aligned}
$$

where equality $(*)$ holds because

$$
\begin{aligned}
\big(A\cap B^c\big)^c\cap(A^c\cap B)^c &= \big(A^c\cup B\big)\cap\big(A\cup B^c\big) \\
&= \Big[\big(A^c\cup B\big)\cap A\Big]\cup\Big[\big(A^c\cup B\big)\cap B^c\Big] \\
&= (A\cap B)\cup(A^c\cap B^c),
\end{aligned}
$$

and equality $(**)$ holds because

$$
\begin{aligned}
(B\cup C)\sqcap\big(B^c\cup C^c\big) &= \big[(B\cup C)\cap B^c\big]\cup\big[(B\cup C)\cap C^c\big] \\
&= \big(B^c\cap C\big)\cup\big(B\cap C^c\big)\,.
\end{aligned}
$$

By (2.9), we have

$$
\begin{aligned}
(E\Delta F)\,\Delta\,(G\Delta H) = E\Delta\big[F\Delta\,(G\Delta H)\big] &= E\Delta\big[F\Delta\,(H\Delta G)\big] \\
&= E\Delta\big[(F\Delta H)\,\Delta G\big] \\
&= E\Delta\big[G\Delta\,(F\Delta H)\big] \\
&= (E\Delta G)\,\Delta\,(F\Delta H)\,.
\end{aligned}
$$

Now it suffices to show that $(\liminf_k A_k)\Delta(\limsup_k A_k) = \big(\limsup_k A_k\big)\smallsetminus\big(\liminf_k A_k\big)$. Notice that

$$
\begin{aligned}
&(\liminf A_k)\Delta(\limsup A_k) \\
&\qquad= \Big[\big(\limsup A_k\big)\smallsetminus\big(\liminf A_k\big)\Big]\cup\Big[\big(\liminf A_k\big)\smallsetminus\big(\limsup A_k\big)\Big] \\
&\qquad= \Big[\big(\limsup A_k\big)\smallsetminus\big(\liminf A_k\big)\Big]\cup\Big[\big(\liminf A_k\big)\cap\big(\liminf A_k^c\big)\Big] \\
&\qquad\overset{***}{=} \big(\limsup A_k\big)\smallsetminus\big(\liminf A_k\big)\,,
\end{aligned}
$$

where $(***)$ holds because $(\liminf_k A_k) \cap (\liminf_k A_k^c) = [A_k \text{ ev.}] \cap [A_k^c \text{ ev.}] = \varnothing$. ◻

► EXERCISE 2.23 (2.2.10). *Let $\Omega = \mathbb{N}$, and let $\mathcal{A} = 2^{\Omega}$. Define μ on $\mathcal{A}$ by $\mu(A) =$ number of points in A if A is finite; define $\mu(A) = +\infty$ if A is infinite.*

a. *μ is a measure on $\mathcal{A}$. This measure is called the* counting measure.

b. *There exists a nonincreasing sequence $\{A_n\}_{n=1}^{\infty}$ of $\mathcal{A}$-sets with $\mu(A_n) = +\infty$ for all $n \in \mathbb{N}$ but $\mu\left(\lim_n A_n\right) = 0$, thus (M9) accordingly fails to hold, hence the assumption that some A_k must have finite measure cannot be dropped.*

PROOF. **(a)** can be found in Vestrup (2003, Example 2, p. 37). For (b), let $A_n = \{n, n+1, \dots\}$ for each $n \in \mathbb{N}$, then $A_n \downarrow \varnothing$, $\mu(A_n) = +\infty$, but $\mu\left(\lim_n A_n\right) = 0$. ◻

► EXERCISE 2.24 (2.2.11). *Let $(\Omega, \mathcal{A}, \mu)$ denote a measure space. Let $\{A_x : x \in \mathbb{R}, x > 0\}$ denote a collection of $\mathcal{A}$-sets.*

a. *Suppose that $0 < x < y$ implies $A_x \subseteq A_y$. Then (i) $\bigcup_{x>0} A_x \in \mathcal{A}$, (ii) $x_1 < x_2$ implies $\mu(A_{x_1}) \leqslant \mu(A_{x_2})$, and (iii) $\mu(A_x) \to \mu(\bigcup_{y>0} A_y)$ as $x \to +\infty$.*

b. *Suppose that $0 < x < y$ implies $A_x \supseteq A_y$. Also, further assume that $\mu(A_z) < \infty$ for some $z > 0$. Then (i) $\bigcap_{x>0} A_x \in \mathcal{A}$, (ii) $x_1 < x_2$ implies $\mu(A_{x_1}) \geqslant \mu(A_{x_2})$, and (iii) $\mu(A_x) \to \mu(\bigcap_{y>0} A_y)$ as $x \to +\infty$. The assumption $\mu(A_z) < +\infty$ for some $z > 0$ cannot be dropped. This and (a) generalize (M8) and (M9) from monotone sequences of sets to monotone [uncountable] collections of sets.*

PROOF. **(a)** Denote $\{A_x : x \in \mathbb{R}, x > 0\}$ as $\mathcal{A}_{\mathbb{R}}$. Define a subset $\mathcal{A}_{\mathbb{N}}$ of $\mathcal{A}_{\mathbb{R}}$ as follows:

$$\mathcal{A}_{\mathbb{N}} = \{A_n \in \mathcal{A} : n \in \mathbb{N}, n > 0\}.$$

Then, for every $x \in \mathbb{R}$ and $x > 0$, there exists $n \in \mathbb{N}$ such that $x \leqslant n$ (by the Archimedan property; see Rudin 1976, Theorem 1.20); that is, $A_x \subseteq A_n$. Thus,

$$\bigcup \mathcal{A}_{\mathbb{R}} = \bigcup \mathcal{A}_{\mathbb{N}} \in \mathcal{A}.$$

(b) Define $\mathcal{B} = \{A_{1/n} : n \in \mathbb{N}, n > 0\}$. Then for every $x \in \mathbb{R}$ and $x > 0$, there exists $n \in \mathbb{N}$ with $n > 0$ such that $x < 1/n$; that is, $A_{1/n} \subseteq A_x$. Thus, $\bigcap \mathcal{A}_{\mathbb{R}} = \bigcap \mathcal{B}$. ◻

► EXERCISE 2.25 (2.2.12). *Let $(\Omega, \mathcal{A}, \mu)$ denote a measure space.*

a. *μ is σ-finite iff there is a nondecreasing sequence $A_1 \subseteq A_2 \subseteq \cdots$ of $\mathcal{A}$-sets with $\mu(A_n) < +\infty$ for all $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} A_n = \Omega$.*

b. *μ is σ-finite iff there is a disjoint sequence $A_1, A_2, \dots$ of $\mathcal{A}$-sets with $\mu(A_n) < +\infty$ for all $n \in \mathbb{N}$ and $\bigcup_{n=1}^{\infty} A_n = \Omega$.*

c. *Let $\mu_1, \ldots, \mu_n$ denote σ-finite measures on $\mathcal{A}$. Then there exists a sequence $\{A_m\}_{m=1}^\infty$ of $\mathcal{A}$-sets such that (i) $\mu_i(A_j) < +\infty$ for all $i = 1, \ldots, n$ and $j \in \mathbb{N}$ and (ii) $\Omega = \bigcup_{m=1}^\infty A_m$. These sets may be chosen to be nondecreasing or disjoint.*

d. *Does (c) hold if we have countably many σ-finite measures on $\mathcal{A}$ as compared to finitely many σ-finite measures on $\mathcal{A}$.*

PROOF. **(a)** The if part is trivial. So assume that μ is σ-finite. Then there exists $\{B_n\} \subseteq \mathcal{A}$ with $\mu(B_n) < +\infty$ for all $n \in \mathbb{N}$ and $\bigcup B_n = \Omega$. Let $A_n = \bigcup_{k=1}^n B_k$. Then $\{A_n\}$ is nondecreasing and $\bigcup B_n = \bigcup A_n = \Omega$.

(b) Again, the if part is trivial. So assume that μ is σ-finite. Let $\{B_n\}$ as in (a). Let $A_1 = B_1$, and $A_n = B_n \setminus (\bigcup_{i=1}^{n-1} B_i)$ for $n \geqslant 2$. Then $\{A_n\}$ is disjoint and $\bigcup B_n = \bigcup A_n = \Omega$.

(c) Let $\mu_1, \ldots, \mu_n$ be σ-finite. Then for each $i = 1, \ldots, n$, there exists $\{A_{ik}\} \subset \mathcal{A}$ such that $\mu_i(A_{ik}) < +\infty$ for all $k \in \mathbb{N}$ and $\bigcup_k A_{ik} = \Omega$. Now let $A_m = \bigcup_{\ell=1}^n A_{\ell m}$. For each i and j,

$$\mu_i(A_j) = \mu_i\left(\bigcup_{\ell=1}^n A_{\ell j}\right) \leqslant \sum_{\ell=1}^n \mu_i(A_j) < +\infty, \tag{2.10}$$

and $\bigcup A_m = \Omega$. It follows from (a) and (b) that $\{A_m\}$ may be chosen as nondecreasing or disjoint.

(d) (c) may not hold if we have countably many σ-finite measures on $\mathcal{A}$ since (2.10) may fail. □

► EXERCISE 2.26 (2.2.13). *Let μ denote a measure on a σ-field $\mathcal{A}$, and let $A_1, A_2, \ldots \in \mathcal{A}$ be such that $\mu(\bigcup_{j=N}^\infty \liminf_k (A_j \cap A_k^c)) < +\infty$ for some $N \in \mathbb{N}$. Use (M10) and parts (a)—(c) of Exercise 2.22 to show the following claims:*

a. $\lim_n \mu(\liminf_k (A_n \cap A_k^c))$ *exists and equals zero.*

b. $\lim_n \mu(A_n \setminus \limsup_k A_k)$ *exists and equals zero.*

c. $\lim_n \mu((\liminf_k A_k) \setminus A_n)$ *exists and equals zero.*

PROOF. **(a)** Since there exists $N \in \mathbb{N}$ such that $\mu(\bigcup_{j=N}^\infty \liminf_k (A_j \setminus A_k)) < +\infty$, it follows from (M10) and Exercise 2.22(a) that

$$
\begin{aligned}
\mu\left(\liminf_n\left[\liminf_k(A_n \smallsetminus A_k)\right]\right) &\leqslant \liminf_n \mu\left(\liminf_k(A_n \smallsetminus A_k)\right)\\
&\leqslant \limsup_n \mu\left(\liminf_k(A_n \smallsetminus A_k)\right)\\
&\leqslant \mu\left(\limsup_n\left[\liminf_k(A_n \smallsetminus A_k)\right]\right)\\
&= \mu(\varnothing)\\
&= 0.
\end{aligned}
$$

Thus, $\lim_n \mu(\liminf_k(A_n \cap A_k^c)) = 0$.

(b) Notice that $A_n \smallsetminus \limsup_k A_k = \liminf_k(A_n \smallsetminus A_k)$ by Exercise 2.22(b). Then (b) follows from (a) immediately.

(c) Using (M10) and Exercise 2.22(c), we get (c). □

▶ Exercise 2.27 (2.2.16). *Let $\mathscr{A}$ be a field on Ω, and suppose that $\mu\colon \mathscr{A} \to \overline{\mathbb{R}}$ satisfies (M1) with $\mu(\Omega) < +\infty$, (M2), (M4) (and hence (M5)), and in addition is continuous from above at $\varnothing$. Then μ is a measure.*

Proof. Let $\{B_n\} \subset \mathscr{A}$ be disjoint, and $\bigcup_n B_n \in \mathscr{A}$. For $n \geqslant 2$, let $C_n = \bigcup_{k=n}^{\infty} B_k$. Then $\{C_n\}$ is nonincreasing and converges to

$$
\bigcap_n C_n = \bigcap_n \bigcup_{k=n}^{\infty} B_k = \limsup_n B_n = \varnothing.
$$

Then $\lim_n \mu(C_n) = 0$; that is,

$$
\begin{aligned}
0 = \lim_n \mu\left(\bigcup_{k=n}^{\infty} B_k\right) &= \lim_n \mu\left(\left(\bigcup_{k=1}^{\infty} B_k\right) \smallsetminus \left(\bigcup_{i=1}^{n-1} B_i\right)\right)\\
&= \mu\left(\bigcup_n B_n\right) - \lim_n \mu\left(\bigcup_{i=1}^{n-1} B_i\right)\\
&= \mu\left(\bigcup_n B_n\right) - \lim_n \sum_{i=1}^{n-1} \mu(B_i)\\
&= \mu\left(\bigcup_n B_n\right) - \sum_{n=1}^{\infty} \mu(B_n),
\end{aligned}
$$

i.e., $\mu(\bigcup_n B_n) = \sum_n \mu(B_n)$. □

▶ Exercise 2.28 (2.2.17). *Let $\Omega = (0,1]$, and let $\mathscr{F}$ consist of $\varnothing$ and the finite disjoint unions of rsc subintervals of $(0,1]$. Then $\mathscr{F}$ is a field. Define μ on $\mathscr{F}$ as follows: $\mu(A) = 1$ if there exists $\varepsilon_A > 0$ with $(1/2, 1/2 + \varepsilon_A] \subseteq A$ and $\mu(A) = 0$*

otherwise. Then μ is well-defined and satisfies (M1), (M2), and (M4), but μ is not countably additive.

Proof. We first show that $\mathcal{F}$ is a field. Suppose that $A = (a_1, a_1'] \cup \cdots (a_m, a_m']$, where the notation is so chosen that $a_1 \leqslant \cdots a_m$. If the $(a_i, a_i']$ are disjoint, then $A^c = (0, a_1] \cup (a_1', a_2] \cup \cdots \cup (a_{m-1}', a_m] \cup (a_m', 1]$ and so lies in $\mathcal{F}$ (some of these intervals may be empty, as a_i' and a_{i+1} may coincide). If $B = (b_1, b_1'] \cup \cdots \cup (b_n, b_n']$, then $(b_j, b_j']$ again disjoint, then

$$A \cap B = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} \Big[(a_i, a_i'] \cap (b_j, b_j'] \Big];$$

each intersection here is again an interval or else the empty set, and the union is disjoint, and hence $A \cap B \in \mathcal{F}$.

Nevertheless, $\mathcal{F}$ is not a σ-field: It does not contain the singleton $\{x\}$, even though each is a countable intersection $\bigcap_n (x - 1/n, x]$ of $\mathcal{F}$-sets.

The set function μ defined above is not countably additive. Counter the rational number on $(0, 1]$ starting 1: $\{1, x_1, x_2, \ldots\}$. This set is countable. Consider the collection $\{(x_1, 1], (x_2, x_1], \ldots\}$. Then $\mu(\bigcup (x_i, x_{i-1}]) = 1$, however, $\sum_{i=1}^{\infty} \mu((x_i, x_{i-1}]) = 0$. ☐

► Exercise 2.29 (2.2.18). *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Suppose that μ is* nonatomic*: $A \in \mathcal{A}$ and $\mu(A) > 0$ imply that there exists $B \subseteq A$ with $B \in \mathcal{A}$ with $0 < \mu(B) < \mu(A)$.*

a. *The measure μ of Example 2 in Section 2.1 is atomic.*

b. *Suppose $A \in \mathcal{A}$ is such that $\mu(A) > 0$, and let $\varepsilon > 0$ be given. Then there exists $B \in \mathcal{A}$ with $B \subseteq A$ and $0 < \mu(B) < \varepsilon$.*

c. *Let $A \in \mathcal{A}$ be such that $\mu(A) > 0$. Given any $0 \leqslant \alpha \leqslant \mu(A)$ there exists a set $B \in \mathcal{A}$ with $B \subseteq A$ and $\mu(B) = \alpha$.*

Proof. **(a)** Let $A = \{\omega_0\}$. Then $A \in 2^{\Omega}$ and $\mu(A) = 1$. The only subsets of A (in $\mathcal{A}$) is $\varnothing$ and A. But $\mu(\varnothing) = 0$ and $\mu(A) = \mu(A)$. So the unit mass at ω_0 is atomic.

(b) Take an arbitrary $A_1 \in \mathcal{A}$ with $A_1 \subseteq A$ and $0 < \mu(A_1) < \mu(A)$. Since $\mu(A) = \mu(A_1) + \mu(A \smallsetminus A_1)$ and $\mu(A \smallsetminus A_1) > 0$ (otherwise $\mu(A_1) = \mu(A)$), we know that either

$$0 < \mu(A_1) \leqslant \mu(A)/2, \tag{2.11}$$

or

$$0 < \mu(A \smallsetminus A_1) \leqslant \mu(A)/2. \tag{2.12}$$

If (2.11) holds, take an arbitrary $A_2 \in \mathcal{A}$ with $A_2 \subseteq A_1$ and $0 < \mu(A_2) < \mu(A_1)$. Then either

$$0 < \mu(A_2) \leqslant \mu(A_1)/2 \leqslant \mu(A)/2^2,$$

or

$$0 < \mu(A_1 \smallsetminus A_2) \leqslant \mu(A_1)/2 \leqslant \mu(A)/2^2.$$

If (2.12) holds, take $A_2 \in \mathcal{A}$ with $A_2 \subseteq A \smallsetminus A_1$ and $0 < \mu(A_2) < \mu(A \smallsetminus A_1)$. Then either

$$0 < \mu(A_2) \leqslant \mu(A \smallsetminus A_1)/2 \leqslant \mu(A)/2^2,$$

or

$$0 < \mu(A_2) \leqslant \mu(A \smallsetminus A_1 \smallsetminus A_2)/2 \leqslant \mu(A \smallsetminus A_1)/2 \leqslant \mu(A)/2^2.$$

Thus, there exists $A_2 \in \mathcal{A}$ with $A_2 \subseteq A$ such that $0 < \mu(A_2) \leqslant \mu(A)/2^2$.

Then by mathematical induction principle, we can show that there exists $A_n \in \mathcal{A}$ with $A_n \subseteq A$ and $0 < \mu(A_n) \leqslant \mu(A)/2^n$. By letting n sufficiently large and setting $B = A_n$, we get the result.

(c) If $\alpha = 0$, set $B = \varnothing$; if $\alpha = \mu(A)$, set $B = A$. So we assume that $0 < \alpha < \mu(A)$.

Let $\mathfrak{C}$ denote the family of collection $\mathcal{D}$ of countable disjoint $\mathcal{A}$-sets contained in A such that $\sum_{D \in \mathcal{D}} \mu(D) \leqslant \alpha$. Notice that $\mathfrak{C}$ is well-defined by (b). For $\mathcal{D}, \mathcal{E} \in \mathfrak{C}$, write $\mathcal{D} \preccurlyeq \mathcal{E}$ iff (i) $\sum_{D \in \mathcal{D}} \mu(D) \leqslant \sum_{E \in \mathcal{E}} \mu(E)$, and (ii) $\mathcal{D} \subseteq \mathcal{E}$.

It is clear that $(\mathfrak{C}, \preccurlyeq)$ is a partially ordered set since $\leqslant$ and $\subseteq$ are partial orderings. For any chain $\mathfrak{D} \subseteq \mathfrak{C}$, there exists an upper bound $\bigcup \mathfrak{D}$. It follows from Zorn's Lemma that there exists a maximal element $\mathcal{F} \in \mathfrak{C}$.

Let $B = \bigcup \mathcal{F}$. We finally show that $\mu(B) = \sum_{F \in \mathcal{F}} \mu(F) = \alpha$. Assume that $\mu(B) < \alpha$. Then $\mu(A \smallsetminus B) > 0$ for otherwise $\mu(A) = \mu(A \smallsetminus B) + \mu(B) = \alpha < \mu(A)$. Take an arbitrary $\varepsilon > 0$ such that $\varepsilon < \alpha - \mu(B)$. It follows from (b) that there exists $C \in \mathcal{A}$ with $C \subseteq A \smallsetminus B$ and $0 < \mu(C) < \varepsilon$. Let $\mathcal{G} = \mathcal{F} \cup \{C\}$. Then all sets in $\mathcal{G}$ are disjoint and

$$\sum_{G \in \mathcal{G}} \mu(G) = \sum_{F \in \mathcal{F}} \mu(F) + \mu(C) < \mu(B) + \varepsilon < \alpha;$$

that is, $\mathcal{G} \in \mathfrak{C}$. Further, $\sum_{G \in \mathcal{G}} \mu(G) = \sum_{F \in \mathcal{F}} \mu(F) + \mu(C) > \sum_{F \in \mathcal{F}} \mu(F)$ since $\mu(C) > 0$, and $\mathcal{F} \subset \mathcal{G}$. Thus, $\mathcal{F} \prec \mathcal{G}$. In Contradict the fact that $\mathcal{F}$ is maximal in $\mathfrak{C}$. □

► EXERCISE 2.30 (2.2.19). *Let $(\Omega, \mathcal{A}, \mu)$ be a measure space. Let $B \in \mathcal{A}$ and $\mathcal{A}_B = \{A \in \mathcal{A} \colon A \subseteq B\}$. Then $\mathcal{A}_B$ is a σ-filed on B, and the restriction of μ to $\mathcal{A}_B$ is a measure on $\mathcal{A}_B$.*

PROOF. Automatically, $B \in \mathcal{A}_B$. If $A \in \mathcal{A}_B$, then $A \in \mathcal{A}$, so $B \smallsetminus A \in \mathcal{A}$ and $B \smallsetminus A \subseteq B$, i.e., $B \smallsetminus A \in \mathcal{A}_B$. Finally, if $\{A_n\} \subseteq \mathcal{A}_B$, then $A_n \in \mathcal{A}$ and $A_n \subseteq B$ for each n. Thus, $\bigcup A_n \in \mathcal{A}$ and $\bigcup A_n \subseteq B$, i.e., $\bigcup A_n \in \mathcal{A}_B$. It is trivial to verify that μ is a measure on $\mathcal{A}_B$. □

2.3 A CLASS OF MEASURES

► EXERCISE 2.31 (2.3.1). *Let $p_1, \dots, p_n > 0$. Fix n real numbers $x_1 < x_2 < \cdots < x_n$, and define*

$$F(x) = \begin{cases} 0 & \textit{if } x < x_1 \\ p_1 + \cdots + p_j & \textit{if there exists } 1 \leqslant j \leqslant n \textit{ such that } x_j \leqslant x < x_{j+1} \\ 1 & \textit{if } x \geqslant x_n. \end{cases}$$

Δ_F is a measure on $\mathcal{A}_1$, and

$$\Delta_F((a,b]) = \begin{cases} \sum \left\{ p_j : j \textit{ is such that } a < x_j \leqslant b \right\} & \textit{if } (a,b] \cap \{x_1, \ldots, x_n\} \neq \varnothing \\ 0 & \textit{otherwise.} \end{cases}$$

This is an important framework in discrete probability theory.

PROOF. We first show that Δ_F takes the given form. If $(a,b] \cap \{x_1, \ldots, x_n\} = \varnothing$, then either $a < b < x_1 < x_2 < \cdots < x_n$ or $x_1 < x_2 < \cdots < x_n < a < b$. Hence, $F(a) = F(b) = 0$ or $F(a) = F(b) = 1$, so $\Delta_F((a,b]) = F(b) - F(a) = 0$. If $(a,b] \cap \{x_1, \ldots, x_n\} \neq \varnothing$, then there exists $i, j = 1, \ldots, n$ such that $a < x_i < x_j \leqslant b$. Hence,

$$\begin{aligned} \Delta_F((a,b]) = F(b) - F(a) &= (p_1 + \cdots + p_j) - (p_1 + \cdots + p_i) \\ &= \sum_{\{j \,:\, a < x_j \leqslant b\}} p_j. \end{aligned}$$

To see that Δ_F is a measure, notice that (M1) and (M2) are satisfied automatically. Let $\{(a_n, b_n]\}_{n=1}^{\infty} \subseteq \mathcal{A}_1$ is disjoint, and assume that $a_1 < b_1 \leqslant a_2 < b_2 \leqslant \cdots \leqslant a_n < b_n \leqslant \cdots$, whence,

$$\begin{aligned} \Delta_F \left(\bigcup_{n=1}^{\infty} (a_n, b_n] \right) &= \sum \left\{ p_k : k \text{ is such that } x_k \in A_n \text{ for some } n \in \mathbb{N} \right\} \\ &= \sum_{n=1}^{\infty} \sum \left\{ p_k : k \text{ is such that } x_k \in A_n \right\} \\ &= \sum_{n=1}^{\infty} \Delta_F \left((a_n, b_n] \right). \end{aligned}$$

□

► EXERCISE 2.32 (2.3.2). *This problem generalizes Example 4. Let $f : \mathbb{R}^k \to \mathbb{R}$ be such that f is continuous and nonnegative. Further suppose that $\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(\boldsymbol{t})\, \mathrm{d}\boldsymbol{t} < +\infty$. Define a function $F : \mathbb{R}^k \to \mathbb{R}$ by*

$$F(\boldsymbol{x}) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_k} f(\boldsymbol{t})\, \mathrm{d}\boldsymbol{t}, \quad \boldsymbol{x} \in \mathbb{R}^k.$$

Then Δ_F is a measure on the semiring $\mathcal{A}_k$, and for all $(\boldsymbol{a}, \boldsymbol{b}] \in \mathcal{A}_k$ we have

$$\Delta_F \left((\boldsymbol{a}, \boldsymbol{b}] \right) = \int_{a_1}^{b_1} \cdots \int_{a_k}^{b_k} f(t_1, \ldots, t_k)\, \mathrm{d}t_k \cdots \mathrm{d}t_1.$$

PROOF. F is continuous. To derive Δ_{F_k}, we use the mathematical induction. If $n = 1$, then $\Delta_{F_1}((a,b]) = F(b) - F(a) = \int_a^b f(t)\,\mathrm{d}t$. Let us assume that the hypothesis hold for $n = k$, and consider $n = k + 1$:

$$
\begin{aligned}
&\Delta_{F_{k+1}}(A) \\
&= \sum_{\boldsymbol{x}\in V(A)} s_A(\boldsymbol{x}) F_{k+1}(\boldsymbol{x}) \\
&= \sum_{\boldsymbol{x}\in V_1(A)} s_A(\boldsymbol{x}) F_{k+1}(\boldsymbol{x}) + \sum_{\boldsymbol{x}\in V_2(A)} s_A(\boldsymbol{x}) F_{k+1}(\boldsymbol{x}) \\
&= \sum_{\boldsymbol{x}^*\in V(A^*)} s_A(\boldsymbol{x}^*, a_{k+1}) F_{k+1}(\boldsymbol{x}^*, a_{k+1}) + \sum_{\boldsymbol{x}^*\in V(A^*)} s_A(\boldsymbol{x}^*, b_{k+1}) F_{k+1}(\boldsymbol{x}^*, b_{k+1}) \\
&= \sum_{\boldsymbol{x}^*\in V(A^*)} (-1)\cdot s_{A^*}(\boldsymbol{x}^*) F_{k+1}(\boldsymbol{x}^*, a_{k+1}) + \sum_{\boldsymbol{x}^*\in V(A^*)} s_{A^*}(\boldsymbol{x}^*) F_{k+1}(\boldsymbol{x}^*, b_{k+1}) \\
&= \sum_{\boldsymbol{x}^*\in V(A^*)} s_{A^*}(\boldsymbol{x}^*)\cdot \left[F_{k+1}(\boldsymbol{x}^*, b_{k+1}) - F_{k+1}(\boldsymbol{x}^*, a_{k+1})\right] \\
&= \sum_{\boldsymbol{x}^*\in V(A^*)} s_{A^*}(\boldsymbol{x}^*)\cdot \int_{a_{k+1}}^{b_{k+1}} F_k(\boldsymbol{x}^*)\,\mathrm{d}t_{k+1} \\
&= \int_{a_{k+1}}^{b_{k+1}} \left[\sum_{\boldsymbol{x}\in V(A^*)} s_{A^*}(\boldsymbol{x}^*) F_k(\boldsymbol{x}^*)\right] \mathrm{d}t_{k+1} \\
&= \int_{a_1}^{b_1} \cdots \int_{a_{k+1}}^{b_{k+1}} f\left(t_1, \dots, t_{k+1}\right) \mathrm{d}t_{k+1} \cdots \mathrm{d}t_1 \\
&\geqslant 0.
\end{aligned}
$$

Hence, $F_k \in \mathfrak{S}_k$. □

► EXERCISE 2.33 (2.3.3). *Let $F_1, \dots, F_k \in \mathfrak{S}$. For each $\boldsymbol{x} \in \mathbb{R}^k$, write $F(\boldsymbol{x}) = \prod_{i=1}^k F_i(x_i)$.*

a. *$F \in \mathfrak{S}_k$, hence Δ_F is a measure on $\mathcal{A}_k$.*

b. *$\Delta_F\left((\boldsymbol{a}, \boldsymbol{b}]\right) = \prod_{i=1}^k \left[F_i(b_i) - F_i(a_i)\right]$ for all $(\boldsymbol{a}, \boldsymbol{b}] \in \mathcal{A}_k$.*

PROOF. The continuity of F is clear. For F_i, we have $\Delta_{F_i}\left((a_i, b_i]\right) = F_i(b_i) - F_i(a_i)$. We can derive the form of Δ_F as in Example 3. □

► EXERCISE 2.34 (2.3.4). *Suppose that $F_i \in \mathfrak{S}_{k_i}$ for $i = 1, \dots, n$. Suppose that $F\colon \mathbb{R}^{\sum_{i=1}^n k_i} \to \mathbb{R}$ is such that*

$$F(\boldsymbol{x}^{(1)}, \dots, \boldsymbol{x}^{(n)}) = \prod_{i=1}^n F_i(\boldsymbol{x}^{(i)})$$

for each $\boldsymbol{x}^{(1)} \in \mathbb{R}^{k_1}, \dots$, and $\boldsymbol{x}^{(n)} \in \mathbb{R}^{k_n}$. Then Δ_F is a measure on $\mathcal{A}_{\sum_{i=1}^n k_i}$. Also, $\Delta_F(A_1 \times \cdots \times A_n) = \prod_{i=1}^n \Delta_{F_i}(A_i)$ for each $A_1 \in \mathcal{A}_1, \dots$, and $A_n \in \mathcal{A}_{k_n}$.

PROOF. We use mathematical induction. If $i = 1$ then $F(\boldsymbol{x}^{(1)}) = F_1(\boldsymbol{x}^{(1)})$, and $\Delta_F(A_1) = \Delta_{F_1}(A_1)$. If $i = 2$ then $F(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) = F_1(\boldsymbol{x}^{(1)}) \times F_2(\boldsymbol{x}^{(2)})$. Consider any $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) \in V(A_1 \times A_2)$. By definition,

$$s_{A_1\times A_2}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) = \begin{cases} +1 & \text{if } |\{i, j : x_i^{(1)} = a_i^{(1)}, x_j^{(2)} = a_j^{(2)}\}| \text{ is even} \\ -1 & \text{if } |\{i, j : x_i^{(1)} = a_i^{(1)}, x_j^{(2)} = a_j^{(2)}\}| \text{ is odd.} \end{cases}$$

$$s_{A_1\times A_2}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})F(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) = s_{A_1\times A_2}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) \cdot \left[F_1(\boldsymbol{x}^{(1)})F_2(\boldsymbol{x}^{(2)})\right]. \tag{2.13}$$

- If $|\{i : x_i^{(1)} = a_i^{(1)}\}| = \#^{(1)}$ is even, and $|\{i : x_i^{(2)} = a_i^{(2)}\}| = \#^{(2)}$ is even, too, then $\{i, j : x_i^{(1)} = a_i^{(1)}, x_j^{(2)} = a_j^{(2)}\}| = \#^{(1,2)}$ is even, and $s_{A_1}(\boldsymbol{x}^{(1)}) = s_{A_2}(\boldsymbol{x}^{(2)}) = s_{A_1\times A_2}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) = +1$. Therefore, by (2.13)

$$\begin{aligned} s_{A_1\times A_2}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})F(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) &= F(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) \\ &= F_1(\boldsymbol{x}^{(1)})F_2(\boldsymbol{x}^{(2)}) \\ &= \left[s_{A_1}(\boldsymbol{x}^{(1)})F_1(\boldsymbol{x}^{(1)})\right] \cdot \left[s_{A_2}F_2(\boldsymbol{x}^{(2)})\right] \end{aligned} \tag{2.14}$$

- If $\#^{(1)}$ and $\#^{(2)}$ are both odd, then $\#^{(1,2)}$ is even, and so

$$\begin{aligned} s_{A_1\times A_2}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})F(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) &= F(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) \\ &= F_1(\boldsymbol{x}^{(1)})F_2(\boldsymbol{x}^{(2)}) \\ &= \left[-F_1(\boldsymbol{x}^{(1)})\right] \cdot \left[-F_2(\boldsymbol{x}^{(2)})\right] \\ &= \left[s_{A_1}(\boldsymbol{x}^{(1)})F_1(\boldsymbol{x}^{(1)})\right] \cdot \left[s_{A_2}F_2(\boldsymbol{x}^{(2)})\right] \end{aligned} \tag{2.15}$$

- If one of the $\#^{(1)}, \#^{(2)}$ is even, and the other is odd, then $\#^{(1,2)}$ is odd. (2.14) holds in this case.

Hence, for any $(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)}) \in A_1 \times A_2$, (2.14) hold. Therefore, we have

$$\begin{aligned} &\sum_{(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})\in V(A_1\times A_2)} s_{A_1\times A_2}(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})F\left(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{2}\right) \\ &= \sum_{(\boldsymbol{x}^{(1)}, \boldsymbol{x}^{(2)})\in V(A_1\times A_2)} \left[s_{A_1}(\boldsymbol{x}^{(1)})F_1(\boldsymbol{x}^{(1)})\right] \times \left[s_{A_2}(\boldsymbol{x}^{(2)})F_2(\boldsymbol{x}^{(2)})\right] \\ &= \sum_{\boldsymbol{x}^{(1)}\in V(A_1)} \left[s_{A_1}(\boldsymbol{x}^{(1)})F_1(\boldsymbol{x}^{(1)})\right] \times \left[\sum_{\boldsymbol{x}^{(2)}\in V(A_2)} s_{A_2}(\boldsymbol{x}^{(2)})F_2(\boldsymbol{x}^{(2)})\right] \\ &= \sum_{\boldsymbol{x}^{(1)}\in V(A_1)} \left[s_{A_1}(\boldsymbol{x}^{(1)})F_1(\boldsymbol{x}^{(1)})\right] \times \Delta_{F_2}(A_2) \\ &= \Delta_{F_1}(A_1) \cdot \Delta_{F_2}(A_2). \end{aligned} \tag{2.16}$$

Now suppose the claim holds for $n = k$, and consider $n = k + 1$. In this case,

$$F(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}, \boldsymbol{x}^{(k+1)}) = \left[\prod_{i=1}^{k} F_i(\boldsymbol{x}^{(i)})\right] F_{k+1}(\boldsymbol{x}^{(k+1)}).$$

Just like Step 2, we have

$$\begin{aligned} & s_{\prod_{i=1}^{k+1} A_i}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k+1)}) F(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k+1)}) \\ &= \left[s_{\prod_{i=1}^{k} A_i}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}) \cdot \prod_{i=1}^{k} F_i(\boldsymbol{x}^{(i)})\right] \times \left[s_{A_{k+1}}(\boldsymbol{x}^{(k+1)}) F_{k+1}(\boldsymbol{x}^{(k+1)})\right] \end{aligned} \tag{2.17}$$

for every $(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k+1)}) \in \prod_{i=1}^{k+1} A_i$. Therefore,

$$\begin{aligned} & \Delta_F(A_1 \times \cdots \times A_{k+1}) \\ &\quad = \sum_{(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k+1)}) \in V\left(\prod_{i=1}^{k+1} A_i\right)} s_{\prod_{i=1}^{k+1} A_i}(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k+1)}) F(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k+1)}) \\ &\quad = \sum_{(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k+1)}) \in V\left(\prod_{i=1}^{k+1} A_i\right)} \left[s_{\prod_{i=1}^{k} A_i}\left(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}\right) \cdot \prod_{i=1}^{k} F_i(\boldsymbol{x}^{(i)})\right] \\ &\qquad \times \left[s_{A_{k+1}}(\boldsymbol{x}^{(k+1)}) F_{k+1}(\boldsymbol{x}^{(k+1)})\right] \\ &\quad = \sum_{(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}) \in V\left(\prod_{i=1}^{k} A_i\right)} \left[s_{\prod_{i=1}^{k} A_i}\left(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k)}\right) \cdot \prod_{i=1}^{k} F_i(\boldsymbol{x}^{(i)})\right] \\ &\qquad \times \left[\sum_{\boldsymbol{x}^{(k+1)} \in V(A_{k+1})} s_{A_{k+1}}(\boldsymbol{x}^{(k+1)}) F_{k+1}(\boldsymbol{x}^{(k+1)})\right] \\ &\quad = \left[\prod_{i=1}^{k} \Delta_{F_i}(\boldsymbol{x}^{(i)})\right] \cdot \Delta_{F_{k+1}}(\boldsymbol{x}^{(k+1)}) \\ &\quad = \prod_{i=1}^{k+1} \Delta_{F_i}(\boldsymbol{x}^{(i)}). \end{aligned}$$

Since $F_i \in \mathfrak{S}_{k_i}$, we have $\Delta_{F_i}(A_i) \geqslant 0$; thus, $\Delta_F(A_1 \times \cdots A_n) = \prod_{i=1}^{n} \Delta_{F_i}(A_i) \geqslant 0$. The continuity of $F(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(k+1)})$ is obvious. Hence,

$$F(\boldsymbol{x}^{(1)}, \ldots, \boldsymbol{x}^{(n)}) \in \mathfrak{S}_{\sum_{i=1}^{n} k_i}. \qquad \square$$

3

EXTENSIONS OF MEASURES

REMARK 3.1 (p. 82). If $A \in \lambda(\mathcal{P})$, then $\mathcal{G}_A$ is a λ-system.

PROOF. Let

$$\mathcal{G}_A = \{C \subseteq \Omega \colon A \cap C \in \lambda(\mathcal{P})\}. \tag{3.1}$$

(λ_1) $A \cap \Omega = A \in \lambda(\mathcal{P}) \Longrightarrow \Omega \in \mathcal{G}_A$. $(\lambda_{2'})$ Suppose $C_1 \subseteq C_2$ with $C_1, C_2 \in \mathcal{G}$. Then we have $A \cap C_1 \subseteq A \cap C_2$ and $A \cap C_1, A \cap C_2 \in \lambda(\mathcal{P})$ by assumption. Since every λ-system is closed under proper differences, we have

$$(A \cap C_2) - (A \cap C_1) = A \cap (C_2 - C_1) \in \lambda(\mathcal{P}),$$

so $C_2 - C_1 \in \mathcal{G}_A$. (λ_3) Let $\{C_n\}_{n=1}^{\infty}$ denote a disjoint collection of $\mathcal{G}_A$-sets, so that $\{A \cap C_n\}_{n=1}^{\infty}$ is a disjoint sequence of $\lambda(\mathcal{P})$-sets. Since $\lambda(\mathcal{P})$ is a λ-system and hence satisfies $(\lambda 3)$, we have

$$A \cap \left(\bigcup_{n=1}^{\infty} C_n\right) = \bigcup_{n=1}^{\infty} (A \cap C_n) \in \lambda(\mathcal{P}),$$

so that $\bigcup_{n=1}^{\infty} C_n \in \mathcal{G}_A$. □

REMARK 3.2 (p. 90). $P \Delta Q = R \Delta S \Longrightarrow P \Delta R = Q \Delta S$.

PROOF. First observe that $A \Delta B = \varnothing$ iff $A = B$. To see this, note that

$$\varnothing = A \Delta B = (A \smallsetminus B) \cup (B \smallsetminus A) \iff [A \smallsetminus B = B \smallsetminus A = \varnothing],$$

but[1]

$$A \smallsetminus B = \varnothing \Longrightarrow A \subseteq B,$$
$$B \smallsetminus A = \varnothing \Longrightarrow B \subseteq A;$$

Thus $A = B$.

For the reverse inclusion, let $A = B$. Then

[1] The proof is as follows: Let $A \smallsetminus B = A \cap B^c = \varnothing$. Let $x \in A$. Then $x \notin B^c \Longrightarrow x \in B$.

$$A \Delta B = (A \setminus B) \cup (B \setminus A) = \varnothing \cup \varnothing = \varnothing.$$

Now we prove the claim. Since $P \Delta Q = R \Delta S$, we have

$$\big(P \Delta Q\big) \, \Delta \, (R \Delta S) = \varnothing. \tag{3.2}$$

It follows from Exercise 2.22(g) that we can rewrite (3.2) as $(P \Delta R) \, \Delta \, \big(Q \Delta S\big) = \varnothing$, and which gives the result: $P \Delta R = Q \Delta S$. □

3.1 Extensions and Restrictions

► Exercise 3.3 (3.1.1). *Let $(\Omega, \mathscr{A}, \mu)$ denote a measure space. Pick $E \in \mathscr{A}$ and define $\mathscr{A}_E = \{F \in \mathscr{A} \colon F \subseteq E\}$. Then $\mathscr{A}_E$ is a σ-field on E, $\mathscr{A}_E = \{A \cap E \colon A \in \mathscr{A}\}$, and the restriction μ_E of μ [from $\mathscr{A}$] to $\mathscr{A}_E$ is a measure. That is, $\big(E, \mathscr{A}_E, \mu_E\big)$ is a measure space and $\mu_E = \mu$ on $\mathscr{A}_E$.*

Proof. Automatically, $E \in \mathscr{A}_E$. If $A \in \mathscr{A}_E$, then $A \in \mathscr{A}$ and $A \subseteq E$; hence $E \setminus A \in \mathscr{A}_E$. If $\{A_n\}_{n=1}^{\infty} \subseteq \mathscr{A}_E$, then $\bigcup_{n=1}^{\infty} A_n \subseteq E$ and $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$, i.e., $\bigcup_{n=1}^{\infty} A_n \in \mathscr{A}_E$. Therefore, $\mathscr{A}_E$ is a σ-field.

We first show

$$\{F \in \mathscr{A} \colon F \subseteq E\} = \mathscr{A}_E \subseteq \mathscr{A}'_E = \{A \cap E \colon A \in \mathscr{A}\}.$$

If $F \in \mathscr{A}_E$ then $F \in \mathscr{A}$ and $F \subseteq E$. Since $F = F \cap E$, we get $F \in \mathscr{A}'_E$. For the converse inclusion direction, let $B \in \mathscr{A}'_E$. Then there exists $A \in \mathscr{A}$ such that $A \cap E = B$. It is obvious that $A \cap E \in \mathscr{A}$ and $A \cap E \subseteq E$, so $A \cap E = B \in \mathscr{A}_E$.

μ_E is a measure [on $\mathscr{A}_E$] because μ is a measure [on $\mathscr{A}$]. [See Exercise 3.4(b).] □

► Exercise 3.4 (3.1.2). *Prove Claim 1 and 2.*

Claim 1 *Assume the notation of the definition. If μ is σ-finite on $\mathscr{G}$, then ν is σ-finite on $\mathscr{G}$ as well.*

Claim 2 *Suppose that $\varnothing \in \mathscr{G}$, and let $\mathscr{G} \subseteq \mathscr{H} \subseteq 2^{\Omega}$. Let $\nu \colon \mathscr{H} \to \overline{\mathbb{R}}$ denote a measure. Then the restriction of ν to $\mathscr{G}$ is a measure.*

Proof. (Claim 1) By definition, μ is the restriction of ν [from $\mathscr{H}$] to $\mathscr{G}$, so $\nu(A) = \mu(A)$ for all $A \in \mathscr{G}$. Since μ is σ-finite on $\mathscr{G}$, there exists a sequence of $\mathscr{G}$-sets, $\{A_n\}_{n=1}^{\infty}$, such that $\Omega = \bigcup_{n=1}^{\infty} A_n$ and $\nu(A_n) = \mu(A_n) < +\infty$ for each $n \in \mathbb{N}$. Hence, ν is σ-finite on $\mathscr{G}$.

(Claim 2) Since μ is the restriction of ν from $\mathscr{H}$ to $\mathscr{G}$, we have $\mu(A) = \nu(A)$ for all $A \in \mathscr{G}$. (M1) To see the nonnegativity, let $A \in \mathscr{G} \subseteq \mathscr{H}$. Since ν is a measure, $\nu(B) \geqslant 0$ for all $B \in \mathscr{H}$; particularly, $\mu(A) = \nu(A) \geqslant 0$ for all $A \in \mathscr{G}$. (M2) $\mu(\varnothing) = \nu(\varnothing) = 0$. (M3) Let $\bigcup_{n=1}^{\infty} A_n \in \mathscr{G}$. Then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \nu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \nu(A_n) = \sum_{n=1}^{\infty} \mu(A_n). \qquad \square$$

3.2 Outer Measures

► Exercise 3.5 (3.2.1). *Let $(\Omega, \mathscr{A}, \mu)$ denote an arbitrary measure space. Define ν on 2^{Ω} by writing $\nu(B) = \inf\{\mu(A) \colon B \subseteq A, A \in \mathscr{A}\}$ for each $B \subseteq \Omega$. Then ν is an outer measure.*

Proof. The nonnegativity of ν is evident since $\mu(\cdot)$ is a measure. To see (O2), observe that $\varnothing \subseteq \varnothing$, so $\nu(\varnothing) \leqslant \mu(\varnothing) = 0$. By (O1), $\nu(\varnothing) = 0$. To see (O3), let $B \subseteq C \subseteq \Omega$. Hence $\{\mu(A) \colon C \subseteq A, A \in \mathscr{A}\} \subseteq \{\mu(A) \colon B \subseteq A, A \in \mathscr{A}\}$, which means that $\inf\{\mu(A) \colon C \subseteq A, A \in \mathscr{A}\} \geqslant \inf\{\mu(A) \colon B \subseteq A, A \in \mathscr{A}\}$, and so $\nu(B) \leqslant \nu(C)$.

To see ν is countable subadditivity, let $\{B_n\}_{n=1}^{\infty} \subseteq 2^{\Omega}$. We just consider the case that $\nu(B_n) < +\infty$ for all $n \in \mathbb{N}$. For each n, there exists $\varepsilon > 0$ and $A_n \in \mathscr{A}$ such that $B_n \subseteq A_n$ and

$$\nu(B_n) + \varepsilon/2^n \geqslant \mu(A_n).$$

Also $\bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} A_n$. Thus,

$$\nu\left(\bigcup_{n=1}^{\infty} B_n\right) \leqslant \mu\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} \mu(A_n) \leqslant \sum_{n=1}^{\infty} \left[\nu(B_n) + \frac{\varepsilon}{2^n}\right] = \sum_{n=1}^{\infty} \nu(B_n) + \varepsilon. \quad \square$$

► Exercise 3.6 (3.2.2). *Let $\nu \colon 2^{\Omega} \to \overline{\mathbb{R}}$ be an outer measure, and suppose in addition that ν is finitely additive: $\nu(A \cup B) = \nu(A) + \nu(B)$, where $A, B \subseteq \Omega$ are disjoint. Then ν is a measure. That is, $(\Omega, 2^{\Omega}, \nu)$ is a measure space.*

Proof. (M1) and (M2) are satisfied automatically. To see (M3) (countable additivity), let $\{A_n\}_{n=1}^{\infty} \subseteq 2^{\Omega}$ be disjoint. Then

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) \geqslant \nu\left(\bigcup_{n=1}^{N} A_n\right) = \sum_{n=1}^{N} \nu(A_n),$$

for every $N \in \mathbb{N}$. Now let $N \uparrow +\infty$ and yield

$$\nu\left(\bigcup_{n=1}^{\infty} A_n\right) \geqslant \sum_{n=1}^{\infty} \nu(A_n). \tag{3.3}$$

Combining (3.3) with (O4) (countable subadditivity) yields the result. $\square$

► Exercise 3.7 (3.2.3). *Suppose that φ and ξ are outer measures [relative to some common background set Ω], and suppose that we define a new function $\nu \colon 2^{\Omega} \to \overline{\mathbb{R}}$ for all $A \subseteq \Omega$ by writing $\nu(A) = \max\{\varphi(A), \xi(A)\}$. Then ν is an outer measure [relative to Ω].*

Proof. (O1) and (O2) are straightforward. To see (O3), let $A \subseteq B \subseteq \Omega$. Then

$$\nu(A) = \max\{\varphi(A), \xi(A)\} \leqslant \max\{\varphi(B), \xi(B)\} = \nu(B).$$

To see (O4), let $\{A_n\}_{n=1}^{\infty} \subseteq 2^{\Omega}$. Then

$$\begin{aligned}\nu\left(\bigcup_{n=1}^{\infty} A_n\right) &= \max\left\{\varphi\left(\bigcup_{n=1}^{\infty} A_n\right), \xi\left(\bigcup_{n=1}^{\infty} A_n\right)\right\} \\ &\leqslant \max\left\{\sum_{n=1}^{\infty} \varphi(A_n), \sum_{n=1}^{\infty} \xi(A_n)\right\} \\ &\leqslant \sum_{n=1}^{\infty} \max\{\varphi(A_n), \xi(A_n)\} \\ &= \sum_{n=1}^{\infty} \nu(A_n).\end{aligned}$$ □

► Exercise 3.8 (3.2.4). *Let ν denote an outer measure, and let $A \subseteq \Omega$. Define a new set function ν_A on 2^{Ω} by writing $\nu_A(B) = \nu(B \cap A)$ for each $B \subseteq \Omega$. Then ν_A is an outer measure relative to Ω.*

Proof. (O1) and (O2) are satisfied automatically. If $B \subseteq C \subseteq \Omega$, then

$$\nu_A(B) = \nu(B \cap A) \leqslant \nu(C \cap A) = \nu_A(C)$$

by the monotonicity of ν. To see (O4), let $\{B_n\} \subseteq 2^{\Omega}$. Then

$$\begin{aligned}\nu_A\left(\bigcup_{n=1}^{\infty} B_n\right) = \nu\left(\left(\bigcup_{n=1}^{\infty} B_n\right) \cap A\right) = \nu\left(\bigcup_{n=1}^{\infty} (B_n \cap A)\right) &\leqslant \sum_{n=1}^{\infty} \nu(B_n \cap A) \\ &= \sum_{n=1}^{\infty} \nu_A(B_n).\end{aligned}$$ □

► Exercise 3.9 (3.2.5). *Let $\{\nu_n\}_{n=1}^{\infty}$ denote a sequence of outer measures [relative to some common Ω], and let $\{a_n\}_{n=1}^{\infty}$ denote a sequence of nonnegative numbers. For each $A \subseteq \Omega$, let $\nu(A) = \sum_{n=1}^{\infty} a_n \cdot \nu_n(A)$. Then ν is an outer measure relative to Ω.*

Proof. (O1) and (O2) are satisfied obviously. If $A \subseteq B \subseteq \Omega$, then

$$\nu(A) = \sum_{n=1}^{\infty} a_n \cdot \nu_n(A) \leqslant \sum_{n=1}^{\infty} a_n \cdot \nu_n(B) = \nu(B).$$

To see (O4), let $\{A_k\} \subseteq 2^{\Omega}$. Then

$$\nu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{n=1}^{\infty} a_n \nu_n \left(\bigcup_{k=1}^{\infty} A_k\right) \leqslant \sum_{n=1}^{\infty} a_n \left[\sum_{k=1}^{\infty} \nu_n(A_k)\right]$$
$$= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} a_n \nu_n(A_k)$$
$$= \sum_{k=1}^{\infty} \nu(A_k). \qquad \square$$

3.3 Carathéodory's Criterion

► Exercise 3.10 (3.3.1). *Show* directly *that if* $A, B \in \mathcal{M}(\nu)$, *then* $A \cup B, A \smallsetminus B \in \mathcal{M}(\nu)$.

Proof. (i) The following method is from Bear (2002). Let $A, B \in \mathcal{M}(\nu)$ and let $T \subseteq \Omega$ be any test set. Let $T = T_1 \cup T_2 \cup T_3 \cup T_4$ as indicated in Figure 3.1. We need to show

$$\nu(T) = \nu\left(T \cap (A \cup B)\right) + \nu\left(T \cap (A \cup B)^c\right), \tag{3.4}$$

or, in terms of Figure 3.1,

$$\nu(T) = \nu(T_1 \cup T_2 \cup T_3) + \nu(T_4). \tag{3.4$'$}$$

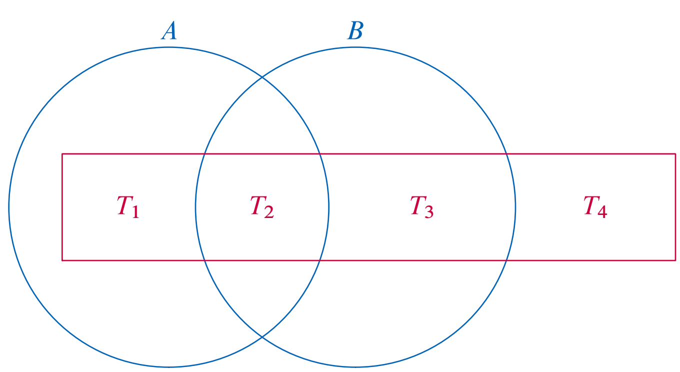

Figure 3.1. $A \cup B \in \mathcal{M}(\nu)$

Cutting the test set $T_1 \cup T_2$ with B gives

$$\nu(T_1 \cup T_2) = \nu(T_2) + \nu(T_1). \tag{3.5}$$

Similarly, cutting $T_3 \cup T_4$ with B gives

$$\nu(T_3 \cup T_4) = \nu(T_3) + \nu(T_4). \tag{3.6}$$

Cutting T with A gives

$$\nu(T) = \nu(T_1 \cup T_2) + \nu(T_3 \cup T_4). \tag{3.7}$$

Combining (3.5), (3.6), and (3.7) we can write

$$\nu(T) = \nu(T_1) + \nu(T_2) + \nu(T_3) + \nu(T_4). \tag{3.8}$$

Now cut $T_1 \cup T_2 \cup T_3$ with A and then use (3.5):

$$\nu(T_1 \cup T_2 \cup T_3) = \nu(T_1 \cup T_2) + \nu(T_3) = \nu(T_1) + \nu(T_2) + \nu(T_3). \tag{3.9}$$

From (3.9) and (3.8) we have the desired equality (3.4′).

(ii) It is clear that $A \in \mathcal{M}(\nu)$ iff $A^c \in \mathcal{M}(\nu)$. Thus, $A, B \in \mathcal{M}(\nu)$ implies that $B \cup A^c \in \mathcal{M}(\nu)$ by the previous result. Since $A \smallsetminus B = (B \cup A^c)^c \in \mathcal{M}(\nu)$, we get $A \smallsetminus B \in \mathcal{M}(\nu)$. □

► Exercise 3.11 (3.3.2). *Suppose that Ω may be written as $\bigcup_{n=1}^{\infty} A_n$, where $\{A_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of subsets of Ω. If $A \subseteq \Omega$ is such that $A \cap A_k \in \mathcal{M}(\nu)$ for all k exceeding some constant k_A, then $A \in \mathcal{M}(\nu)$.*

Proof. By the *Outer Measure Theorem*, $\mathcal{M}(\nu)$ is a σ-field on Ω. Therefore,

$$\bigcup_{k=k_A+1}^{\infty} (A \cap A_k) = A \cap \left(\bigcup_{k=k_A+1}^{\infty} A_k \right) \in \mathcal{M}(\nu).$$

But since $\{A_k\}_{k=1}^{\infty}$ is a nondecreasing sequence, we have

$$\bigcup_{k=k_A+1}^{\infty} A_k = \bigcup_{k=1}^{\infty} A_k = \Omega,$$

which means that

$$A \cap \left(\bigcup_{k=k_A+1}^{\infty} A_k \right) = A \cap \Omega = A \in \mathcal{M}(\nu).$$ □

► Exercise 3.12 (3.3.3). *Let ν denote an outer measure such that $\nu(\Omega) < +\infty$, and further suppose that if $A \subseteq \Omega$ with $\nu(A) < +\infty$, then there exists $B \in \mathcal{M}(\nu)$ such that $A \subseteq B$ and $\nu(A) = \nu(B)$. Then $E \in \mathcal{M}(\nu)$ iff $\nu(\Omega) = \nu(E) + \nu(E^c)$.*

Proof. If $E \in \mathcal{M}(\nu)$, then $\nu(T) = \nu(T \cap E) + \nu(T \cap E^c)$; in particular, this holds for $T = \Omega$, so $\nu(\Omega) = \nu(E) + \nu(E^c)$.

For the other direction, suppose $\nu(\Omega) = \nu(E) + \nu(E^c)$. Since $\nu(\Omega) < +\infty$, we get $\nu(E), \mu(E^c) < +\infty$ by the monotonicity of ν. Then there exist $B', B'' \in \mathcal{M}(\nu)$ such that $E^c \subseteq B'$, $\nu(E^c) = \nu(B')$, and $E \subseteq B''$, $\nu(E) = \nu(B'')$. Let $B = (B')^c \in \mathcal{M}(\nu)$. Then $B \subseteq E$ and

$$\nu(B) = \nu((B')^c) = \nu(\Omega) - \nu(B') = \left[\nu(E) + \nu(E^c)\right] - \nu(E^c) = \nu(E).$$

Hence, there exist $B, B'' \in \mathcal{M}(\nu)$ such that $B \subseteq E \subseteq B''$, and $\nu(B) = \nu(E) = \nu(B'')$.

Notice that E is the union of B and a subset of $B'' \smallsetminus B$. If we can show that every subset of $B'' \smallsetminus B$ is in $\mathcal{M}(\nu)$, then $E \in \mathcal{M}(\nu)$ (since $\mathcal{M}(\nu)$ is a σ-field). For every $C \subseteq B'' \smallsetminus B$, we have

$$\nu(C) \leqslant \nu(B'' \smallsetminus B) = \nu(B'') - \nu(B) = 0.$$

Therefore, $\nu(C) = 0$, i.e., $C \in \mathcal{M}(\nu)$, and so $E \in \mathcal{M}(\nu)$. □

► EXERCISE 3.13 (3.3.4). *Suppose that ν is an arbitrary outer measure, and let $A, B \subseteq \Omega$ with $A \in \mathcal{M}(\nu)$. Show that $\nu(A \cup B) + \nu(A \cap B) = \nu(A) + \nu(B)$.*

PROOF. $A \in \mathcal{M}(\nu)$ implies that

$$\nu(A \cup B) = \nu\left((A \cup B) \cap A\right) + \nu\left((A \cup B) \cap A^c\right) = \nu(A) + \nu\left(B \cap A^c\right), \tag{3.10}$$

and

$$\nu(B) = \nu(B \cap A) + \nu\left(B \cap A^c\right). \tag{3.11}$$

Combining (3.10) with (3.11) we get $\nu(A \cup B) + \nu(A \cap B) = \nu(A) + \nu(B)$. □

► EXERCISE 3.14 (3.3.5). *Let ν denote an outer measure, and let $\{A_n\}_{n=1}^{\infty}$ denote a nondecreasing sequence of $\mathcal{M}(\nu)$-sets. Show that $\nu(\lim(A \cap A_n)) = \lim \nu(A \cap A_n)$ for any $A \subseteq \Omega$. State and prove an analogous result for nonincerasing sequences of $\mathcal{M}(\nu)$-sets.*

PROOF. Let $\{A_n\}_{n=1}^{\infty}$ be a nondecreasing sequence of $\mathcal{M}(\nu)$-sets. Then $\{A \cap A_n\}$ forms a nondecreasing sequence, so $\lim(A \cap A_n) = \bigcup(A \cap A_n) = A \cap (\bigcup A_n)$.

Let $B_1 = A_1$ and $B_n = A_n \smallsetminus A_{n-1}$ for $n \geqslant 2$. Then $\{B_n\} \subseteq \mathcal{M}(\nu)$ is disjoint and $\bigcup A_n = \bigcup B_n$. Thus

$$\begin{aligned}
\nu\left(\lim(A \cap A_n)\right) = \nu\left(A \cap \left(\bigcup B_n\right)\right) &= \sum_{n=1}^{\infty} \nu(A \cap B_n) \\
&= \lim_n \sum_{i=1}^{n} \nu(A \cap B_n) \\
&= \lim_n \nu\left(A \cap \left(\bigcup_{i=1}^{n} B_i\right)\right) \\
&= \lim_n \nu(A \cap A_n).
\end{aligned}$$

If $\{A_n\}$ is a nonincreasing sequence of $\mathcal{M}(\nu)$ sets, then $\{A_n^c\}$ is a nondecreasing sequence of $\mathcal{M}(\nu)$ sets. Thus $\nu(\lim(A \smallsetminus A_n)) = \lim \nu(A \smallsetminus A_n)$. □

► EXERCISE 3.15 (3.3.6). *Let ν denote an outer measure such that the following holds: if $A \subseteq \Omega$ with $\nu(A) < +\infty$, then there is $B \in \mathcal{M}(\nu)$ with $A \subseteq B$ and*

$\nu(A) = \nu(B)$. Then, for any nondecreasing sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of Ω, we have $\nu(\lim A_n) = \lim \nu(A_n)$.

PROOF. If there exists A_k such that $\nu(A_k) = +\infty$, then $\nu(\lim A_n) = \nu(\bigcup A_n) \geqslant \nu(A_k) = +\infty$, and so $\nu(\lim A_n) = +\infty$; on the other hand, $\nu(A_1) \leqslant \nu(A_2) \leqslant \cdots$ and $\nu(A_k) = +\infty$ imply that $\lim \nu(A_n) = +\infty$.

Now let $\nu(A_n) < +\infty$ for all $n \in \mathbb{N}$. Then there exists $B_n \in \mathcal{M}(\nu)$ such that $A_n \subseteq B_n$ and $\nu(A_n) = \nu(B_n)$ for each $n \in \mathbb{N}$. We first show that we can choose $\{B_n\}$ so that it is nonincreasing.

Consider B_n and B_{n+1}. If $B_n \supset B_{n+1}$, then $\nu(B_n) = \nu(B_{n+1})$ since $\nu(B_n) = \nu(A_n) \leqslant \nu(A_{n+1}) = \nu(B_{n+1})$ always holds. But then $\nu(A_n) = \nu(A_{n+1})$ and so we can just let $B_{n+1} = B_n$ after we having chosen B_n.

Thus, $\nu(\lim B_n) = \lim \nu(B_n)$ exists. Since $\{A_n\}_{n=1}^{\infty}$ is nondecreasing, we have $\nu(\bigcup_{i=1}^{n} A_i) = \nu(A_n) = \nu(B_n)$ for all $n \in \mathbb{N}$. Take the limit and we get $\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \nu(\lim B_n) = \lim \nu(B_n) = \lim \nu(A_n)$. □

► EXERCISE 3.16 (3.3.7). *In each of the following parts, (i) describe the outer measure μ^* on 2^{Ω} induced by the given μ, (ii) describe the collection $\mathcal{M}(\mu^*)$ and determine if $\mathcal{M}(\mu^*)$ is a σ-field, and (iii) check to see whether $\mu^* = \mu$ on the given collection $\mathcal{A}$.*

a. $\Omega = \{1, 2, 3\}$, $\mathcal{A} = \big\{\varnothing, \{1\}, \{2, 3\}, \Omega\big\}$, *and μ is a measure on $\mathcal{A}$ such that $\mu(\Omega) = 1$ and $\mu\left(\{1\}\right) = 0$.*

SOLUTION. (a) Since μ is a measure on $\mathcal{A}$, by (finite) additivity, $\mu(\varnothing) = 0$ and $\mu\left(\{2, 3\}\right) = \mu(\Omega) - \mu\left(\{1\}\right) = 1$. Then, $\mu^*(\varnothing) = 0$, $\mu^*\left(\{1\}\right) = 0$, $\mu^*\left(\{2, 3\}\right) = 1$, $\mu^*(\Omega) = 1$, $\mu^*\left(\{2\}\right) = \mu\left(\{2, 3\}\right) = 1$, $\mu^*(3) = \mu\left(\{2, 3\}\right) = 1$, $\mu^*\left(\{1, 2\}\right) = \mu^*\left(\{1, 3\}\right) = \mu(\Omega) = 1$. □

3.4 EXISTENCE OF EXTENSIONS

► EXERCISE 3.17 (3.4.1). *Let $k \in \mathbb{N}$ and refer to the measure $\lambda_k : \mathcal{B}^k \to \overline{\mathbb{R}}$ that assigns the value $\prod_{i=1}^{k} (b_i - a_i)$ to every $(\boldsymbol{a}, \boldsymbol{b}] \in \mathcal{A}_k$, as given in this section's example. For this measure λ_k, we have the following;*

a. $\lambda_k\left((\boldsymbol{a}, \boldsymbol{b}]\right) = \lambda_k\left((\boldsymbol{a}, \boldsymbol{b})\right) = \lambda_k\left([\boldsymbol{a}, \boldsymbol{b})\right) = \lambda_k\left([\boldsymbol{a}, \boldsymbol{b}]\right) = \prod_{i=1}^{k} (b_i - a_i)$.

b. *λ_k applies to any k-dimensional rectangle that contains a k-dimensional open set and is unbounded in at least one dimension gives $+\infty$.*

c. *λ_k applied to any bounded $\mathcal{B}^k$-set yields a finite number.*

d. *It might be thought that if $A \subseteq B$ with $B \in \mathcal{B}^k$ with $\lambda_k(B) = 0$, then $\lambda_k(A)$ must exists and equal 0. Show that if there exists an uncountable set $C \in \mathcal{B}^k$ with $\lambda_k(C) = 0$, then $\lambda_k(A)$ need not even exist, let along equal zero, and thus λ_k violates our intuition in this regard.*

PROOF. **(a)** Observe first that $\{\boldsymbol{x}\} = \lim_n \left(\boldsymbol{x} - \mathbf{1}/n, \boldsymbol{x}\right]$ for any $\boldsymbol{x} \in \mathbb{R}^k$. Therefore,

$$\lambda_k\left(\{\boldsymbol{x}\}\right) = \lambda_k\left(\lim_n \left(\boldsymbol{x} - \mathbf{1}/n, \boldsymbol{x}\right]\right) = \lim_n \lambda_k\left(\boldsymbol{x} - \mathbf{1}/n, \boldsymbol{x}\right] = \lim_n 1/n^k = 0.$$

Now by (M5),

$$\lambda_k\left(\boldsymbol{a}, \boldsymbol{b}\right) = \lambda_k\left(\left(\boldsymbol{a}, \boldsymbol{b}\right] \smallsetminus \{\boldsymbol{b}\}\right) = \lambda_k\left(\boldsymbol{a}, \boldsymbol{b}\right] - \lambda_k\left(\{\boldsymbol{b}\}\right) = \lambda_k\left(\boldsymbol{a}, \boldsymbol{b}\right].$$

(b) Write $(\boldsymbol{a}, \boldsymbol{b}] = (a_1, b_1] \times \cdots (a_k, b_k]$, and assume that $b_1 - a_1 = \infty$. Since $(\boldsymbol{a}, \boldsymbol{b}]$ contains an open set, $b_i - a_i > 0$ for each $i = 1, \ldots, k$. Therefore, $\lambda(\boldsymbol{a}, \boldsymbol{b}] = \infty$.

(c) Let $A \in \mathscr{B}^k$ be bounded. Then there exists a bounded $(\boldsymbol{a}, \boldsymbol{b}]$ containing A. Hence $\lambda_k(A) \leqslant \lambda_k(\boldsymbol{a}, \boldsymbol{b}] < \infty$.

(d) By the Continuum Hypothesis, if C is countable then $|C| \geqslant \mathfrak{c}$. Hence $\left|2^C\right| \geqslant 2^{\mathfrak{c}} > \mathfrak{c}$. However, $|\mathscr{B}^k| = \mathfrak{c}$. □

► EXERCISE 3.18 (3.4.2). *This problem reviews the Extension Theorem.*

a. *Where or how is the fact that $\mu(\varnothing) = 0$ used?*

b. *What happens if $\mathscr{A} = 2^\Omega$?*

SOLUTION. (a) $\varnothing \in \mathcal{M}\left(\mu\right)$ and (b) The Extension Theorem holds if and only if μ is a measure on $\mathscr{A}$. □

► EXERCISE 3.19 (3.4.3). *Consider the Extension Theorem framework. If we have $\mu(A) < +\infty$ for each $A \in \mathscr{A}$, it might not be the case that the measure extension $\mu^*_{\sigma(\mathscr{A})}$ assigns finite measure to every set in $\sigma(\mathscr{A})$. However, if μ is σ-finite on $\mathscr{A}$, then the measure extension $\mu^*_{\sigma(\mathscr{A})}$ is σ-finite on $\sigma(\mathscr{A})$, and the measure extension $\mu^*_{\mathcal{M}(\mu^*)}$ is σ-finite on $\mathcal{M}(\mu^*)$.*

PROOF. Let $\Omega = \mathbb{R}$ and $\mathscr{A}$ consist of $\varnothing$ and all bounded rsc intervals $(a, b]$. Let $\mu(a, b] = b - a$ for all $(a, b] \in \mathscr{A}$. Then $\mu(A) < +\infty$ for each $A \in \mathscr{A}$. However, $\mathbb{R} \in \sigma\left(\mathscr{A}\right)$, and $\mu^*_{\sigma(\mathscr{A})}(\mathbb{R}) = +\infty$. The other two claims are obvious. □

► EXERCISE 3.20 (3.4.4). *There is a measure $\nu \colon \mathscr{B} \to \overline{\mathbb{R}}$ with $\nu(\mathbb{R}) = 1$ and*

$$\nu(a, b] = \int_a^b (2\pi)^{-1/2}\, e^{-z^2/2}\, \mathrm{d}z,$$

where the integral is the familiar Riemann integral from calculus.

PROOF. According to the approach of Georgakis (1994), let $y = zs$, $\mathrm{d}y = z\,\mathrm{d}s$, then

$$
\begin{aligned}
\left(\int_a^b e^{-z^2/2}\,\mathrm{d}z\right)^2 &= \int_a^b \left(\int_a^b e^{-(z^2+y^2)/2}\,\mathrm{d}y\right)\mathrm{d}z \\
&= \int_a^b \left(\int_a^b e^{-z^2(1+s^2)/2} z\,\mathrm{d}s\right)\mathrm{d}z \\
&= \int_a^b \left(\int_a^b e^{-z^2(1+s^2)/2} z\,\mathrm{d}z\right)\mathrm{d}s \\
&= \int_a^b \left[\frac{1}{-\left(1+s^2\right)} e^{-z^2(1+s^2)/2}\bigg|_a^b\right]\mathrm{d}s \\
&= \int_a^b \frac{1}{1+s^2} e^{-a^2(1+s^2)/2}\,\mathrm{d}s - \int_a^b \frac{1}{1+s^2} e^{-b^2(1+s^2)/2}\,\mathrm{d}s. \qquad \square
\end{aligned}
$$

► Exercise 3.21 (3.4.5). *Consider the Extension Theorem framework again.*

a. *If $A \subseteq B \subseteq \Omega$ with $B \in \mathcal{M}(\mu^*)$ and $\mu^*_{\mathcal{M}(\mu^*)}(B) = 0$, then $A \in \mathcal{M}(\nu^*)$ and $\mu^*_{\mathcal{M}(\mu^*)}(A) = 0$.*

b. *If in (a) we replace every occurrence of $\mathcal{M}(\mu^*)$ [including instances where it appears as a subscript] with $\sigma(\mathcal{A})$, then the claim is not necessarily true.*

Proof. **(a)** Assume the hypotheses. To see $A \in \mathcal{M}(\mu^*)$, note that for any $T \subseteq \Omega$ with $\mu^*(T) < +\infty$, we have $\mu^*(T \cap A) \leqslant \mu^*(T \cap B) \leqslant \mu^*(B) = 0$ by monotonicity of μ^*. Therefore, $\mu^*(T) \geqslant \mu^*(T \cap A) + \mu^*(T \cap A^c) = \mu^*(T \cap A^c)$ always holds.

(b) The same reason as in Exercise 3.17(d). $\square$

► Exercise 3.22 (3.4.6). *Let $\mathcal{F}$ denote a field on Ω, and let $\mu\colon \mathcal{F} \to \overline{\mathbb{R}}$ denote a measure. Let $\mu^*\colon 2^{\Omega} \to \overline{\mathbb{R}}$ be given by*

$$
\mu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \mu(A_n)\colon \{A_n\} \text{ is a } \mathcal{F}\text{-covering of } A\right\}, \quad A \subseteq \Omega.
$$

Then μ^* is an outer measure, and the restriction of μ^* to the σ-field $\mathcal{M}(\mu^*)$ is a measure. With these facts, we have that $\mu^* = \mu$ on $\mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{M}(\mu^*)$. Finally, there exists a measure extension of μ to $\sigma(\mathcal{F})$.

Proof. Notice that $\mu(\varnothing) = 0$ and $\mu(A) \geqslant 0$ for all $A \in \mathcal{F}$ since μ is a measure on $\mathcal{F}$. So μ^* is an outer measure by Example 1 in Section 3.2. The other parts are standard. $\square$

► Exercise 3.23 (3.4.7). *Let $\mathcal{F}$ denote a field on Ω. Suppose that $\nu\colon \mathcal{F} \to \overline{\mathbb{R}}$ is such that (i) $\nu(A) \geqslant 0$ for all $A \in \mathcal{F}$, (ii) ν is finitely additive, and (iii) if $\{A_n\}_{n=1}^{\infty}$ is a nonincreasing sequence of $\mathcal{F}$-sets with* $\lim A_n = \varnothing$*, the* $\lim \nu(A_n) = 0$*. Define $\nu^*\colon 2^{\Omega} \to \overline{\mathbb{R}}$ for all $A \subseteq \Omega$ by writing*

$$
\nu^*(A) = \inf\left\{\sum_{n=1}^{\infty} \nu(A_n)\colon \{A_n\}_{n=1}^{\infty} \text{ is an } \mathcal{F}\text{-covering of } A\right\}.
$$

a. *ν^* is an outer measure.*

b. *$\mathcal{M}(\nu^*)$ is a σ-field on Ω.*

c. *The restriction of ν^* to $\mathcal{M}(\nu^*)$ is a measure on $\mathcal{M}(\nu^*)$.*

d. *$\mathcal{F} \subseteq \mathcal{M}(\nu^*)$.*

e. *There exists a measure extension of ν to $\sigma(\mathcal{F})$.*

PROOF. **(a)** It suffices to show that $\nu(\varnothing) = 0$ by Example 1 of Section 3.2. Take a sequence $\{\varnothing, \varnothing, \dots\}$. Then $0 = \lim \nu(\varnothing) = \nu(\varnothing)$.

(b) —(e) are from the Outer Measure Theorem. □

3.5 Uniqueness of Measures and Extensions

► EXERCISE 3.24 (3.5.1). *If μ_1 and μ_2 are finite measures with domain $\sigma(\mathcal{P})$ (where $\mathcal{P}$ denotes a π-system on Ω), if Ω can be expressed as an amc union of $\mathcal{P}$-sets, and if $\mu_1 = \mu_2$ on $\mathcal{P}$, then $\mu_1 = \mu_2$ on $\sigma(\mathcal{P})$.*

PROOF. Assume the hypotheses. Then σ_1 is σ-finite with respect to $\mathcal{P}$ and $\mu_1 = \mu_2$ on $\mathcal{P}$. By the Uniqueness Theorem, $\mu_1 = \mu_2$ on $\sigma(\mathcal{P})$. □

► EXERCISE 3.25 (3.5.2). *Let μ_1 and μ_2 denote finite measures with domain $\sigma(\mathcal{P})$, where $\mathcal{P}$ is a π-system on Ω, and further suppose that $\mu_1 = \mu_2$ on $\mathcal{P}$. Then $\mu_1 = \mu_2$.*

PROOF. I am not sure about this exercise. If $\Omega \in \mathcal{P}$, then by letting

$$\mathcal{L} = \{A \in \sigma(\mathcal{P}) \colon \mu_1(A) = \mu_2(A)\},$$

we can easily to show that $\mathcal{L}$ is a λ-system with $\mathcal{P} \subseteq \mathcal{L}$. Then the result is trivial. □

► EXERCISE 3.26 (3.5.3). *Let $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$, let $\mathcal{A}$ consist of $\varnothing$, $\{\omega_1, \omega_2\}$, $\{\omega_1, \omega_3\}$, $\{\omega_2, \omega_4\}$, $\{\omega_3, \omega_4\}$, and Ω, and let $\mu \colon \mathcal{A} \to \mathbb{R}$ be defined as follows: $\mu(\Omega) = 6$, $\mu(\varnothing) = 0$, and $\mu(\{\omega_1, \omega_2\}) = \mu(\{\omega_1, \omega_3\}) = \mu(\{\omega_2, \omega_4\}) = \mu(\{\omega_3, \omega_4\}) = 3$.*

a. *$\mathcal{A}$ is neither a π-system nor a semiring, and $\sigma(\mathcal{A}) = 2^{\Omega}$.*

b. *μ is a measure.*

c. *Define two new distinct measures ν and ξ on the σ-field 2^{Ω} by the following: $\nu(\{\omega_1\}) = \nu(\{\omega_4\}) = 1$, $\nu(\{\omega_2\}) = \nu(\{\omega_3\}) = 2$, $\xi(\{\omega_2\}) = \xi(\{\omega_3\}) = 1$, and $\xi(\{\omega_1\}) = \xi(\{\omega_4\}) = 2$. Then ν and ξ are distinct measure extensions of μ from $\mathcal{A}$ to $\sigma(\mathcal{A}) = 2^{\Omega}$.*

d. *Let $\mu^*\colon 2^\Omega \to \mathbb{R}$ denote the outer measure induced by μ. Then $\mu^* = \mu$ on $\mathscr{A}$ and μ^* is a measure.*

e. *μ^*, ν, and ξ are distinct measures.*

Proof. **(a)** $\mathscr{A}$ is not a π-system because $\{\omega_1, \omega_2\} \cap \{\omega_1, \omega_3\} = \{\omega_1\} \notin \mathscr{A}$, and so $\mathscr{A}$ is not a semiring. $\sigma(\mathscr{A}) = 2^\Omega$ since every singleton can be expressed as a intersection of $\mathscr{A}$-sets.

(b) Easy to check.

(c) For example, $\nu(\{\omega_1, \omega\}) = \nu(\{\omega_1\}) + \nu(\{\omega_2\}) = 1 + 2 = \mu(\{\omega_1, \omega_2\})$.

(d) For example, $\mu^*(\{\omega\}) = \inf\{\mu(\{\omega_1, \omega_2\}), \mu(\{\omega_1, \omega_3\})\} = 3$.

(e) Trivial to see that they are distinct. □

► Exercise 3.27 (3.5.5). *We assume the setup of Exercise 3.22. The aim of this exercise is to show that if ν is a measure with domain $\sigma(\mathscr{F})$ such that $\nu = \mu$ on the field $\mathscr{F}$, then ν coincides with the measure extension of μ to $\sigma(\mathscr{F})$ guaranteed by Exercise 3.22.*

a. *If $B \in \sigma(\mathscr{F})$, then $\nu(B) \leqslant \mu^*(B)$.*

b. *If $F \in \sigma(\mathscr{F})$ and $\mu^*(F) < +\infty$, then $\nu(F) = \mu^*(F)$.*

c. *If μ is σ-finite on $\mathscr{F}$, then $\nu(E) = \mu^*(E)$ for all $E \in \sigma(\mathscr{F})$. This gives the uniqueness of the measure extension whose existence is guaranteed by Exercise 3.22.*

Proof. **(a)** Let $\mathscr{F}_\sigma$ denote the family of all countable unions of $\mathscr{F}$-sets. If $A \in \mathscr{F}_\sigma$, i.e., $A = \bigcup A_n$ with $A_n \in \mathscr{F}$ for all n, then, by letting $B_1 = A_1$ and $B_n = A_n \smallsetminus (\bigcup_{i \leqslant n} A_i)$ for $n \geqslant 2$, we can rewrite A as a disjoint union of $\mathscr{F}$-sets $\{B_n\}$. Thus,

$$\nu(A) = \nu\left(\bigcup B_n\right) = \sum \nu(B_n) = \sum \mu(B_n) = \sum \mu^*(B_n) = \mu^*\left(\bigcup B_n\right) = \mu^*(A).$$

Now take an arbitrary $B \in \sigma(\mathscr{F})$, and we show that

$$\begin{aligned}\mu^*(B) &= \inf\left\{\sum \mu(A_n)\colon \{A_n\} \text{ is an } \mathscr{F}\text{-covering of } B\right\} \\ &= \inf\left\{\mu^*(A)\colon B \subseteq A \in \mathscr{F}_\sigma\right\} \qquad \text{defined as } \beta.\end{aligned}$$

Firstly, $B \subseteq A$ implies that $\mu^*(B) \leqslant \mu^*(A)$ and so $\mu^*(B) \leqslant \beta$. Secondly, for all $A \in \mathscr{F}_\sigma$, there exists $\{A_n\} \subseteq \mathscr{F}_\sigma$ such that $A = \bigcup A_n$, and so we get $\mu^*(A) = \mu^*(\bigcup A_n) \leqslant \sum \mu^*(A_n) = \sum \mu(A_n)$; thus, $\beta \leqslant \mu^*(B)$. Therefore,

$$\mu^*(B) = \inf\left\{\mu^*(A)\colon B \subseteq A \in \mathscr{F}_\sigma\right\} = \inf\{\nu(A)\colon B \subseteq A \in \mathscr{F}_\sigma\} \geqslant \nu(B).$$

(b) It suffices to show that $\nu(F) \geqslant \mu^*(F)$ by part (a). Since $\mu^*(F) = \inf\{\mu^*(A) : F \subseteq A \in \mathcal{F}_\sigma\}$, for a given $\varepsilon > 0$, there exists $C \in F_\sigma$ with $F \subseteq C$, such that

$$\mu^*(F) + \varepsilon > \mu^*(C).$$

Hence,

$$\begin{aligned}\mu^*(F) \leqslant \mu^*(C) = \nu(C) = \nu(F) + \nu(C \smallsetminus F) &\leqslant \nu(F) + \mu^*(C \smallsetminus F)\\ &= \nu(F) + \mu^*(C) - \mu^*(F)\\ &< \mu(F) + \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we get $\mu^*(F) \leqslant \nu(F)$ whenever $F \in \sigma(\mathcal{F})$ and $\mu^*(F) < +\infty$.

(c) If μ is σ-finite on $\mathcal{F}$, then there exists $\{A_n\} \subseteq \mathcal{F}$ such that $\Omega = \bigcup A_n$ and $\mu(A_n) < +\infty$ for all n. Without loss of generality, we can assume that $\{A_n\}$ is disjoint $\mathcal{F}$-sets. Then by (b), for every $E \in \sigma(\mathcal{F})$ we get

$$\nu(E) = \nu\left(\bigcup (E \cap A_n)\right) = \sum \nu(E \cap A_n) = \sum \mu^*(E \cap A_n) = \mu^*(E). \qquad \square$$

► Exercise 3.28 (3.5.6). *A lattice on Ω is a collection $\mathcal{L} \subseteq 2^\Omega$ such that (i) $\Omega \in \mathcal{L}$, (ii) $\varnothing \in \mathcal{L}$, (iii) $\mathcal{L}$ is closed under (finite) unions, and (iv) $\mathcal{L}$ is a π-system. We also define the following two collections: $\mathcal{D} = \{B \smallsetminus A : A, B \in \mathcal{L}, A \subseteq B\}$, and $\mathcal{U}$ will denote the collection of all finite disjoint unions of $\mathcal{D}$-sets.*

a. *$\mathcal{D}$ is a π-system.*

b. *$\mathcal{U}$ is a π-system.*

c. *$\mathcal{U}$ is closed under complementation.*

d. *$\mathcal{U}$ coincides with the minimal field containing the lattice $\mathcal{L}$.*

e. *Let $\mathcal{A}$ denote a σ-field on Ω that contains $\mathcal{L}$. Suppose that μ and ν are measures with domain $\mathcal{A}$ such that $\mu = \nu$ on $\mathcal{L}$. Furthermore, suppose that $\Omega = \bigcup A_n$, where $A_n \in \mathcal{L}$ and $\mu(A_n) < +\infty$ for each $n \in \mathbb{N}$. Then $\mu = \nu$ on $\sigma(\mathcal{L})$.*

Proof. **(a)** Write $D_i = B_i \smallsetminus A_i$ with $A_i, B_i \in \mathcal{L}$ and $A_i \subseteq B_i$, for $i = 1, 2$. Then

$$\begin{aligned}D_1 \cap D_2 &= (B_1 \smallsetminus A_1) \cap (B_2 \smallsetminus A_2)\\ &= (B_1 \cap A_1^c) \cap (B_2 \cap A_2^c)\\ &= (B_1 \cap B_2) \smallsetminus (A_1 \cup A_2)\\ &= (B_1 \cap B_2) \smallsetminus \left[(B_1 \cap B_2) \cap (A_1 \cup A_2)\right]\\ &\in \mathcal{D}.\end{aligned}$$

(b) Let $U_1 = D_1 \cup \cdots \cup D_m$ and $U_2 = E_1 \cup \cdots \cup E_n$, where $D_1, \dots, D_m \in \mathcal{D}$ are disjoint, and $E_1, \dots, E_n \in \mathcal{D}$ disjoint. Then

$$U_1 \cap U_2 = \bigcup_{i=1}^{m} \bigcup_{j=1}^{n} (D_i \cap E_j).$$

Since $D_i \cap E_j \in \mathcal{D}$ ($\mathcal{D}$ is a π-system), $U_1 \cap U_2$ is a finite disjoint unions of $\mathcal{D}$-sets, and so is in $\mathcal{U}$.

(c) Pick an arbitrary $U \in \mathcal{U}$. Then there exists disjoint $D_1, \dots, D_n \in \mathcal{D}$ such that $U = \bigcup_{i=1}^{n} D_i$. If $n = 1$, then $U = D_1 = B_1 \smallsetminus A_1$, where $A_1, B_1 \in \mathcal{L}$ and $A_1 \subseteq B_1$. Thus, $U^c = (A \smallsetminus \varnothing) \cup (\Omega \smallsetminus B) \in \mathcal{U}$. Let us assume that $U^c \in \mathcal{U}$ when $U = D_1 \cup \cdots \cup D_n$ and consider $n + 1$. Then

$$\left(\bigcup_{i=1}^{n+1} D_i\right)^c = \left(\bigcup_{i=1}^{n} D_i\right)^c \cap D_{n+1}^c \in \mathcal{U},$$

since $(\bigcup_{i=1}^{n} D_i)^c \in \mathcal{U}$ by the induction hypothesis, $D_{n+1}^c \in \mathcal{U}$ as in the case of $n + 1$, and $\mathcal{U}$ is a π-system.

(d) Notice that $\Omega = \Omega \smallsetminus \varnothing \in \mathcal{U}$, so $\mathcal{U}$ is a field by (b) and (c). If $A \in \mathcal{L}$, then $A = A \smallsetminus \varnothing \in \mathcal{U}$, so $\mathcal{L} \subseteq \mathcal{U}$. Thus, $f(\mathcal{L}) \subseteq \mathcal{U}$. It is easy to see that $\mathcal{U} \subseteq f(\mathcal{L})$.

(e) Let $\Omega = \bigcup_{n=1}^{\infty} A_n$ with $A_n \in \mathcal{L}$ and $\mu(A_n) < +\infty$ for each A_n. Let $U \in \mathcal{U}$; then $U = D_1 \cup \cdots \cup D_m$ for some disjoint $D_1, \dots, D_m \in \mathcal{D}$. Hence,

$$\begin{aligned}\mu(U) = \mu\left(\bigcup_{n=1}^{\infty} (A_n \cap U)\right) &= \sum_{n=1}^{\infty} \mu(A_n \cap U) \\ &= \sum_{n=1}^{\infty} \sum_{i=1}^{m} \mu(A_n \cap D_i)\end{aligned}$$

□

3.6 The Completion Theorem

► Exercise 3.29 (3.6.1). *Let $k \geqslant 2$, and let $\lambda_k \colon \mathcal{B}^k \to \overline{\mathbb{R}}$ denote the unique measure with domain $\mathcal{B}^k$ that assigns the value $\prod_{i=1}^{k} (b_i - a_i)$ to each k-dimensional rsc rectangle $(\boldsymbol{a}, \boldsymbol{b}]$. Let $A = \{\boldsymbol{x} \in \mathbb{R}^k : x_2 = \cdots = x_k = 0\}$. Then $A \in \mathcal{B}^k$, A is uncountable, and $\lambda_k(A) = 0$, hence $\left|\overline{B^k}\right| = 2^{\mathfrak{c}}$.*

Proof. $A \in \mathcal{B}^k$ since $A = \lim_n [(-n, n] \times (-1/n, 1/n]^{k-1}]$; A is uncountable since $\mathbb{R}$ is uncountable; $\lambda_k(A) = 0$ since $\lambda_k(A) = (+\infty) \times 0 \times \cdots \times 0 = 0$.

Since $\overline{\mathcal{B}^k}$ is complete (the completion of $\mathcal{B}^k$), and $A \in \mathcal{B}^k \subseteq \overline{\mathcal{B}^k}$, we know that every subset of A is in $\overline{\mathcal{B}^k}$. There are $2^{\mathfrak{c}}$ subsets of A, so $\left|\overline{\mathcal{B}^k}\right| \geqslant 2^{\mathfrak{c}}$; on the other hand, there are 2^c subsets on $\mathbb{R}$, i.e., $\left|\overline{\mathcal{B}^k}\right| \leqslant 2^c$. It follows from the Cantor-Bernstein theorem that $\left|\overline{\mathcal{B}^k}\right| = 2^c$. □

► EXERCISE 3.30 (3.6.2). *Let $(\Omega, \mathcal{F}, \nu)$ denote a measure space. If $A, B \in \mathcal{F}$ with $A \subseteq E \subseteq B$ and $\nu(B \setminus A) = 0$, then $E \in \overline{\mathcal{F}}$ and $\overline{\nu}(E) = \nu(A) = \nu(B)$.*

PROOF. We first show $E \in \overline{\mathcal{F}}$. Since $A, B \in \mathcal{F}$, and $\mathcal{F}$ is a σ-field, we get $B \setminus A \in \mathcal{F}$. Now we can write E as

$$E = A \cup \left[(B \setminus A) \setminus (B \setminus E)\right].$$

Since $A \in \mathcal{F}$, $(B \setminus A) \setminus (B \setminus E) \subseteq B \setminus A \in \mathcal{F}$, and $\nu(B \setminus A) = 0$, we have $(B \setminus A) \setminus (B \setminus E) \in \mathcal{N}_0(\nu)$; thus $E \in \overline{\mathcal{F}}$.

To show $\overline{\nu}(E) = \nu(A) = \nu(B)$, we only need to show that $\nu(A) = \nu(B)$ since $\overline{\nu}(E) = \nu(A)$ by definition. If $\nu(A) < +\infty$ or $\nu(B) < +\infty$, then $0 = \nu(B \setminus A) = \nu(B) - \nu(A)$ implies that $\nu(B) = \nu(A)$. If $\nu(A) = +\infty$, then by the monotonicity of a measure, $\nu(B) \geqslant \nu(A) = +\infty$, and so $\nu(B) = +\infty = \nu(A)$ □

► EXERCISE 3.31 (3.6.3). *Let $(\Omega, \mathcal{F}, \nu)$ denote a measure space. Furthermore, let $\mathcal{F}_1$ denote a sub-σ-field of $\mathcal{F}$. Then there exists a minimal σ-field $\mathcal{F}_2$ such that $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \mathcal{F}$ and $\mathcal{N}(\nu) \subseteq \mathcal{F}_2$. Also, $A \in \mathcal{F}_2$ iff there exists $B \in \mathcal{F}_1$ with $A \Delta B \in \mathcal{N}(\nu)$.*

PROOF. Let $\mathcal{F}_2 = \sigma(\mathcal{F}_1, \mathcal{N}(\nu))$. It is clear that $\mathcal{F}_1, \mathcal{N}(\nu) \subseteq \mathcal{F}_2$. Since $\mathcal{F}_1, \mathcal{N}(\nu) \subseteq \mathcal{F}$, we have that

$$\mathcal{F}_2 = \sigma(\mathcal{F}_1, \mathcal{N}(\nu)) \subseteq \sigma(\mathcal{F}) = \mathcal{F}.$$

We next show that $(\Omega, \mathcal{F}_2, \nu)$ is complete (where ν is restricted on $\mathcal{F}_2$). □

3.7 THE RELATIONSHIP BETWEEN $\sigma(\mathcal{A})$ AND $\mathcal{M}(\mu^*)$

► EXERCISE 3.32 (3.7.1). *Let Ω be uncountable, let $\mathcal{A}$ denote the σ-filed $\{A \subseteq \Omega\colon A$ is amc or A^c is ams$\}$, and define $\mu\colon \mathcal{A} \to \overline{\mathbb{R}}$ by stipulating that $\mu(A)$ denotes the number of points in A if A is finite and $\mu(A) = +\infty$ if A is infinite.*

a. *$(\Omega, \mathcal{A}, \mu)$ is a non-σ-finite measure space.*

b. *$(\Omega, \mathcal{A}, \mu)$ is complete.*

c. *Letting μ^* denote the outer measure induced by μ, the (σ-field) $\mathcal{M}(\mu^*)$ coincides with 2^Ω.*

PROOF. **(a)** For every sequence $\{A_n\} \subseteq \mathcal{A}$ with $\mu(A_n) < +\infty$, i.e., A_n is finite for all n, their union $\bigcup_n A_n$ is amc. Hence, $(\Omega, \mathcal{A}, \mu)$ is non-σ-finite.

(b) Let $A \subseteq B \subseteq \Omega$ with $B \in \mathcal{A}$ and $\mu(B) = 0$. Then B must be empty and so $A = \varnothing = B \in \mathcal{A}$.

(c) Take an arbitrary $T \subseteq \Omega$ with $\mu^*(T) < +\infty$; that is, T is finite. Then for every subset $A \subseteq \Omega$, we have that $T \cap A$ and $T \cap A^c$ are both finite and $|T| = |T \cap A| + |T \cap A^c|$. Hence, $\mu^*(T) = \mu^*(T \cap A) + \mu^*(T \cap A^c)$, i.e., $A \in \mathcal{M}(\mu^*)$. Thus, $\mathcal{M}(\mu^*) = 2^{\Omega}$.

Note that here $\bar{\mathcal{A}} = \mathcal{A} \neq \mathcal{M}(\mu^*) = 2^{\Omega}$, so the σ-finiteness is essential. □

3.8 Approximations

▶ Exercise 3.33 (3.8.1). *The assumption $\nu(B) < +\infty$ is not superfluous in Claim 4, and the assumption $\nu(A) < +\infty$ is not superfluous in Claim 6.*

Proof. Let $\mathcal{A}$ be the semiring consisting of $\varnothing$ and all bounded rsc $(a, b]$. Let us consider $(\mathbb{R}, \mathcal{B}, \lambda)$. For Claim 4, take $B = \mathbb{R}$. It is evident that for any finite disjoint $\mathcal{A}$-sets $(a_1, b_1], \dots, (a_n, b_n]$, we have $\lambda(\mathbb{R} \Delta \bigcup_{i=1}^n (a_i, b_i]) = +\infty$. For Claim 6, let $A = \mathbb{R}$. then for any bounded set $E \in \mathcal{B}$, there exists $(a, b] \in \mathcal{A}$ containing E, so $\lambda(E) \leqslant \lambda(a, b] = b - a < +\infty = \lambda(\mathbb{R})$. □

▶ Exercise 3.34 (3.8.2). *Let $\nu \colon \mathcal{B}^k \to \overline{\mathbb{R}}$ be nonnegative and finitely additive with $\nu(\mathbb{R}^k) < +\infty$. Suppose that $\nu(A) = \sup\{\nu(K) \colon K \subseteq A, K \text{ compact}\}$ for each $A \in \mathcal{B}^k$. Then ν is a finite measure.*

Proof. It suffices to show that ν is countably additive. □

3.9 A Further Description of $\mathcal{M}(\mu^*)$

▶ Exercise 3.35 (3.9.1). *Countable superadditivity: If $A_1, A_2, \dots \subseteq \Omega$ are disjoint, then $\mu_*(\bigcup_{n=1}^\infty A_n) \geqslant \sum_{n=1}^\infty \mu_*(A_n)$.*

Proof. Fix an arbitrary $\varepsilon > 0$. For every A_n, find $C_n \in \sigma(\mathcal{A})$ with $C_n \subseteq A_n$ such that

$$\mu^*_{\sigma(\mathcal{A})}(C_n) + \varepsilon/2^n > \mu_*(A_n).$$

Since $\bigcup_{n=1}^\infty C_n \subseteq \bigcup_{n=1}^\infty A_n$, we have

$$\sum_{n=1}^\infty \mu_*(A_n) < \sum_{n=1}^\infty \mu^*_{\sigma(\mathcal{A})}(C_n) + \varepsilon = \mu^*_{\sigma(\mathcal{A})}\left(\bigcup_{n=1}^\infty C_n\right) + \varepsilon \leqslant \mu_*\left(\bigcup_{n=1}^\infty A_n\right) + \varepsilon,$$

where the first equality holds since $\{C_n\}$ is disjoint $\sigma(\mathcal{A})$-sets. Since $\varepsilon > 0$ is arbitrary, we get the countable superadditivity. □

▶ Exercise 3.36 (3.9.2). *For any $A \subseteq \Omega$, $\mu^*(A) = \inf\{\mu^*_{\mathcal{M}(\mu^*)}(B) \colon A \subseteq B \in \mathcal{M}(\mu^*)\} = \mu^{**}(A)$.*

Proof. Define

$$\mathcal{C} = \left\{\mu^*_{\sigma(\mathcal{A})}(B) : A \subseteq B \in \sigma(\mathcal{A})\right\} \quad \text{and} \quad \mathcal{D} = \left\{\mu^*_{\mathcal{M}(\mu^*)}(B) : A \subseteq B \in \mathcal{M}(\mu^*)\right\}.$$

It is clear that $\mathcal{C} \subseteq \mathcal{D}$, so $\inf \mathcal{C} \geqslant \inf \mathcal{D}$. Next, pick $d \in \mathcal{D}$. Then d must be of the form $\mu^*_{\mathcal{M}(\mu^*)}(B)$, where $A \subseteq B \in \mathcal{M}(\mu^*)$. Write $B = C \cup D$, where $C \in \sigma(\mathcal{A})$, $D \subseteq N$ and N is a $\mu^*_{\sigma(\mathcal{A})}$-null set. Thus, there exists $C \cup N \in \sigma(\mathcal{A})$ such that $A \subseteq B \subseteq C \cup N$ and

$$\mu^*_{\sigma(\mathcal{A})}(C \cup N) \leqslant \mu^*_{\sigma(\mathcal{A})}(C) + \mu^*_{\sigma(\mathcal{A})}(N) = \mu^*_{\mathcal{M}(\mu^*)}(C) \leqslant \mu^*_{\mathcal{M}(\mu^*)}(B).$$

Denote $\mu^*_{\sigma(\mathcal{A})}(C \cup N) = c$. Hence, for every $d \in \mathcal{D}$, there exists $c \in \mathcal{C}$ with $c \leqslant d$. It follows that $\inf \mathcal{C} \leqslant \inf \mathcal{D}$. □

► EXERCISE 3.37 (3.9.3). *For any $A \subseteq \Omega$, $\mu_*(A) = \sup\{\mu^*_{\mathcal{M}(\mu^*)}(B) : B \subseteq A, B \in \mathcal{M}(\mu^*)\}$.*

PROOF. Define

$$\mathcal{C} = \left\{\mu^*_{\sigma(\mathcal{A})}(B) : B \subseteq A, B \in \sigma(\mathcal{A})\right\}, \quad \mathcal{D} = \left\{\mu^*_{\mathcal{M}(\mu^*)}(B) : B \subseteq A, B \in \mathcal{M}(\mu^*)\right\}.$$

First, $\mathcal{C} \subseteq \mathcal{D}$ implies that $\sup \mathcal{C} \leqslant \sup \mathcal{D}$. Next, pick $d \in \mathcal{D}$. Then d must be of the form $\mu^*_{\mathcal{M}(\mu^*)}(B)$, where $B \subseteq A$ and $B \in \mathcal{M}(\mu^*)$. Write $B = C \cup D$ with $C \in \sigma(\mathcal{A})$, $D \subseteq N$, and N is a $\mu^*_{\sigma(\mathcal{A})}$-null set. Thus, there exists $C \in \sigma(\mathcal{A})$ such that $C \subseteq B \subseteq A$, and

$$\mu^*_{\sigma(\mathcal{A})}(C) = \mu^*_{\mathcal{M}(\mu^*)}(C \cup D) = \mu^*_{\mathcal{M}(\mu^*)}(B).$$

Denote $c = \mu^*_{\sigma(\mathcal{A})}(C)$. So for every $d \in \mathcal{D}$ there exists $c \in \mathcal{C}$ such that $c = d$. Therefore, $\sup \mathcal{C} \geqslant \sup \mathcal{D}$. □

► EXERCISE 3.38 (3.9.4). *For any $A \subseteq \Omega$, there is $E \in \sigma(\mathcal{A})$ such that $E \subseteq A$ and $\mu^*_{\sigma(\mathcal{A})}(E) = \mu_*(A)$.*

PROOF. For every $n \in \mathbb{N}$, there exists $E_n \in \sigma(\mathcal{A})$ with $E_n \subseteq A$ such that

$$\mu^*_{\sigma(\mathcal{A})}(E_n) \geqslant \mu_*(A) - 1/n.$$

Let $E = \bigcup_{n=1}^{\infty} E_n$. Then $E \in \sigma(\mathcal{A})$, $E \subseteq A$, and for all n we have

$$\mu^*_{\sigma(\mathcal{A})}(E) \geqslant \mu^*_{\sigma(\mathcal{A})}(A_n) \geqslant \mu_*(A) - 1/n;$$

hence, $\mu^*_{\sigma(\mathcal{A})}(E) \geqslant \mu_*(A)$. Since $\mu^*_{\sigma(\mathcal{A})}(E) \leqslant \mu_*(A)$ holds, we get the result. □

► EXERCISE 3.39 (3.9.5). *The infimum that defines $\mu^*(A)$ and the supremum that defines $\mu_*(A)$ are achieved for each $A \subseteq \Omega$. That is, there exist $C, B \in \sigma(\mathcal{A})$ with $C \subseteq A \subseteq B$ and $\mu^*_{\sigma(\mathcal{A})}(C) = \mu_*(A)$ and $\mu^*_{\sigma(\mathcal{A})}(B) = \mu^*(A)$.*

PROOF. We have proved the inner measure in the previous exercise, so we focus on the outer measure. For every $n \in \mathbb{N}$, there exists $B_n \in \sigma(\mathcal{A})$ with

$A \subseteq B_n$ and $\mu^*(A) + 1/n \geqslant \mu^*_{\sigma(\mathcal{A})}(B_n)$. Let $B = \bigcap_{n=1}^\infty B_n$. Then $B \in \sigma(\mathcal{A})$, $A \subseteq B$, and for all n,

$$\mu^*_{\sigma(\mathcal{A})}(B) \leqslant \mu^*_{\sigma(\mathcal{A})}(B_n) \leqslant \mu^*(A) + 1/n;$$

that is, $\mu^*_{\sigma(\mathcal{A})}(B) \leqslant \mu^*(A)$. Since the other direction is clear, we get the result. □

► EXERCISE 3.40 (3.9.6). *Let $A \subseteq \Omega$ and let $\{A_n\}$ denote a disjoint sequence of $\mathcal{M}(\mu^*)$-sets. Then we have $\mu_*(\bigcup_{n=1}^\infty (A \cap A_n)) = \sum_{n=1}^\infty \mu_*(A_n \cap A)$.*

PROOF. There exists $E \in \mathcal{M}(\mu^*)$ with $E \subseteq \bigcup_{n=1}^\infty (A \cap A_n) = A \cap (\bigcup_{n=1}^\infty A_n)$ and $\mu^*_{\mathcal{M}(\mu^*)}(E) = \mu_*(\bigcup_{n=1}^\infty (A \cap A_n))$ by Exercise 3.38 (since $\overline{\sigma(\mathcal{A})} = \mathcal{M}(\mu^*)$). Thus,

$$\mu_*\left(\bigcup_{n=1}^\infty (A \cap A_n)\right) = \mu^*_{\mathcal{M}(\mu^*)}(E) = \sum_{n=1}^\infty \mu^*_{\mathcal{M}(\mu^*)}(E \cap A_n) \leqslant \sum_{n=1}^\infty \mu_*(A \cap A_n).$$

Then the desired result follows from Exercise 3.35. □

► EXERCISE 3.41 (3.9.7). *If $A, B \subseteq \Omega$ are disjoint, then $\mu_*(A \cup B) \leqslant \mu_*(A) + \mu^*(B) \leqslant \mu^*(A \cup B)$.*

PROOF. Let $F \in \sigma(\mathcal{A})$ with $B \subseteq F$ with $\mu^*(B) = \mu^*_{\sigma(\mathcal{A})}(F)$. Let $E \in \sigma(\mathcal{A})$ with $E \subseteq A \cup B$ such that $\mu_*(A \cup B) = \mu^*_{\sigma(\mathcal{A})}(E)$. Since $E \setminus F \subseteq E \setminus B \subseteq (A \cup B) \setminus B = A$, it follows that

$$\mu_*(A \cup B) = \mu^*_{\sigma(\mathcal{A})}(E) \leqslant \mu^*_{\sigma(\mathcal{A})}(E \setminus F) + \mu^*_{\sigma(\mathcal{A})}(F) \leqslant \mu_*(A) + \mu^*(B).$$

Dually, let $H \in \sigma(\mathcal{A})$ with $H \subseteq A$ and $\mu^*_{\sigma(\mathcal{A})}(H) = \mu_*(A)$. Let $G \in \sigma(\mathcal{A})$ with $A \cup B \subseteq G$ and $\mu^*_{\sigma(\mathcal{A})}(G) = \mu^*(A \cup B)$. Since $B \subseteq G \setminus H$, it follows that

$$\mu^*(A \cup B) = \mu^*_{\sigma(\mathcal{A})}(G) = \mu^*_{\sigma(\mathcal{A})}(H) + \mu^*_{\sigma(\mathcal{A})}(G \setminus H) \geqslant \mu_*(A) + \mu^*(B). \qquad \square$$

► EXERCISE 3.42 (3.9.8). *If $A \in \mathcal{M}(\mu^*)$ and $B \subseteq \Omega$, then $\mu^*_{\mathcal{M}(\mu^*)}(A) = \mu_*(B \cap A) + \mu^*(B^c \cap A)$.*

PROOF. Applying Exercise 3.41 to $B \cap A$ and $B^c \cap A$, we obtain

$$\mu_*(A) \leqslant \mu_*(B \cap A) + \mu^*(B^c \cap A) \leqslant \mu^*(A).$$

Since $A \in \mathcal{M}(\mu^*)$, we have $\mu_*(A) = \mu^*(A) = \mu^*_{\mathcal{M}(\mu^*)}(A)$, and thus we get the result. □

4

LEBESGUE MEASURE

4.1 Lebesgue Measure: Existence and Uniqueness

► Exercise 4.1 (4.1.1). *Let $x \in \mathbb{R}$ and $k \geqslant 2$. Then $\lambda(\{x\}) = \lambda_k(\{x\} \times \mathbb{R}^{k-1}) = 0$. Next, for any $j \in \{1, \ldots, k-1\}$ and $\boldsymbol{x} \in \mathbb{R}^j$ we have $\lambda_k(\{\boldsymbol{x}\} \times \mathbb{R}^{k-j}) = 0$.*

Proof. Since the sequence $\{\{x\} \times (-n, n]^{k-1}\}$ is increasing and converges to $\{x\} \times \mathbb{R}^{n-1}$, we have

$$\lambda_k(\{x\} \times \mathbb{R}^{k-1}) = \lambda_k(\lim_n \{x\} \times (-n, n]^{k-1}) = 0 = \lambda(\{x\}).$$

The other claim can be proved in the same way. □

► Exercise 4.2 (1.4.2). *Enumerate the rationals in $(0,1]$ by $\{q_1, q_2, \ldots\}$. Given arbitrarily small $\varepsilon > 0$, remove the interval $A_n = (q_n - \varepsilon/2^{n+1}, q_n + \varepsilon/2^{n+1}) \cap (0,1]$. Let $A = \bigcup_{n=1}^{\infty} A_n$. Then $\overline{\lambda}(A) \leqslant \varepsilon$, despite the fact that A is an open dense subset of $(0,1]$. Also, we have $\overline{\lambda}((0,1] \smallsetminus A) \geqslant 1 - \varepsilon$, even though $(0,1] \smallsetminus A$ is a nowhere dense subset of $(0,1]$.*

Proof. For every A_n we have $0 < \overline{\lambda}(A_n) \leqslant \overline{\lambda}(q_n - \varepsilon/2^{n+1}, q_n + \varepsilon/2^{n+1}) = \varepsilon/2^n$; hence,

$$\overline{\lambda}(A) = \overline{\lambda}\left(\bigcup_{n=1}^{\infty} A_n\right) \leqslant \sum_{n=1}^{\infty} \overline{\lambda}\,(A_n) \leqslant \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon,$$

and so $\overline{\lambda}((0,1] \smallsetminus A) = 1 - \overline{\lambda}(A) \geqslant 1 - \varepsilon$. □

► Exercise 4.3 (4.1.3). *There cannot exist a closed subset of $(0,1]$ whose interior is empty, yet has $\overline{\lambda}$-measure of one.*

Proof. □

► Exercise 4.4 (4.14). *λ_k is nonatomic: any $A \in \mathscr{B}^k$ with $\lambda_k(A) > 0$ has a proper subset $B \in \mathscr{B}^k$ with $0 < \lambda_k(B) < \lambda_k(A)$. This forces $\overline{\lambda}_k$ to be nonatomic as well.*

PROOF. Take any $A \in \mathscr{L}^k$ with $\overline{\lambda}_k(A) > 0$. Since $\mathscr{L}^k = \overline{\mathscr{B}^k}$, the completion of $\mathscr{B}^k$, there exists $C, M \in \mathscr{B}^k$, where M is a subset of λ_k-null set, such that $A = C \cup B$. Therefore, $\lambda_k(B) = \overline{\lambda}_k(A) > 0$. Since λ_k is nonatomic, there exists $B \in \mathscr{B}^k$ with $0 < \lambda_k(B) < \lambda_k(C) = \overline{\lambda}_k(A)$, that is, $\overline{\lambda}_k$ is nonatomic. □

► EXERCISE 4.5 (4.1.5). *Let $k \geqslant 2$. There exists an uncountable set $U \in \mathscr{B}^k$ with $\lambda_k(U) = 0$.*

PROOF. Let $U = \{x\} \times \mathbb{R}$, where $x \in \mathbb{R}$. Then U is uncountable, and $\lambda(U) = 0$. □

► EXERCISE 4.6 (4.1.6). $|\mathscr{L}^k| = 2^{\mathfrak{c}}$ *and* $|\mathscr{B}^k| = \mathfrak{c}$ *for each* $k \in \mathbb{N}$.

PROOF. Since $|\mathbb{R}^k| = \mathfrak{c}$ and $\mathscr{L}^k \subseteq 2^{\mathbb{R}^k}$, we first have $|\mathscr{L}^k| \leqslant 2^{\mathfrak{c}}$; on the other hand, there exists an uncountable set $U \subseteq \mathbb{R}^k$ such that $\lambda_k(U) = 0$ (if $k = 1$, consider the Cantor set; if $k \geqslant 2$, consider the set U in the preceding exercise), so $|U| = \mathfrak{c}$ by the Continuum Hypothesis. Since $\mathscr{L}^k$ is complete, we have $|\mathscr{L}^k| \geqslant |2^U| = 2^{\mathfrak{c}}$. It follows from the Cantor-Bernstein Theorem that $|\mathscr{L}^k| = 2^{\mathfrak{c}}$. □

► EXERCISE 4.7 (4.1.7). *Assume that $\mathscr{L} = 2^{\mathbb{R}}$; in particular every one of the $2^{\mathfrak{c}}$ subsets of $[0,1]$ is a Lebesgue set. Let $B = \{\overline{\lambda}(A) \colon A \subseteq [0,1], \overline{\lambda}(A) \notin A\}$. Consideration of the set B (which is in $\mathscr{L}$ be assumption) leads to contradiction.*

PROOF. $B \subseteq [0,1]$ so $\overline{\lambda}(B)$ exists. We now have a contradiction: $\overline{\lambda}(B) \in B$ iff $\overline{\lambda}(B) \notin B$. □

► EXERCISE 4.8 (4.1.8). *Let $k \geqslant 2$. Every line is in $\mathscr{B}^k$, has λ_k-measure zero, and hence has $\overline{\lambda}_k$-measure.*

PROOF. Let ℓ be the line. Take two points $\boldsymbol{a}$ and $\boldsymbol{b}$ in ℓ, and denote $[\boldsymbol{a}, \boldsymbol{b}]$ as the segment on ℓ. Then $[\boldsymbol{a}, \boldsymbol{b}]$ is closed in $\mathbb{R}^k$ and so is in $\mathscr{B}^k$. Enumerate the points with rational coordinates on $[\boldsymbol{a}, \boldsymbol{b}]$; then it is easy to see that $\overline{\lambda}_k([\boldsymbol{a}, \boldsymbol{b}]) = 0$.

Write ℓ as an increasing limit of line segments containing $[\boldsymbol{a}, \boldsymbol{b}]$. Then we get the result. □

► EXERCISE 4.9 (4.1.9). *Let $\mathscr{B}_{(0,1]} = \sigma(\{(a,b] \colon (a,b] \subseteq (0,1]\})$.*

a. $\mathscr{B}_{(0,1]} = \{B \subseteq (0,1] \colon B \in \mathscr{B}\}$ *and* $\overline{\mathscr{B}_{(0,1]}} = \{B \subseteq (0,1] \colon B \in \mathscr{L}\}$.

b. *Construct Lebesgue measure on both $\mathscr{B}_{(0,1]}$ and $\overline{\mathscr{B}_{(0,1]}}$. Call these measures $\lambda_{(0,1]}$ and $\overline{\lambda}_{(0,1]}$, and denote $\overline{\mathscr{B}_{(0,1]}}$ by $\mathscr{L}_{(0,1]}$.*

c. *$\lambda_{(0,1]}$ as constructed is the measure restriction of λ from $\mathscr{B}$ to $\mathscr{B}_{(0,1]}$ and $\overline{\lambda}_{(0,1]}$ as constructed is the measure restriction of $\overline{\lambda}$ from $\mathscr{L}$ to $\mathscr{L}_{(0,1]}$.*

PROOF. See, for example, Resnick (1999, Theorem 1.8.1). □

4.2 Lebesgue Sets

No exercise.

4.3 Translation Invariance of Lebesgue Measure

► Exercise 4.10 (4.3.2). *Let $A \in \mathscr{L}$ be such that $\overline{\lambda}(A) > 0$, and let $c \in [0,1)$. There exists an open interval U such that $\overline{\lambda}(A \cap U) \geqslant c\lambda(U)$.*[1]

Proof. It follows from the Approximation Theorem for Lebesgue measure that

$$\overline{\lambda}(A) = \inf\{\lambda(G) \colon G \text{ open}, A \subseteq G\}.$$

Then for any $\varepsilon > 0$, there exists an open set G containing A such that $\lambda(G) < \overline{\lambda}(A) + \varepsilon\lambda(G)$, i.e., $(1-\varepsilon)\lambda(G) < \overline{\lambda}(A)$. Thus, for an arbitrary $c \in [0,1)$, there exists an open set G containing A such that

$$c\lambda(G) \leqslant \overline{\lambda}(A).$$

Write G as an countable disjoint unions of open intervals: $G = \bigcup G_n$. Then $\overline{\lambda}(A) = \overline{\lambda}(A \cap G)$ since $A \subseteq G$. We thus obtain

$$\begin{aligned} c\lambda(G) = c\lambda\left(\bigcup_{n=1}^{\infty} G_n\right) = \sum_{n=1}^{\infty} c\lambda(G_n) \leqslant \overline{\lambda}(A) &= \overline{\lambda}\left(\bigcup_{n=1}^{\infty}(A \cap G_n)\right) \\ &= \sum_{n=1}^{\infty} \overline{\lambda}(A \cap G_n). \end{aligned}$$

Hence, for some $N \in \mathbb{N}$, we must have $c\lambda(G_N) \leqslant \overline{\lambda}(A \cap G_N)$. Let $U = G_N$ and we are done. □

► Exercise 4.11 (4.3.3). *Let $A \in \mathscr{L}$ contain an open interval. Then there exists $a > 0$ such that $(-a,a)$ is contained in $D(A) = \{x - y \colon x, y \in A\}$.*

Proof. Let $(b,c) \subseteq A$; then $(b-c, c-b) \subseteq D(A)$. Let $a = c - b$ and so $(-a,a) \subseteq D(A)$. □

► Exercise 4.12 (4.3.4). *Let $A \in \mathscr{L}$ be such that $\overline{\lambda}(A) > 0$. Then there exists $a > 0$ such that $(-a,a)$ is contained in $D(A) = \{x - y \colon x, y \in A\}$.*

Proof. It follows from Exercise 4.10 that there exists an open interval $U \subseteq \mathbb{R}$ such that

$$\overline{\lambda}(A \cap U) \geqslant 3\lambda(U)/4.$$

We next show that a can be taken as $\lambda(U)/2$.

[1] Exercise 4.10—4.16 are from Halmos (1974).

(i) For an arbitrary $x \in (-\lambda(U)/2, \lambda(U)/2)$, the set $U \cup (U \oplus x)$ is an open interval containing $(A \cap U) \cup ((A \cap U) \oplus x)$, and

$$\lambda(U \cup (U \oplus x)) < \lambda(U) + \lambda(U)/2 = \frac{3}{2}\lambda(U).$$

(ii) $(A \cap U) \cup ((A \cap U) \oplus x)$ is an interval. Suppose that $(A \cap U) \cap ((A \cap U) \oplus x) = \varnothing$; then

$$\begin{aligned}\overline{\lambda}((A \cap U) \cup ((A \cap U) \oplus x)) &= \overline{\lambda}(A \cap U) + \overline{\lambda}((A \cap U) \oplus x)\\ &= 2\overline{\lambda}(A \cap U)\\ &\geqslant \frac{3}{2}\lambda(U),\end{aligned}$$

which contradicts the fact that $\overline{\lambda}((A \cap U) \cup ((A \cap U) \oplus x)) \leqslant \lambda(U \cup (U \oplus x)) < 3\lambda(U)/2$.

(iii) Thus, for every $x \in (-\lambda(U)/2, \lambda(U)/2)$, there exists $y \in (A \cap U) \cap ((A \cap U) \oplus x)$; that is, there exists $y, z \in A$ such that $y = z + x$. But then $x = y - z$ and so $x \in D(A)$. Therefore, if we let $a = \lambda(U)/2$, then $(-a, a) \subseteq D(A)$. □

► EXERCISE 4.13 (4.3.5). *Let A be a dense subset of $\mathbb{R}$. Then $cA = \{ca : a \in A\}$ is dense for any $c \neq 0$.*

PROOF. Take an arbitrary point $x \in \mathbb{R}$ and an arbitrary open interval $(x-\varepsilon, x+\varepsilon)$. Now consider $(x - \varepsilon/c, x + \varepsilon/c)$. Since A is dense, there exists $a \in A$ such that $a \in (x - \varepsilon/c, x + \varepsilon/c)$. Thus, $ca \in (x - \varepsilon, x + \varepsilon)$ and $ca \in cA$, i.e., cA is dense in $\mathbb{R}$. □

► EXERCISE 4.14 (4.3.6). *Let ξ be an irrational number.*

a. *Let $A = \{n + m\xi : n, m \in \mathbb{Z}\}$. Then A is a dense subset of $\mathbb{R}$.*

b. *Let $B = \{n + m\xi : n, m \in \mathbb{Z}, n$ even$\}$. Then B is a sense subset of $\mathbb{R}$.*

c. *Let $C = \{n + m\xi : n, m \in \mathbb{Z}, n$ odd$\}$. Then C is a sense subset of $\mathbb{R}$.*

PROOF. **(a)** For every positive integer i there exists a unique integer n_i (which may be positive, negative, or zero) such that $0 \leqslant n_i + i\xi < 1$; we write $x_i = n_i + i\xi$. If U is any open interval, then there is a positive integer k such that $\mu(U) > 1/k$. Among the $k+1$ numbers, $x_1, \ldots, x_{k+1}$, in the unit interval, there must be at least two, say x_i and x_j, such that $|x_i - x_j| < 1/k$. It follows that some integral multiple of $x_i - x_j$, i.e. some element of A, belongs to the interval U, and this concludes the proof of the assertion concerning A.

(b) If ξ is irrational, then $\xi/2$ is also irrational. Then $D = \{n + (m/2)\xi : n, m \in \mathbb{Z}\}$ is dense by (a); then $2D = \{n + m\xi : n, m \in \mathbb{Z}\}$ is dense by Exercise 4.13.

(c) Notice that $C = B \oplus 1$, and translates of dense sets are obviously dense. □

► EXERCISE 4.15 (4.3.7). *For $x, y \in \mathbb{R}$ write $x \sim y$ iff $x - y \in A = \{n + m\xi : n, m \in \mathbb{Z}\}$, where ξ is a fixed irrational number as in the previous exercise. Then $\sim$ is an equivalence relation, and hence $\mathbb{R}$ may be partitioned into disjoint equivalence classes.*

PROOF. We first show that $\sim$ is reflexive. For every $x \in \mathbb{R}$, we have $x - x = 0 = 0 + 0\xi \in A$, i.e. $x \sim x$. We next show that $\sim$ is symmetric. If $x \sim y$, then $x - y = n + m\xi$ and so $y - x = -n - m\xi \in A$, i.e. $y \sim x$. Finally, we verify that $\sim$ is transitive. Let $x \sim y \sim z$. Then $x - y = n + m\xi$ and $y - z = p + q\xi$, where $n, m, p, q \in \mathbb{Z}$. Thus, $x - z = (n + p) + (m + q)\xi \in A$, i.e. $x \sim z$. □

► EXERCISE 4.16 (4.3.8). *We now invoke (AC) to form a set E_0 consisting of exactly one element from each of the equivalence classes in the previous exercise. We will now show that $E_0 \notin \mathcal{L}$.*

a. *There exist Borel subsets of E_0.*

b. *Let $F \subseteq E_0$ be a Borel set. Then $D(F)$ cannot contain any nonzero elements of A, where A is the set in Exercise 4.14.*

c. *By (b), there cannot exist an open interval containing the origin that is contained in $D(F)$, hence $\lambda(F) = 0$.*

d. *From (c), we have $\lambda_*(E_0) = 0$.*

e. *If a_1 and a_2 are distinct elements of $A = \{n + m\xi : m, n \in \mathbb{Z}\}$, then $E_0 \oplus a_1$ and $E_0 \oplus a_2$ are disjoint.*

f. *$\mathbb{R} = \bigcup\{E_0 \oplus a : a \in A\}$, the countable union being disjoint.*

g. *If $E_0 \in \mathcal{L}$, then $\overline{\lambda}(E_0 \oplus a) = 0$ for each $a \in A$, hence $\overline{\lambda}(\mathbb{R}) = 0$. Therefore, since the assumption that $E_0 \in \mathcal{L}$ leads to an absurdity, it must be the case that $E_0 \notin \mathcal{L}$, and hence there exists a subset of $\mathbb{R}$ that fails to be a Lebesgue set.*

PROOF. **(a)** Every singleton is a Borel set.

(b) If there exists $x \neq 0$ and $x \in D(F) \cap A$, there there exists $y, z \in F$ such that $y - z \in A$ and $y \neq z$. But then $y \sim z$ and $y \neq z$, which contradicts the construction of E_0.

(c) If $\lambda(F) > 0$, then there exists $a > 0$ such that $(-a, a) \subseteq D(F)$ by Exercise 4.12. Then there exists $x \in A$ such that $x \in (a/2, a)$ since A is dense, which contradicts (b). Thus, $\lambda(F) = 0$.

(d) $\lambda_*(F) = 0$ follows from the definition immediately.

(e) If $z = x_1 + a_1 = x_2 + a_2$, where $x_1, x_2 \in E_0$, then $x_1 - x_2 = a_2 - a_1 \in A$. Then $x_1 \sim x_2$ and $x_1 \neq x_2$ (since $a_1 \neq a_2$). A contradiction.

(f) It suffices to show that $\mathbb{R} \subseteq \bigcup\{E_0 \oplus a : a \in A\}$. Take an arbitrary $r \in \mathbb{R}$. Since $\sim$ is an equivalence relation on $\mathbb{R}$, there is an equivalence class $[x]_\sim$ containing r. In particular, $r \sim x$, i.e. $r - x = n + m\xi$ for some $n, m \in \mathbb{Z}$. Hence, $r = x + n + m\xi \in E_0 \oplus (n + m\xi)$. The union is countable since A is countable.

(g) If $E_0 \in \mathcal{L}$, then $E_0 \oplus a \in \mathcal{L}$ for all $a \in A$, and

$$\overline{\lambda}(E_0 \oplus a) = \overline{\lambda}(E_0) = \lambda_*(E_0) = 0.$$

But then

$$\overline{\lambda}(\mathbb{R}) = \overline{\lambda}\left(\bigcup_{a \in A}(E_0 \oplus a)\right) = \sum_{a \in A} \overline{\lambda}(E_0 \oplus a) = 0.$$

A contradiction. $\square$

5

MEASURABLE FUNCTIONS

5.1 Measurability

► Exercise 5.1 (5.1.1). *Let $f : \Omega \to \overline{\mathbb{R}}$ be $\mathcal{F}/\mathcal{B}^*$-measurable. Let $y \in \overline{\mathbb{R}}$, and let $h : \Omega \to \overline{\mathbb{R}}$ be such that*

$$h(\omega) = \begin{cases} \sqrt{f(\omega)} & \text{if } f(\omega) \geqslant 0 \\ y & \text{if } f(\omega) < 0. \end{cases}$$

Then h is $\mathcal{F}/\mathcal{B}^*$-measurable.

Proof. Define $\varphi : \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ by letting

$$\varphi(x) = \begin{cases} \sqrt{x} & \text{if } x \geqslant 0 \\ y & \text{if } x < 0. \end{cases}$$

We first show that φ is $\mathcal{B}^*/\mathcal{B}^*$-measurable by demonstrating that $\varphi^{-1}(t, \infty] \in \mathcal{B}^*$ for each $t \in \mathbb{R}$. If $y < 0$, then

$$\varphi^{-1}(t, \infty] = \begin{cases} \overline{\mathbb{R}} & \text{if } t < y \\ [0, \infty] & \text{if } t \in [y, 0] \\ (t^2, \infty] & \text{if } t > 0. \end{cases}$$

If $y \geqslant 0$, then

$$\varphi^{-1}(t, \infty] = \begin{cases} \overline{\mathbb{R}} & \text{if } t < 0 \\ [-\infty, 0) \cup (t^2, \infty] & \text{if } t \in [0, y) \\ (t^2, \infty] & \text{if } t \geqslant y. \end{cases}$$

Therefore, φ is $\mathcal{B}^*/\mathcal{B}^*$-measurable, and so $h = \varphi \circ f$ is $\mathcal{F}/\mathcal{B}^*$-measurable. □

► Exercise 5.2 (5.1.2). *There exists a continuous function $f : \mathbb{R} \to \mathbb{R}$ and a subset $A \subseteq \mathbb{R}$ such that $A \in \mathcal{L}$ but $f^{-1}(A) \notin \mathcal{L}$.*

PROOF. Do according with the hints. □

► EXERCISE 5.3 (5.1.5). *Suppose that $f : \Omega \to \overline{\mathbb{R}}$ is $\mathcal{F}/\mathcal{B}^*$-measurable.*

a. *If $\mathcal{F} = 2^{\Omega}$, f can be any function from Ω into $\overline{\mathbb{R}}$.*

b. *If $\mathcal{F} = \{\varnothing, \Omega\}$, then f must be constant.*

c. *If $\mathcal{F} = \sigma(\{A_1, \ldots, A_n\})$, where $A_1, \ldots, A_n$ are disjoint subsets of Ω such that $\Omega = \bigcup_{i=1}^{n} A_i$, then f must have the form $f = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}$, where $c_1, \ldots, c_n \in \overline{\mathbb{R}}$.*

PROOF. **(a)** is trivial. For (b), if f takes two different values, say, y_1 and y_2 and $y_1 < y_2$, then $f^{-1}[y_2, \infty] \notin \{\varnothing, \Omega\}$; that is, f is not $\mathcal{F}/\mathcal{B}^*$-measurable. For (c), note that $f^{-1}(c_i) \in \sigma(\{A_1, \ldots, A_n\})$. □

► EXERCISE 5.4 (5.1.6). *If $f : \Omega \to \overline{\mathbb{R}}$ is such that $f^{-1}(\{x\}) \in \mathcal{F}$ for every $x \in \overline{\mathbb{R}}$, then f is not necessarily $\mathcal{F}/\mathcal{B}^*$-measurable.*

PROOF. Let $\Omega = \mathbb{R}$ and $\mathcal{F} = \mathcal{L}$. Let $A \notin \mathcal{L}$, and let

$$f(x) = \begin{cases} x & \text{if } x \in A, \\ -x & \text{if } x \notin A. \end{cases}$$

Then $f^{-1}(\{x\}) \in \mathcal{L}$ for any $x \in \mathbb{R}$, but f fails to be $\mathcal{L}/\mathcal{B}^*$-measurable. □

► EXERCISE 5.5 (5.1.7). *If $A \subseteq \mathbb{R}$ is any type of interval and $f : A \to \overline{\mathbb{R}}$ is monotone, then f is both Borel and Lebesgue measurable.*

PROOF. Without loss of generality, we suppose that f is increasing in the sense that $x_1 < x_2$ implies that $f(x_1) \leqslant f(x_2)$. Then for any $r \in \overline{\mathbb{R}}$, $f^{-1}\left([r, +\infty]\right) = [x, +\infty)$, where $x = \inf\{x \in \mathbb{R} : f(x) \geqslant r\}$. Hence, f^{-1} is Boreal, and so is Lebesgue measurable. □

► EXERCISE 5.6 (5.1.11). *Let $f : \Omega \to \overline{\mathbb{R}}$, and suppose that $\Omega = \bigcup_{n=1}^{\infty} A_n$, where $A_1, A_2, \ldots$ are disjoint $\mathcal{F}$-sets [$\mathcal{F}$ is a σ-field on Ω]. Let $\mathcal{F}_n = \{A \in \mathcal{F} : A \subset A_n\}$ for each $n \in \mathbb{N}$. Then $\mathcal{F}_n$ is a σ-field for each $n \in \mathbb{N}$. Let f_n denote the restriction of f from Ω to A_n, $n \in \mathbb{N}$. Then f is $\mathcal{F}/\mathcal{B}^*$-measurable iff f_n is $\mathcal{F}_n/\mathcal{B}^*$-measurable for each $n \in \mathbb{N}$.*

PROOF. Assume that each f_n is $\mathcal{F}_n/\mathcal{B}^*$-measurable. Let $B \in \mathcal{B}^*$. Then

$$f^{-1}(B) = \bigcup_{n=1}^{\infty} f_n^{-1}(B) \in \mathcal{F}$$

since each $f_n^{-1}(B) \in \mathcal{F}_n \subset \mathcal{F}$. Now assume that f is $\mathcal{F}/\mathcal{B}^*$-measurable. Take any f_n and $B \in \mathcal{B}^*$. Then

$$f_n^{-1}(B) = A_n \cap f^{-1}(B) \in \mathcal{F}.$$ □

► EXERCISE 5.7 (5.1.12). *Show that the function* $\varphi\colon \overline{\mathbb{R}} \to \overline{\mathbb{R}}$ *given in Example 4 is* $\mathcal{B}^*/\mathcal{B}^*$*-measurable by suitably appealing to (MF6).*

PROOF. Let $A_1 = [-\infty, 0)$, $A_2 = \{0\}$, and $A_3 = (0, +\infty]$. Let $f_i = \varphi \restriction A_i$ for $i = 1, 2, 3$. Since both f_1 and f_3 are continuous, they are $\mathcal{B}^*/\mathcal{B}^*$-measurable; since f_2 is constant, it is $\mathcal{B}^*/\mathcal{B}^*$-measurable. By (MF6), φ is $\mathcal{B}^*/\mathcal{B}^*$-measurable. □

► EXERCISE 5.8 (5.1.13). *The minimal* σ*-field* $\mathcal{F}$ *on* Ω *such that* $f : \Omega \to \overline{\mathbb{R}}$ *is* $\mathcal{F}/\mathcal{B}^*$*-measurable is* $f^{-1}\left(\mathcal{B}^*\right)$*.*

PROOF. It suffices to show that $f^{-1}\left(\mathcal{B}^*\right)$ is a σ-field on Ω since by $\mathcal{F}/\mathcal{B}^*$-measurability of f, any σ-filed $\mathcal{F}$ includes $f^{-1}\left(\mathcal{B}^*\right)$. First, $\Omega \in f^{-1}\left(\mathcal{B}^*\right)$ since $f^{-1}\left(\overline{\mathbb{R}}\right) = \Omega$. If $A \in f^{-1}\left(\mathcal{B}^*\right)$, there exists $B \in \mathcal{B}^*$ such that $f^{-1}(B) = A$, then $f^{-1}\left(\overline{\mathbb{R}} \smallsetminus B\right) = \Omega \smallsetminus A$ implies that $f^{-1}\left(\mathcal{B}^*\right)$ is closed under complements. To see that $f^{-1}\left(\mathcal{B}^*\right)$ is closed under countable union, let $\{A_n\}_{n=1}^{\infty} \subseteq f^{-1}\left(\mathcal{B}^*\right)$. So there exists $B_n \in \mathcal{B}^*$ for each $n \in \mathbb{N}$ with $f^{-1}(B_n) = A_n$. Therefore,

$$f^{-1}\left(\bigcup_{n=1}^{\infty} B_n\right) = \bigcup_{n=1}^{\infty} f^{-1}(B_n) = \bigcup_{n=1}^{\infty} A_n$$

implies $f^{-1}\left(\mathcal{B}^*\right)$ is closed under countable unions. □

► EXERCISE 5.9 (5.1.14). *The word* continuous *in (MF4) may be replaced by either of* lower semicontinuous *and* upper semicontinuous*.*

PROOF. For a detailed discussion of semicontinuous functions, see Ash (2009, Section 8.4). Let $f : A \to \overline{\mathbb{R}}$ be low semicontinuous (LSC), then $f^{-1}(t, \infty]$ is open for any $t \in \overline{\mathbb{R}}$. Therefore, $f^{-1}(t, \infty] \in \mathcal{F}$ and so f is $\mathcal{F}/\mathcal{B}^*$-measurable. Now let f be upper semicontinuous (USC), then $-f$ is LSC and so is $\mathcal{F}/\mathcal{B}^*$-measurable; then $f = -\left(-f\right)$ is $\mathcal{F}/\mathcal{B}^*$-measurable. □

5.2 COMBINING MEASURABLE FUNCTIONS

► EXERCISE 5.10 (5.2.1). *If* $f : \Omega \to \overline{\mathbb{R}}$ *is* $\mathcal{F}/\mathcal{B}^*$*-measurable, then* $|f|$ *is* $\mathcal{F}/\mathcal{B}^*$*-measurable. However, if* $|f|$ *is* $\mathcal{F}/\mathcal{B}^*$*-measurable, then* f *is not necessarily* $\mathcal{F}/\mathcal{B}^*$*-measurable.*

PROOF. Since $|f| = f^+ - f^-$, and f is $\mathcal{F}/\mathcal{B}^*$-measurable if and only if f^+ and f^- are measurable, we know that $|f|$ is $\mathcal{F}/\mathcal{B}^*$-measurable. To see that the converse is not true take $A \notin \mathcal{F}$ and let

$$f(\omega) = \mathbb{1}_A(\omega) - \mathbb{1}_{A^c}(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ -1 & \text{if } \omega \in A^c. \end{cases}$$

It is not $\mathcal{F}/\mathcal{B}^*$-measurable since $f^{-1}(0,+\infty] = A \notin \mathcal{F}$. But $|f| = 1$ is $\mathcal{F}/\mathcal{B}^*$-measurable. □

► EXERCISE 5.11 (5.2.2). *Let $n \in \mathbb{N}$, and let $f_1, \dots, f_n$ denote $\mathcal{F}/\mathcal{B}^*$-measurable functions with common domain $A \in \mathcal{F}$.*

a. *Both* $\max\{f_1, f_2\}$ *and* $\min\{f_1, f_2\}$ *are $\mathcal{F}/\mathcal{B}^*$-measurable functions.*

b. *Both* $\max\{f_1, \dots, f_n\}$ *and* $\min\{f_1, \dots, f_n\}$ *are $\mathcal{F}/\mathcal{B}^*$-measurable functions.*

PROOF. **(a)** Let $g = \max\{f_1, f_2\}$. For an arbitrary $x \in \mathbb{R}$, we have

$$\big\{\omega \in A : \max\{f_1, f_2\}(\omega) < x\big\} = \{\omega \in A : f_1(\omega) < x\} \cap \{\omega \in A : f_2(\omega) < x\} \in \mathcal{F},$$

and

$$\big\{\omega \in A : \min\{f_1, f_2\} > x\big\} = \{\omega \in A \colon f_1(\omega) > x\} \cap \{\omega \in A : f_2(\omega) > x\} \in \mathcal{F}.$$

(b) We do the max case. Let $g_n = \max\{f_1, \dots, f_n\}$. The claim holds for $n = 1$ and 2 by (a). Assume that it is true for $n \in \mathbb{N}$. Then for $n + 1$, we have

$$\{\omega \in A : g_{n+1}(\omega) < x\} = \left(\bigcap_{i=1}^{n} \{\omega \in A : f_i(\omega) < x\}\right) \cap \{\omega \in A : f_{n+1}(\omega) < x\} \in \mathcal{F}$$

by the induction hypothesis. □

► EXERCISE 5.12 (5.2.3). *Let $(\Omega, \mathcal{F}, \mu)$ denote a measure space, and let $f : \Omega \to \overline{\mathbb{R}}$ denote a $\mathcal{F}/\mathcal{B}^*$-measurable mapping. Let $\nu \colon \mathcal{B}^* \to \overline{\mathcal{R}}$ be such that $\nu(B) = \mu(f^{-1}(B))$ for every $B \in \mathcal{B}^*$. That is, $\nu = \mu \circ f^{-1}$. Then $(\overline{\mathbb{R}}, \mathcal{B}^*, \nu)$ is a measure space. Furthermore, even if μ is σ-finite, ν is not necessarily σ-finite.*

PROOF. It is clear that ν is well defined since f is $\mathcal{F}/\mathcal{B}^*$-measurable. To see ν is a measure on $\mathcal{B}^*$, note that (i) $\nu(B) = \mu(f^{-1}(B)) \geqslant 0$ for every $B \in \mathcal{B}^*$, (ii) $\nu(\varnothing) = \mu(\varnothing) = 0$, and (iii) For a disjoint sequence $\{B_n\} \subseteq \mathcal{B}^*$, we know that the sequence $\{f^{-1}(B_n)\} \subseteq \mathcal{F}$ is disjoint; then the countable additivity follows. □

► EXERCISE 5.13 (5.2.4). *This exercise concerns itself with (MF9).*

a. *Prove part (b) of (MF9) by suitably adapting the proof of (a).*

b. *Prove part (b) of (MF9) by using (a) of (MF9) and (MF7).*

c. *Show that $\{\omega \in A : f(\omega) + g(\omega) < x\} = \bigcup_{r_1, r_2 \in \mathbb{Q}; r_1 + r_2 < x} \{\omega \in A : f(\omega) < r_1\} \cap \{\omega \in A : g(\omega) < r_2\}$.*

d. *Repeat part (c) for $f - g$ by proving an analogous identity.*

e. *Let $y \in \overline{\mathbb{R}}$, $n \geqslant 2$, and for $i = 1, \dots, n$, let $f_i \colon A \to \overline{\mathbb{R}}$ denote a $\mathcal{F}/\mathcal{B}^*$-measurable function. Let $h \colon A \to \overline{\mathbb{R}}$ be defined for all $\omega \in A$ by the rule*

$$h(\omega) = \begin{cases} f_1(\omega) + \cdots + f_n(\omega) & \textit{if } f_1(\omega) + \cdots + f_n(\omega) \textit{ is defined} \\ y & \textit{if } f_1(\omega) + \cdots + f_n(\omega) \textit{ is undefined.} \end{cases}$$

Show that h is $\mathcal{F}/\mathcal{B}^*$-measurable.

PROOF. **(a)** Let $y \in \overline{\mathbb{R}}$, and let $x \in \mathbb{R}$. Define

$$A_y = \begin{cases} \left[f^{-1}(-\infty) \cap g^{-1}(-\infty)\right] \cup \left[f^{-1}(\infty) \cap g^{-1}(\infty)\right] & \text{if } y > x \\ \varnothing & \text{if } y \leqslant x. \end{cases}$$

Observe that (i) $A_y \subseteq A$, and (ii) the assumption of $\mathcal{F}/\mathcal{B}^*$-measurability for f and g forces $A_y \in \mathcal{F}$ (and hence $A \smallsetminus A_y \in \mathcal{F}$). Next,

$$\begin{aligned} h^{-1}(x, \infty] &= \{\omega \in A : h(\omega) > x\} \\ &= \left\{\omega \in A \smallsetminus A_y : h(\omega) > x\right\} \cup \left\{\omega \in A_y : h(\omega) > x\right\} \\ &= \left\{\omega \in A \smallsetminus A_y : f(\omega) - g(\omega) > x\right\} \cup A_y \\ &= \left\{\omega \in A \smallsetminus A_y : f(\omega) > x + g(\omega)\right\} \cup A_y \\ &\in \mathcal{F}. \end{aligned}$$

(b) Note that $-g$ is $\mathcal{F}/\mathcal{B}^*$-measurable since g is. Then $f-g$ is $\mathcal{F}/\mathcal{B}^*$-measurable. It follows form (MF7)(c) that h is $\mathcal{F}/\mathcal{B}^*$-measurable.

(c) Let $L = \{\omega \in A : f(\omega) + g(\omega) < x\}$, and $R = \bigcup_{r_1, r_2 \in \mathbb{Q}; r_1 + r_2 < x} \{\omega \in A : f(\omega) < r_1\} \cap \{\omega \in A : g(\omega) < r_2\}$. If $\omega \in L$, then $f(\omega) + g(\omega) < x$. Take $\varepsilon \in \mathbb{R}_{++}$ such that

$$[f(\omega) + \varepsilon] + [g(\omega) + \varepsilon] = x.$$

Such an ε exists since $f(\omega), g(\omega), x \in \mathbb{R}$. Then there exists $r_1 \in \mathbb{Q}$ such that $f(\omega) < r_1 < f(\omega) + \varepsilon$ since $\mathbb{Q}$ is dense in $\mathbb{R}$; similarly, there exists $r_2 \in \mathbb{Q}$ such that $g(\omega) < r_2 < g(\omega) + \varepsilon$. Thus,

$$r_1 + r_2 < f(\omega) + \varepsilon + g(\omega) + \varepsilon = x;$$

that is, $\omega \in R$. The other direction is evident.

(d) For $f - g$, we have

$$\{\omega \in A : f(\omega) - g(\omega) < x\} = \bigcup_{\substack{r_1, r_2 \in \mathbb{Q} \\ r_1 - r_2 < x}} \{\omega \in A : f(\omega) < r_1\} \cap \{\omega \in A : -g(\omega) < -r_2\}.$$

Let L denote the left hand side of the above display, and let R denote the right hand side. If $\omega \in L$, then $f(\omega) - g(\omega) < x$. Pice $\varepsilon \in \mathbb{R}_{++}$ such that $[f(\omega) + \varepsilon] - [g(\omega) - \varepsilon] = x$. Pick $r_1, r_2 \in \mathbb{Q}$ such that $f(\omega) < r_1 < f(\omega) + \varepsilon$ and $-g(\omega) < -r_2 < -g(\omega) + \varepsilon$. Then $r_1 - r_2 < x$, i.e., $\omega \in R$. The other direction is evident.

(e) The claim holds for $n = 2$. Let us assume that it holds for $n \in \mathbb{N}$. We now consider the $n + 1$ case. Define

$$A_y = \begin{cases} \{\omega \in A : f_1(\omega) + \cdots + f_{n+1}(\omega) \text{ is undefined}\} & \text{if } y > x \\ \varnothing & \text{if } y \leqslant x. \end{cases}$$

It is clear that $A_y \in \mathcal{F}$. Next,

$$\begin{aligned} h^{-1}(x, \infty] &= \{\omega \in A : h(\omega) > x\} \\ &= \{\omega \in A \smallsetminus A_y : f_1(\omega) + \cdots + f_{n+1}(\omega) > x\} \cup A_y. \end{aligned}$$

It suffices to show that $\{\omega \in A \smallsetminus A_y : f_1(\omega) + \cdots + f_{n+1}(\omega) > x\} \in \mathcal{F}$. Notice that if $f_1(\omega) + \cdots + f_{n+1}(\omega)$ is defined, then $f_1(\omega) + \cdots + f_n(\omega)$ is defined, too. Thus, $f_1 + \cdots + f_n$ is $\mathcal{F}/\mathcal{B}^*$-measurable on A_y. It follows from (MF8) that $h^{-1}(x, \infty] \in \mathcal{F}$. □

► EXERCISE 5.14 (5.2.5). *This exercise concerns itself with (MF10).*

a. *Directly prove (a) of (MF10) assuming that f and g are real valued instead of extended real valued.*

b. *Prove (b) of (MF10) assuming that f and g are real valued and that g is nonzero on A.*

c. *Using the previous part, now prove (b) of (MF10) in full generality.*

PROOF. **(a)** For every $x \in \mathbb{R}$, we have

$$(fg)^{-1}(x, \infty] = \left\{\omega \in A : \left[f(\omega) + g(\omega)\right]^2 - \left[f(\omega) - g(\omega)\right]^2 > 4x\right\} \in \mathcal{F};$$

that is, fg is $\mathcal{F}/\mathcal{B}^*$-measurable.

(b) Write $f/g = f \times (1/g)$. Then $1/g$ is real valued and $\mathcal{F}/\mathcal{B}^*$-measurable. It follows from (a) that f/g is $\mathcal{F}/\mathcal{B}^*$-measurable.

(c) Take an arbitrary $x \in \mathbb{R}$. Define

$$A_y = \begin{cases} \{\omega \in A : f(\omega)/g(\omega) \text{ is undefined}\} & \text{if } y > x \\ \varnothing & \text{if } y \leqslant x. \end{cases}$$

Notice that

$$\{\omega \in A : f(\omega)/g(\omega) \text{ is undefined}\} = [f^{-1}(\pm\infty) \cap g^{-1}(\pm\infty)] \cup g^{-1}(0);$$

thus, $A_y \in \mathcal{F}$. Next,

$$\begin{aligned} h^{-1}(x, \infty] &= \{\omega \in A : f(\omega)/g(\omega) > x\} \\ &= \{\omega \in A \smallsetminus A_y : f(\omega)/g(\omega) > x\} \cup A_y. \end{aligned}$$

So it suffices to show that $\{\omega \in A \smallsetminus A_y : f(\omega)/g(\omega) > x\} \in \mathcal{F}$. It can be proved case by case. □

5.3 Sequences of Measurable Functions

► EXERCISE 5.15 (5.3.1). *Prove (a)-(e) of (MF13).*

PROOF. **(a)** Pick an arbitrary $\omega \in A$. If $f(\omega) \geqslant 0$, we have $f^+(\omega) = f(\omega)$ and $f^-(\omega) = 0$. Then

$$f^+(\omega) + f^-(\omega) = f(\omega).$$

Next, if $f(\omega) < 0$, we get $f^+(\omega) = 0$ and $f^-(\omega) = -f(\omega)$. Then

$$f^+(\omega) + f^-(\omega) = -f(\omega).$$

Hence, $|f| = f^+ + f^-$.

Now if $f = f^+$ and $f^- = 0$, then $|f| = f^+ + f^- = f$, i.e., $f \geqslant 0$. The other claim is similar.

(b) If $c \geqslant 0$, then $cf(\omega) \geqslant 0$ iff $f(\omega) \geqslant 0$. Hence,

$$(cf)^+(\omega) = \begin{cases} cf(\omega) & \text{if } cf(\omega) \geqslant 0 \\ 0 & \text{if } cf(\omega) < 0 \end{cases} = \begin{cases} cf(\omega) & \text{if } f(\omega) \geqslant 0 \\ 0 & \text{if } f(\omega) < 0 \end{cases} = cf^+(\omega).$$

If $c < 0$, then $cf(\omega) \geqslant 0$ iff $f(\omega) \leqslant 0$. Thus,

$$(cf)^+(\omega) = \begin{cases} cf(\omega) & \text{if } f(\omega) \leqslant 0 \\ 0 & \text{if } f(\omega) > 0 \end{cases} = -cf^-(\omega).$$

(c) Suppose that $f(\omega) = -g(\omega) > 0$; then $(f+g)^+(\omega) = (f+g)^-(\omega) = 0$, but $f^+(\omega) + g^+(\omega) = f^+(\omega) > 0$, and $f^-(\omega) + g^-(\omega) = -g(\omega) > 0$.

To see $0 \leqslant (f+g)^+ \leqslant f^+ + g^+$ (the first inequality always holds), observe that for every $\omega \in A$,

$$(f+g)^+(\omega) = \begin{cases} f(\omega) + g(\omega) & \text{if } f(\omega) + g(\omega) \geqslant 0 \\ 0 & \text{if } f(\omega) + g(\omega) < 0, \end{cases}$$

and

$$f^+(\omega) + g^+(\omega) = \begin{cases} f(\omega) + g(\omega) & \text{if } f(\omega) \geqslant 0, g(\omega) \geqslant 0 \\ f(\omega) & \text{if } f(\omega) \geqslant 0, g(\omega) < 0 \\ g(\omega) & \text{if } f(\omega) < 0, g(\omega) \geqslant 0 \\ 0 & \text{if } f(\omega) < 0, g(\omega) < 0. \end{cases}$$

For instance, if $f(\omega) \geqslant 0$, $g(\omega) < 0$, and $f(\omega) + g(\omega) \geqslant 0$, then $(f+g)^+(\omega) = f(\omega) + g(\omega) < f(\omega) = f^+(\omega) + g^+(\omega)$. All other cases can be analyzed similarly.

(d) If $|g| \leqslant f$, i.e., $g^+ + g^- \leqslant f$, then $0 \leqslant g^+, g^- \leqslant f$.

(e) For every $\omega \in A$, we have $f(\omega) = g(\omega) - h(\omega) \leqslant g(\omega)$; thus $f^+ \leqslant g^+ = g$. Similarly, for every $\omega \in A$, we have $f(\omega) \geqslant -h(\omega)$, and so $f^- \leqslant (-h)^- = h$. □

► EXERCISE 5.16 (5.3.2). *The class of $\mathcal{F}/\mathcal{B}^*$-measurable functions is not necessarily closed under uncountable suprema and infima. The following outline gives a simple instantiation of this claim. Let $\Omega = \mathbb{R}$ and $\mathcal{F} = \mathcal{B}$.*

a. *Let E denote a non-Borel set as constructed in Section 4.5. Argue that E cannot be at most countable.*

b. *For each $x \in E$, define $f_x \colon \mathbb{R} \to \overline{\mathbb{R}}$ by writing $f_x(\omega) = \mathbb{1}_{\{x\}}(\omega)$ for each $\omega \in \mathbb{R}$. Then f_x is $\mathcal{F}/\mathcal{B}^*$-measurable for each $x \in E$, but $\sup_{x \in E} f_x = \mathbb{1}_E$, hence $\sup_{x \in E} f_x$ is not $\mathcal{F}/\mathcal{B}^*$-measurable.*

PROOF. **(a)** Every singleton set $\{x\} \subset \mathbb{R}$ is a Borel set; thus, if E is at most countable, it would be a Borel set.

(b) For every $x \in E$, the function f_x is $\mathcal{B}/\mathcal{B}^*$-measurable by (MF3) since $\{x\} \in \mathcal{B}$. However, $\sup_{x \in E} f_x = \mathbb{1}_E$ is not $\mathcal{B}/\mathcal{B}^*$-measurable since $E \notin \mathcal{B}$. □

5.4 ALMOST EVERYWHERE

► EXERCISE 5.17 (5.4.1). *In (MF15), the completeness of $(\Omega, \mathcal{F}, \mu)$ is not a redundant assumption.*

PROOF. Note that the proof of (MF15) dependents on (MF14), which depends on the completeness of the measure space. □

► EXERCISE 5.18 (5.4.2). *Suppose that $f \colon \mathbb{R} \to \mathbb{R}$ is a differentiable function. Then f' is a Borel measurable function.*

PROOF. Let $f_n(x) = f(x + 1/n)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$. Then f_n is a Borel measurable function for each $n \in \mathbb{N}$ since f is continuous. Therefore,

$$\frac{f(x + 1/n) - f(x)}{1/n} = n\left[f_n(x) - f(x)\right]$$

is Borel measurable for each $n \in \mathbb{N}$. Thus $f' = \lim_n n(f_n - f)$ is Borel measurable. □

5.5 SIMPLE FUNCTIONS

► EXERCISE 5.19 (5.5.1). *Refer to (MF18).*

a. *If $f \colon \Omega \to \overline{\mathbb{R}}$ is a general $\mathcal{F}/\mathcal{B}^*$-measurable function, then there exists a sequence $\{s_n\}_{n=1}^{\infty}$ of $\mathcal{F}/\mathcal{B}^*$-measurable and finite-valued simple functions such that $s_n \to f$, having the additional property that $0 \leqslant |s_1| \leqslant |s_2| \leqslant \cdots$.*

b. *If in (a) the function f is also bounded, then $s_n \to f$ uniformly on A.*

c. *The following converse to (MF18) holds: if* $f\colon A \to \overline{\mathbb{R}}$ *is such that there exists a sequence* $\{s_n\}$ *of simple* $\mathcal{F}/\mathcal{B}^*$*-measurable functions with* $s_n \to f$ *then* f *is* $\mathcal{F}/\mathcal{B}^*$*-measurable.*

Proof. **(a)** Write $f = f^+ - f^-$. Then there exist nondecreasing, nonnegative $\mathcal{F}/\mathcal{B}^*$-measurable and finite-valued simple functions $\{s_n^+\}$ and $\{s_n^-\}$ such that $s_n^+ \to f^+$ and $s_n^- \to f^-$. Let $s_n = s_n^+ - s_n^-$ for all n, and consider $\{s_n\}$.

(b) We first consider a nonnegative $\mathcal{F}/\mathcal{B}^*$-measurable bounded function $f\colon A \to [0,\infty)$. Fix an $\varepsilon > 0$. For every $n \in \mathbb{N}$ and $k = 1, \dots, n2^n$, define

$$A_{n,k} = \left\{\omega \in A : \frac{k-1}{2^n} \leqslant f(\omega) < \frac{k}{2^n}\right\} \quad \text{and} \quad B_n = \{\omega \in A : f(\omega) \geqslant n\}.$$

Take $N_1 \in \mathbb{N}$ so that $f(\omega) - s_{N_1}(\omega) < 1/2^{N_1} \leqslant \varepsilon$ for all $\omega \in \bigcup_{k=1}^{n2^n} A_{N_1,k}$. Now pick $N \in \mathbb{N}$ such that $N \geqslant N_1$ and $f(\omega) - N < \varepsilon$ for all $\omega \in B_N$. This proves that $s_n \to f$ uniformly. This result can be easily extended.

(c) Follows from (MF11). □

► Exercise 5.20 (5.5.2). *Consider the measure space* $(\mathbb{R}^k, \mathcal{L}^k, \overline{\lambda}_k)$. *Let* $f\colon \mathbb{R}^k \to \overline{\mathbb{R}}$ *denote a Lebesgue measurable function. We will show that there exists a Borel measurable function* $g\colon \mathbb{R}^k \to \overline{\mathbb{R}}$ *with* $|g| \leqslant |f|$ *and* $f = g$ $\overline{\lambda}_k$–a. e.

a. *Let* $f \geqslant 0$. *There exists a sequence* $\{s_n\}$ *of nonnegative finite-valued Lebesgue measurable simple functions such that* $0 \leqslant s_1 \leqslant s_2 \leqslant \dots \leqslant f$ *and* $s_n \to f$.

   Pick $m \in \mathbb{N}$, *and write* $s_m = \sum_{j=1}^{n_m} c_{mj} \mathbb{1}_{A_{mj}}$, *where* $0 \leqslant c_{m1}, \dots, c_{mn_m} < \infty$ *and* $A_{m1}, \dots, A_{mn_m}$ *are disjoint* $\mathcal{L}^k$*-sets with* $\bigcup_{j=1}^{n_m} A_{mj} = \mathbb{R}^k$. *Write the set* A_{mj} *as* $B_{mj} \cup C_{mj}$, *where* $B_{mj} \in \mathcal{B}^k$ *and* C_{mj} *is contained in some* λ_k*-null set* N_{mj}. *Define* $s_m^* = \sum_{j=1}^{n_m} c_{mj} \mathbb{1}_{B_{mj}}$.

b. *For each* $m \in \mathbb{N}$, s_m^* *is a Borel measurable simple function such that* $0 \leqslant s_m^* \leqslant s_m$ *and* $s_m^* = s_m$ $\overline{\lambda}_k$–a. e.

   Define $N = \bigcup_{m=1}^{\infty} N_m$, *where* $N_m = \{\boldsymbol{x} \in \mathbb{R}^k : s_m(\boldsymbol{x}) \neq s_m^*(\boldsymbol{x})\}$ *for each* $m \in \mathbb{N}$, *and let* $g = \sup_{m\in\mathbb{N}} s_m^*$.

c. N *is* $\overline{\lambda}_k$*-null,* $0 \leqslant g \leqslant f$, *and* $g = f$ $\overline{\lambda}_k$–a. e.

d. g *is Borel measurable, hence the proof is complete in the nonnegative case.*

e. *The claim holds when* $f\colon \mathbb{R}^k \to \overline{\mathbb{R}}$ *is an arbitrary Lebesgue measurable function.*

Proof. **(a)** Follows from (MF18) immediately.

(b) s_m^* is $\mathcal{B}^k/\mathcal{B}^*$-measurable since $B_{mj} \in \mathcal{B}^k$ for every $j = 1, \dots, n_m$ (by (MF16)). Define

$$N_m = \left\{\boldsymbol{x} \in \mathbb{R}^k : s_m^*(\boldsymbol{x}) \neq s_m(\boldsymbol{x})\right\}.$$

Then $D \subseteq \bigcup_{j=1}^{n_m} C_{mj}$ and so $\overline{\lambda}_k(D) \leqslant \overline{\lambda}_k(\bigcup_{j=1}^{n_m} C_{mj}) = 0$; that is, $s_m^* = s_m$ $\overline{\lambda}_k$–a. e. The other claims are trivial.

(c) It follows from (b) that $\overline{\lambda}_k(N_m) = 0$ for every $m \in \mathbb{N}$. Thus, $\overline{\lambda}_k(N) \leqslant \sum_{m=1}^{\infty} \overline{\lambda}_k(N_m) = 0$.

Fix an arbitrary $\boldsymbol{x} \in \mathbb{R}^k$. Then $s_m^*(\boldsymbol{x}) \leqslant s_m(\boldsymbol{x}) \leqslant f(\boldsymbol{x})$. Hence, $g(\boldsymbol{x}) = \sup_m s_m^*(\boldsymbol{x}) \leqslant f(\boldsymbol{x})$, i.e., $g \leqslant f$. Notice that $g(\boldsymbol{x}) \neq f(\boldsymbol{x})$ only probably on N, and since $\overline{\lambda}_k(N) = 0$, we conclude that $g = f\ \overline{\lambda}_k$ -a. e.

(d) It follows from (b) that s_m^* is $\mathscr{B}^k/\mathscr{B}^*$-measurable. Then g is $\mathscr{B}^k/\mathscr{B}^*$-measurable since $g = \sup_{m\in\mathbb{N}} s_m^*$.

(e) Write $f = f^+ - f^-$. Then there exist nonnegative $\mathscr{B}^k/\mathscr{B}^*$-measurable functions g_1 and g_2 such that $0 \leqslant g_1 \leqslant f^+$, $g_1 = f^+\ \overline{\lambda}_k$ -a. e., and $0 \leqslant g_2 \leqslant f^-$, $g_2 = f^-\ \overline{\lambda}_k$ -a. e. Let $g = g_1 - g_2$. Then $g = f\ \overline{\lambda}_k$ -a. e., and

$$|g| = |g_1 - g_2| \leqslant |g_1| + |g_2| = g_1 + g_2 \leqslant f^+ + f^- = |f|. \qquad \square$$

► Exercise 5.21 (5.5.3). *In Exercise 5.20, show that the Borel measurable function $g\colon \mathbb{R}^k \to \overline{\mathbb{R}}$ may be chosen such that $|g| \geqslant |f|$ and $f = g\ \overline{\lambda}_k$ -a. e.*

Proof. Just let $A_{mj} = B_{mj} \smallsetminus C_{mj}$, and $\bigcup_{j=1}^{n_m} B_{mj} = \mathbb{R}^k$. Then $s_m^* \geqslant s_m$ for every m, and so $g \geqslant f$ when $f \geqslant 0$. As in Exercise 5.20(e), we also get the other results. $\square$

5.6 Some Convergence Concepts

► Exercise 5.22 (5.6.1). *This result concerns uniqueness.*

a. *If $f_n \to f\ \mu$ -a. e. and $f_n \to g\ \mu$ -a. e., then $f = g\ \mu$ -a. e..*

b. *If $f_n \to f$ in μ-measure and $f_n \to g$ in μ-measure, then $f = g\ \mu$ -a. e..*

Proof. **(a)** Write $[f(\omega) \neq g(\omega)]$ as the union of four $\mathscr{F}$-sets:

$$\begin{aligned}
A_1 &= [f(\omega) \neq g(\omega), f_n(\omega) \to f(\omega), g_n(\omega) \to g(\omega)],\\
A_2 &= [f(\omega) \neq g(\omega), f_n(\omega) \to f(\omega), g_n(\omega) \not\to g(\omega)],\\
A_3 &= [f(\omega) \neq g(\omega), f_n(\omega) \not\to f(\omega), g_n(\omega) \to g(\omega)],\\
A_4 &= [f(\omega) \neq g(\omega), f_n(\omega) \not\to f(\omega), g_n(\omega) \not\to g(\omega)].
\end{aligned}$$

Since limits of sequences of numbers are unique, $A_1 = \varnothing$. Observe that each of A_2, A_3, and A_4 are contained in ν-null sets; for example, $A_2 \subseteq [g_n(\omega) \not\to g(\omega)]$. Thus, $\mu[f(\omega) \neq g(\omega)] = 0$.

(b) Fix $m \in \mathbb{N}$. Suppose that there exists $\omega \in A$ such that $|f(\omega) - g(\omega)| > 1/m$. For an arbitrary $n \in \mathbb{N}$, if $|f_n(\omega) - f(\omega)| < 1/(2m)$, then

$$|f_n(\omega) - g(\omega)| \geqslant |f(\omega) - g(\omega)| - |f_n(\omega) - f(\omega)| > 1/(2m).$$

Thus,

$$[|f(\omega) - g(\omega)| > 1/m] \subseteq [|f_n(\omega) - f(\omega)| \geqslant 1/(2m)] \cup [|f_n(\omega) - g(\omega)| > 1/(2m)].$$

For any fixed $\varepsilon > 0$, $N \in \mathbb{N}$ such that $\mu[|f_n(\omega) - f(\omega)| \geqslant 1/(2m)] < \varepsilon/2^{m+1}$ and $\mu[|f_n(\omega) - g(\omega)| > 1/(2m)] < \varepsilon/2^{m+1}$. Then $\mu[|f(\omega) - g(\omega)| > 1/m] < \varepsilon/2^m$. Since

$$\begin{aligned}
\mu[f(\omega) \neq g(\omega)] &= \mu\left(\bigcup_{m=1}^{\infty}[|f(\omega) - g(\omega)| > 1/m]\right) \\
&\leqslant \sum_{m=1}^{\infty} \mu[|f(\omega) - g(\omega)| > 1/m] \\
&\leqslant \sum_{m=1}^{\infty} \varepsilon/2^m \\
&= \varepsilon,
\end{aligned}$$

we have $f = g$ μ-a. e. □

► EXERCISE 5.23 (5.6.2). *Suppose that* $f_n \xrightarrow{\mu} f$ *and* $g_n \xrightarrow{\mu} g$.

a. $f_n - f \xrightarrow{\mu} 0$ *and* $|f_n| \xrightarrow{\mu} |f|$.

b. *If* $a, b \in \mathbb{R}$, *then* $af_n + b \xrightarrow{\mu} af + b$.

c. *If* $a, b \in \mathbb{R}$, *then* $af_n + bg_n \xrightarrow{\mu} af + bg$.

d. $f_n^+ \xrightarrow{\mu} f^+$ *and* $f_n^- \xrightarrow{\mu} f^-$.

e. *If* $\mu(A) < \infty$ *and* $\eta > 0$, *there is* $M > 0$ *with* $\mu[|g(\omega)| > M] < \eta$.

f. *If* $\mu(A) < \infty$, *then* $f_n g \xrightarrow{\mu} fg$.

g. *If* $\mu(A) < \infty$, *then* $f_n g_n \xrightarrow{\mu} fg$.

h. *If* $\mu(A) = \infty$, *then* $f_n g_n$ *does not necessarily converge to* fg *in* μ*-measure.*

i. *It is not necessarily the case that* $f_n/g_n \xrightarrow{\mu} f/g$, *even if* $g(\omega) \neq 0$ *and* $g_n(\omega) \neq 0$ *for every* $\omega \in A$ *and* $n \in \mathbb{N}$. *However, if* $\mu(A) < \infty$, *the result follows.*

PROOF. **(a)** $f_n \xrightarrow{\mu} f$ iff for any $\varepsilon > 0$, we have $\lim_{n\to\infty} \mu[|f_n(\omega) - f(\omega)| > \varepsilon] = 0$; that is

$$\lim_{n\to\infty} \mu\left[|(f_n(\omega) - f(\omega)) - 0| > \varepsilon\right] = 0,$$

i.e., $f_n - f \xrightarrow{\mu} 0$.

To see the second implication, note that $||f_n(\omega)| - |f(\omega)|| \leqslant |f_n(\omega) - f(\omega)|$, and so

$$\left[||f_n(\omega)| - |f(\omega)|| > \varepsilon\right] \subseteq \left[|f_n(\omega) - f(\omega)| > \varepsilon\right].$$

Therefore, $\mu[|f_n(\omega) - f(\omega)| > \varepsilon] = 0$ implies that $\mu[||f_n(\omega)| - |f(\omega)|] > \varepsilon] = 0$; that is, $|f_n| \xrightarrow{\mu} |f|$.

(b) It is clear that

$$\begin{aligned}\left[|f_n(\omega) - f(\omega)| > \varepsilon\right] &= \left[|af_n(\omega) - af(\omega)| > |a|\varepsilon\right] \\ &= \left[|(af_n(\omega) - b) - (af(\omega) - b)| > |a|\,\varepsilon\right].\end{aligned}$$

(c) It suffices to show that $f_n + g_n \xrightarrow{\mu} f + g$. By the triangle inequality, we have

$$|(f_n + g_n)(\omega) - (f + g)(\omega)| \leqslant |f_n(\omega) - f(\omega)| + |g_n(\omega) - g(\omega)|.$$

Therefore, $[|(f_n + g_n)(\omega) - (f + g)(\omega)| > \varepsilon] \subset [|f_n(\omega) - f(\omega)| > \varepsilon/2] \cup [|g_n(\omega) - g(\omega)| > \varepsilon/2]$, and so $f_n + g_n \xrightarrow{\mu} f + g$. Then, by (b), it is evident that $af_n + bg_n \xrightarrow{\mu} af + bg$.

(d) Observe that

$$\begin{aligned}|f_n^+(\omega) - f^+(\omega)| &= |\max\{f_n(\omega), 0\} - \max\{f(\omega), 0\}| \\ &\leqslant |\max\{f_n(\omega), 0\} - f(\omega)| \\ &= |f(\omega) - \max\{f_n(\omega), 0\}| \\ &\leqslant |f(\omega) - f_n(\omega)|.\end{aligned}$$

Thus, $[|f_n^+(\omega) - f^+(\omega)| > \varepsilon] \subseteq [|f_n(\omega) - f(\omega)| > \varepsilon]$, and so $f_n^+ \xrightarrow{\mu} f^+$. Similarly, $f_n^- \xrightarrow{\mu} f^-$.

(e) Let $A_m = [|g(\omega)| > m]$ for every $m \in \mathbb{N}$. Observe that $\{A_m\} \subseteq \mathscr{F}$ and $A_m \downarrow \varnothing$ (since $g(\omega) \in \mathbb{R}$). By monotonicity, we get $\mu(A_1) \leqslant \mu(A) < \infty$; thus, $\lim_m \mu(A_m) = \mu(\lim_m A_m) = \mu(\varnothing) = 0$. Given $\eta > 0$, there exists $M \in \mathbb{N}$ such that $\mu(A_M) < \eta$; that is, $\mu[|g(\omega)| > M] < \eta$.

(f) Observe that for an arbitrary $M \in \mathbb{N}$,

$$\begin{aligned}\left[|f_n(\omega)g(\omega) - f(\omega)g(\omega)| > \varepsilon\right] &= \left[|f_n(\omega) - f(\omega)| \cdot |g(\omega)| > \varepsilon\right] \\ &= \left[|f_n(\omega) - f(\omega)| \cdot |g(\omega)| > \varepsilon, |g(\omega) > M\right] \\ &\quad \cup \left[|f_n(\omega) - f(\omega)| \cdot |g(\omega)| > \varepsilon, |g(\omega)| \leqslant M\right] \\ &\subseteq \left[|g(\omega)| > M\right] \cup \left[|f_n(\omega) - f(\omega)| > \varepsilon/M\right].\end{aligned}$$

It follows from (e) that for an arbitrary $\delta > 0$, there exists M such that $\mu[|g(\omega)| > M] < \delta/2$. Since $f_n \xrightarrow{\mu} f$, there exists $N_0 \in \mathbb{N}$ such that $\mu[|f_n(\omega) - f(\omega)| > \varepsilon/M] < \delta/2$. Let $N = \max\{M, N_0\}$. Then $\mu[|f_n(\omega)g(\omega) - f(\omega)g(\omega)| > \varepsilon] < \delta$; that is, $f_n g \xrightarrow{\mu} fg$.

(h) Observe that how the assumption $\mu(A) < \infty$ is used.

(i) Notice that

$$[|g(\omega) > M] = [|1/g(\omega)| < 1/M]. \qquad \square$$

► EXERCISE 5.24 (5.5.3). *Suppose that $f_n \xrightarrow{\text{a.e.}} f$ and $g_n \xrightarrow{\text{a.e.}} g$.*

a. $f_n - f \xrightarrow{\text{a.e.}} 0$ *and* $|f_n| \xrightarrow{\text{a.e.}} |f|$.

b. *If* $a, b \in \mathbb{R}$, *then* $af_n + b \xrightarrow{\text{a.e.}} af + b$.

c. *If* $a, b \in \mathbb{R}$, *then* $af_n + bg_n \xrightarrow{\text{a.e.}} af + bg$.

d. $f_n^+ \xrightarrow{\text{a.e.}} f^+$ *and* $f_n^- \xrightarrow{\text{a.e.}} f^-$.

e. $f_n g_n \xrightarrow{\text{a.e.}} fg$.

f. *If* $g, g_n \neq 0$ μ-a.e. *for each* $n \in \mathbb{N}$, *then* $f_n/g_n \xrightarrow{\text{a.e.}} f/g$.

PROOF. **(a)** We have

$$\mu[f_n(\omega) - f(\omega) \not\to 0] = \mu[f_n(\omega) \not\to f(\omega)] = 0.$$

Similarly for $|f_n|$.

(b) If a, the claim holds trivially. So assume that $a \neq 0$. Then

$$\mu[af_n(\omega) + b \not\to af(\omega) + b] = \mu[f_n(\omega) \not\to f(\omega)] = 0.$$

(c) For every $\omega \in A$ with $af_n(\omega) + bg(\omega) \not\to af(\omega) + bg(\omega)$, if $f_n(\omega) \to f(\omega)$, then $g_n(\omega) \not\to g(\omega)$. Therefore,

$$[af_n(\omega) + bg(\omega) \not\to af(\omega) + bg(\omega)] \subseteq [f_n(\omega) \not\to f(\omega)] \cup [g_n(\omega) \not\to g(\omega)],$$

and so $af_n + bg_n \xrightarrow{\text{a.e.}} af + bg$.

(d) Take an arbitrary $\omega \in A$ such that $f_n(\omega) \to f(\omega)$. If $f(\omega) > 0$, then there exists $N \in \mathbb{N}$ such that $f_n(\omega) > 0$ for all $n \geqslant N$, and then $f_n^+(\omega) \to f^+(\omega)$. If $f(\omega) < 0$, then there exists $N \in \mathbb{N}$ such that $f_n(\omega) < 0$ for all $n \geqslant N$, and then $f_n^+(\omega) = 0$ for all $n \geqslant N$. Thus, $f_n^+(\omega) \to 0 = f^+(\omega)$. Finally, if $f(\omega) = 0$, then there exists $N \in \mathbb{N}$ such that $|f_n(\omega) - f(\omega)| < \varepsilon$. In this case, for all $n \geqslant N$, either $f_n^+(\omega) = f_n(\omega) > 0$ and $f_n(\omega) < \varepsilon$, or $f_n^+(\omega) = 0$; thus, $f_n^+(\omega) \to f^+(\omega) = 0$.

We thus proved that $f_n^+(\omega) \to f^+(\omega)$ whenever $f_n(\omega) \to f(\omega)$. In other words, we have

$$[f_n^+(\omega) \not\to f^+(\omega)] \subseteq [f_n(\omega) \not\to f(\omega)];$$

that is, $f_n^+ \xrightarrow{\text{a.e.}} f^+$. Similarly, we get $f_n^- \xrightarrow{\text{a.e.}} f^-$.

(e) $[f_n(\omega)g_n(\omega) \not\to f(\omega)g(\omega)] \subseteq [f_n(\omega) \not\to f(\omega)] \cup [g_n(\omega) \not\to g(\omega)]$.

(f) $[f_n(\omega)/g_n(\omega) \not\to f(\omega)/g(\omega)] \subseteq [f_n(\omega) \not\to f(\omega)] \cup [g_n(\omega) \not\to g(\omega)]$. □

► EXERCISE 5.25 (5.5.4). *Suppose for each* $n \in \mathbb{N}$ *that* $f_n = g_n$ μ-a.e..

a. *If* $f_n \xrightarrow{\text{a.e.}} f$, *then* $g_n \xrightarrow{\text{a.e.}} f$.

b. *If* $f_n \xrightarrow{\mu} f$, *then* $g_n \xrightarrow{\mu} f$.

PROOF. **(a)** We have

$$[g_n(\omega) \not\to f(\omega)] \subseteq \left(\bigcup_{n=1}^{\infty}[g_n(\omega) \neq f_n(\omega)]\right) \cup [f_n(\omega) \not\to f(\omega)].$$

Thus, $g_n \xrightarrow{\text{a.e.}} f$ if $g_n \xrightarrow{\text{a.e.}} f$.

(b) We have

$$[|g_n(\omega) - f(\omega)| > \varepsilon] \subseteq \left(\bigcup_{n=1}^{\infty}[g_n(\omega) \neq f_n(\omega)]\right) \cup [|f_n(\omega) - f(\omega)| > \varepsilon];$$

thus, $g_n \xrightarrow{\mu} f$. □

► EXERCISE 5.26 (5.6.5). *Prove the following statements connecting convergence in μ-measure with convergence μ-a. e.*

a. $f_n \xrightarrow{\mu} f$ *iff* $f_{n_j} \xrightarrow{\mu} f$ *for every subsequence* $\{n_j\}$.

b. $f_n \xrightarrow{\mu} f$ *iff each subsequence of* $\{f_n\}$ *has a sub-subsequence that converges to* f μ-a. e..

PROOF. See Resnick (1999, Theorem 6.3.1). Here is the basic procedure of the proof: $f_n \xrightarrow{\mu} f$ iff $\{f_n\}$ is Cauchy in measure, i.e., $\mu[|f_r - f_s|] \to 0$ as $r, s \to \infty$. Then there exists a subsequence $\{f_{n_k}\}$ which converges a.s. □

► EXERCISE 5.27 (5.6.6). *Suppose that* $\varphi\colon \mathbb{R} \to \mathbb{R}$ *is continuous.*

a. *If* $f_n \xrightarrow{\text{a.e.}} f$, *then* $\varphi \circ f_n \xrightarrow{\text{a.e.}} \varphi \circ f$.

b. *If* $f_n \xrightarrow{\mu} f$, *then* $\varphi \circ f_n \xrightarrow{\mu} \varphi \circ f$.

PROOF. **(a)** There exists a null set $N \in \mathcal{F}$ with $\mu(N) = 0$ such that if $\omega \in N^c$, then $f_n(\omega) \to f(\omega)$. By continuity of φ, if $\omega \in N^c$, then $\varphi\left(f_n(\omega)\right) \to \varphi\left(f(\omega)\right)$.

(b) Let $\{\varphi \circ f_{n_k}\}$ be some subsequence of $\{\varphi \circ f_n\}$. It suffices to find an a.s. convergence subsequence $\{\varphi \circ f_{n_{k(i)}}\}$. But we know that $\{f_{n_k}\}$ has some a.s. convergent subsequence $\{f_{n_{k(i)}}\}$ such that $f_{n_{k(i)}} \xrightarrow{\text{a.e.}} f$. Thus, $\varphi \circ f_{n_{k(i)}} \xrightarrow{\text{a.e.}} \varphi \circ f$ by (a). □

5.7 CONTINUITY AND MEASURABILITY

5.8 A GENERALIZED DEFINITION OF MEASURABILITY

► EXERCISE 5.28 (5.8.1). *Regarding the measure* ν*:*

a. *ν is really a measure.*

b. *If μ is finite, then so is ν.*

c. *If μ is σ-finite, ν need not be σ-finite.*

PROOF. **(a)** (i) $\nu(A') \geqslant 0$ for all $A' \in \mathscr{F}'$ since $\nu(A') = \mu(f^{-1}(A')) \geqslant 0$ for all $A' \in \mathscr{F}'$. (Ii) $\nu(\varnothing) = (\mu \circ f^{-1})(\varnothing) = \mu(\varnothing) = 0$. (iii) Let $\{A'_n\} \subseteq \mathscr{F}'$ be disjoint. Then

$$\nu\left(\bigcup A'_n\right) = \mu\left(f^{-1}\left(\bigcup A'_n\right)\right) = \mu\left(\bigcup f^{-1}(A'_n)\right) = \sum \mu\left(f^{-1}(A'_n)\right)$$
$$= \sum \nu(A'_n).$$

(b) If $\mu(\Omega) < \infty$, then $\nu(\Omega') = \mu(f^{-1}(\Omega')) = \mu(\Omega) < \infty$.

(c) Suppose that there exists a unique sequence $\{A_n\}$ of $\mathscr{F}$-sets such that $\Omega = \bigcup A_n$ and $\mu(A_n) < \infty$ for all $n \in \mathbb{N}$; suppose that there does not exist a sequence $\{A'_n\}$ of $\mathscr{F}'$-sets such that $f^{-1}(A'_n) = A_n$ for all $n \in \mathbb{N}$. Then ν is not σ-finite. □

► EXERCISE 5.29 (5.8.2). *Modify (MF14) and prove it in the more general setting of this section.*

PROOF. (MF14) can be modified in the following way:

(MF14′) Let $(\Omega, \mathscr{F}, \mu)$ denote a complete measure space. Pick nonempty A in $\mathscr{F}$, and let f be defined μ-a.e. on A and $\mathscr{F}/\mathscr{F}'$-measurable. If g is defined μ-a.e. on A and $f = g$ μ-a.e. on A, then g is $\mathscr{F}/\mathscr{F}'$-measurable.

PROOF OF (MF′). Let $B = \{\omega \in \operatorname{dom}(f) \cap \operatorname{dom}(g) : f(\omega) = g(\omega)\}$, so that $\mu(A \smallsetminus B) = 0$. Take an arbitrary $A' \in \mathscr{F}'$, and observe that

$$g^{-1}(A') = \left[g^{-1}(A') \cap B\right] \cup \left[g^{-1}(A') \cap (A \smallsetminus B)\right]$$
$$= \left[f^{-1}(A') \cap B\right] \cup \left[g^{-1}(A') \cap (A \smallsetminus B)\right].$$

Since $f^{-1}(A') \in \mathscr{F}$ and $B \in \mathscr{F}$, we have $f^{-1}(A') \cap B \in \mathscr{F}$. Next, $g^{-1}(A') \cap (A \smallsetminus B)$ is a subset of $A \smallsetminus B$, and $\mu(A \smallsetminus B) = 0$. Since $(\Omega, \mathscr{F}, \mu)$ is complete, we have $g^{-1}(A') \cap (A \smallsetminus B) \in \mathscr{F}$. Thus, $g^{-1}(A') \in \mathscr{F}$ and so g is $\mathscr{F}/\mathscr{F}'$-measurable. □

6

THE LEBESGUE INTEGRAL

6.1 Stage One: Simple Functions

► Exercise 6.1 (6.1.1). *Let $E \in \mathcal{F}$.*

a. $\mu(E) = 0$ *iff* $\mathcal{I}_E^{\mathfrak{s}}(s) = 0$ *for every* $s \in \mathfrak{S}$.

b. *For any* $c \geqslant 0$, $\mathcal{I}_E^{\mathfrak{s}}(c) = c\mu(E)$.

Proof. **(a)** Write $s = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$. If $\mu(E) = 0$, then $\mathcal{I}_E^{\mathfrak{s}}(s) = \sum_{i=1}^n c_i \mu(A_i \cap E) = \sum_{i=1}^n c_i 0 = 0$ since $\mu(A_i \cap E) \leqslant \mu(E) = 0$ implies that $\mu(A_i \cap E) = 0$. For the converse direction, observe that if $\mu(E) > 0$, then $\mathcal{I}_E^{\mathfrak{s}}(\mathbb{1}_E) = \mu(E) > 0$, where, of course, $\mathbb{1}_E \in \mathfrak{S}$.

(b) Write $c \in \mathfrak{S}$ as $c = c\mathbb{1}_\Omega$. Thus, $\mathcal{I}_E^{\mathfrak{s}}(c) = c\mu(\Omega \cap E) = c\mu(E)$. □

► Exercise 6.2 (6.1.2). *Let $t, s, s_1, s_2, \ldots \in \mathfrak{S}$. Why can't we say that $\mathcal{I}_E^{\mathfrak{s}}(as + bt) = a\mathcal{I}_E^{\mathfrak{s}}(s) + b\mathcal{I}_E^{\mathfrak{s}}(t)$ for every $a, b \in \mathbb{R}$, as compared to saying that the result holds for every $0 \leqslant a, b < \infty$? Also, why can't we necessarily write $\mathcal{I}_E^{\mathfrak{s}}(\sum_{i=1}^\infty c_i s_i) = \sum_{i=1}^\infty c_i \mathcal{I}_E^{\mathfrak{s}}(s_i)$? [What is the domain of $\mathcal{I}_E^{\mathfrak{s}}$?]*

Proof. This is because the domain of $\mathcal{I}_E^{\mathfrak{s}}$ is $\mathfrak{S}$: the collection of *finite*-valued *nonnegative* $\mathcal{F}/\mathcal{B}^*$-measurable simple functions with domain Ω. Hence, if $a < 0$ and $b < 0$, then $as + bt \notin \mathfrak{S}$. We can't necessarily write $\mathcal{I}_E^{\mathfrak{s}}(\sum_{i=1}^\infty c_i s_i) = \sum_{i=1}^\infty c_i \mathcal{I}_E^{\mathfrak{s}}(s_i)$ because it is possible that $\sum_{i=1}^\infty c_i s_i(\omega) = \infty$ for some $\omega \in \Omega$. □

► Exercise 6.3 (6.1.3). *Let $E \in \mathcal{F}$ be such that $\mu(E) > 0$. Then $\mathcal{I}_E^{\mathfrak{s}}(s) = 0$ iff $s = 0$ μ-a. e. on E. In particular, $\mathcal{I}^{\mathfrak{s}}(s) = 0$ iff $s = 0$ μ-a. e. [on Ω].*

Proof. Assume first that $s = 0$ μ-a. e. on E. Let $E_1 = \{\omega \in E : s(\omega) = 0\}$ and $E_2 = E \smallsetminus E_1$. Then $\mu(E_2) = 0$, and so

$$\mathcal{I}_E^{\mathfrak{s}}(s) = \mathcal{I}_{E_1 \cup E_2}^{\mathfrak{s}}(s) = \mathcal{I}_{E_1}^{\mathfrak{s}}(s) = 0.$$

Conversely, assume that $\mathcal{I}_E^{\mathfrak{s}}(s) = 0$. Let $A_n = \{\omega \in E : s(\omega) \geqslant 1/n\}$ for each $n \in \mathbb{N}$. Then, for every $n \in \mathbb{N}$, we have $A_n \in \mathcal{F}$ and $\frac{1}{n}\mathbb{1}_{A_n} \leqslant s$ on E;

hence $\mathcal{I}_E^{\mathfrak{s}}(\frac{1}{n}\mathbb{1}_{A_n}) \leqslant \mathcal{I}_E^{\mathfrak{s}}(s)$ by (S4), whence $\mu(A_n)/n \leqslant 0$ by (S6). It follows that $\mu(A_n) = 0$ for all $n \in \mathbb{N}$. Since

$$\{\omega \in E : s(\omega) > 0\} = \bigcup_{n=1}^{\infty} A_n,$$

we have $\mu(\{\omega \in E : s(\omega) > 0\}) = 0$; hence $s = 0$ μ-a.e.. Replacing E by Ω we get the second claim. $\square$

► Exercise 6.4 (6.1.4). *Let $E \in \mathcal{F}$ and $s \in \mathfrak{S}$.*

a. *If $\mu(E) < \infty$, then $\mathcal{I}_E^{\mathfrak{s}}(s) < \infty$, but the converse is not necessarily true.*

b. *If $\mu(E) = \infty$ and $\mathcal{I}_E^{\mathfrak{s}}(s) = \infty$, then $\mu(\{\omega \in E : s(\omega) > 0\}) > 0$, but the converse is not necessarily true.*

c. *Let $\mu(E) = \infty$. Then $\mathcal{I}_E^{\mathfrak{s}}(s) < \infty$ iff $\mu(\{\omega \in E : s(\omega) > 0\}) < \infty$.*

Proof. **(a)** Let $s = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}$, where $0 \leqslant c_i < \infty$ and $\bigcup_{i=1}^{n} A_i = \Omega$. If $\mu(E) < \infty$, then $\mu(A_i \cap E) < \mu(E) < \infty$, for all i. Thus, $\mathcal{I}_E^{\mathfrak{s}}(s) = \sum_{i=1}^{n} c_i \mu(A_i \cap E) < \infty$ since the finite summation of finite terms is finite.

To see the converse is not necessarily true, consider $t = \mathbb{1}_{E^c}$. Then $\mathcal{I}_E^{\mathfrak{s}}(t) = 0$, and which holds no matter whether $\mu(E) = \infty$ or not as long as we assume that $0 \times \infty = 0$.

(b) If $s = 0$ μ-a.e. on E, then $\mathcal{I}_E^{\mathfrak{s}}(s) = 0$ by the previous exercise; hence, $\mu(\{\omega \in E : s(\omega) > 0\}) > 0$.

To see the converse is not necessarily true, let $t = \mathbb{1}_E$. Then $\mu(\{\omega \in E : t(\omega) > 0\}) = \mu(E)$. By letting $0 < \mu(E) < \infty$, we see that $\mu(\{\omega \in E : t(\omega) > 0\}) > 0$, but $\mu(E) < \infty$ and $\mathcal{I}_E^{\mathfrak{s}}(t) = \mu(E) < \infty$.

(c) If $\mu(E) = \infty$, then

$$\begin{aligned}\sum_{i=1}^{n} c_i \mu(A_i \cap E) < \infty &\iff \mu(A_i \cap E) < \infty \text{ for all } i \text{ with } c_i > 0 \\ &\iff \mu(\{\omega \in E : s(\omega) > 0\}) < \infty.\end{aligned}$$ $\square$

► Exercise 6.5 (6.1.5). *Suppose that $\{E_n\}_{n=1}^{\infty}$ is a nonincreasing sequence of $\mathcal{F}$-sets. Also, let $s \in \mathfrak{S}$.*

a. *It is not necessarily the case that $\lim_n \mathcal{I}_{E_n}^{\mathfrak{s}}(s) = \mathcal{I}_{\lim_n E_n}^{\mathfrak{s}}(s)$.*

b. *If $\mathcal{I}_{E_n}^{\mathfrak{s}}(s) < \infty$ for some $n \in \mathbb{N}$, then $\lim_n \mathcal{I}_{E_n}^{\mathfrak{s}}(s) = \mathcal{I}_{\lim_n E_n}^{\mathfrak{s}}(s)$.*

c. *If $\{E_n\}_{n=1}^{\infty}$ is no longer nonincreasing but is still such that $\lim_n E_n$ exists, state conditions under which $\lim_n \mathcal{I}_{E_n}^{\mathfrak{s}}(s) = \mathcal{I}_{\lim_n E_n}^{\mathfrak{s}}(s)$.*

Proof. **(a)** Let $(\Omega, \mathcal{F}, \mu) = (\mathbb{N}, 2^{\mathbb{N}}, \mu)$, where μ is the counting measure; let $s = \mathbb{1}_{\Omega}$. Let $E_n = \{j \in \mathbb{N} : j \geqslant n\}$. Then $E_n \downarrow \varnothing$, $\mu(E_n) = \infty$ and $\mu(\bigcap_{n=1}^{\infty} E_n) = 0$. In this case, $\mathcal{I}_{E_n}^{\mathfrak{s}}(\mathbb{1}_{\Omega}) = \infty$ for each n; but $\mathcal{I}_{\lim_n E_n}^{\mathfrak{s}}(\mathbb{1}_{\Omega}) = 0$. This argument and the following example is modified from Folland (1999, p. 26).

(b) The same as (M9). This example shows that some finiteness assumption is necessary.

(c) See Vestrup (2003, p. 47-48). □

▶ EXERCISE 6.6 (6.1.6). *Let $s \in \mathfrak{S}$ and recall (S7), where ν_s is the measure on $\mathcal{F}$ with $\nu_s(B) = \mathcal{I}_B^{\mathfrak{s}}(s)$ for every $B \in \mathcal{F}$. Then, for any $E \in \mathcal{F}$ and $t \in \mathfrak{S}$, we have $\mathcal{I}_E^{\mathfrak{s}}(t;\nu_s) = \mathcal{I}_E^{\mathfrak{s}}(ts;\mu)$.*

PROOF. Let $s = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$ and $t = \sum_{j=1}^m d_j \mathbb{1}_{B_j}$. Then

$$\begin{aligned}
\mathcal{I}_E^{\mathfrak{s}}(t;\nu_s) = \sum_{j=1}^m d_j \nu_s(B_j \cap E) = \sum_{j=1}^m d_j \mathcal{I}_{B_j \cap E}^{\mathfrak{s}}(s;\mu) &= \sum_{j=1}^m d_j \mathcal{I}_E^{\mathfrak{s}}(s\mathbb{1}_{B_j};\mu) \\
&= \mathcal{I}_E^{\mathfrak{s}}\left(\sum_{j=1}^m d_j s \mathbb{1}_{B_j};\mu\right) \\
&= \mathcal{I}_E^{\mathfrak{s}}(ts;\mu).
\end{aligned}$$ □

▶ EXERCISE 6.7 (6.1.7). *Let $A \in \mathcal{F}$ be nonempty, and let $s \in \mathfrak{S}$. Let $\mathcal{F}_A = \{E \subseteq A : E \in \mathcal{F}\}$, and let μ_A denote the restriction of μ from $\mathcal{F}$ to $\mathcal{F}_A$. Finally, let s_A denote the restriction of s from Ω to A.*

a. *$(A, \mathcal{F}_A, \mu_A)$ is a measure space.*

b. *If $\mathcal{J}$ denotes this section's functional* relative to $(A, \mathcal{F}_A, \mu_A)$, *then we have $\mathcal{I}_E^{\mathfrak{s}}(s;\mu) = \mathcal{J}_E(s_A;\mu_A)$ for every $E \in \mathcal{F}_A$.*

PROOF. **(a)** We only need to show that $\mathcal{F}_A$ is a σ-field. (i) $A \in \mathcal{F}_A$ because $A \in \mathcal{F}$ and $A \subseteq A$. (ii) If $E \in \mathcal{F}_A$, then $E \subseteq A$ and $E \in \mathcal{F}$. Thus, $A \smallsetminus E \subseteq A$ and $A \smallsetminus E \in \mathcal{F}$, i.e., $A \smallsetminus E \in \mathcal{F}_A$. (iii) Let $\{E_n\} \subseteq \mathcal{F}_A$. Then $E_n \subseteq A$ and $E_n \in \mathcal{F}$ for all n. Thus, $\bigcup E_n \subseteq A$ and $\bigcup E_n \in \mathcal{F}$; that is, $\bigcup E_n \in \mathcal{F}_A$.

(b) Let $s = \sum_{i=1}^n c_i \mathbb{1}_{A_i}$. Then $s_A = \sum_{i=1}^n c_i \mathbb{1}_{A_i \cap A;A}$, and so

$$\mathcal{J}_E(s_A;\mu_A) = \sum_{i=1}^n c_i \mu_A(A_i \cap A \cap E) = \sum_{i=1}^n c_i \mu(A_i \cap E) = \mathcal{I}_E^{\mathfrak{s}}(s;\mu)$$

since $\mu_A = \mu$ on $\mathcal{F}_A$ and $E \subseteq A$. □

▶ EXERCISE 6.8 (6.1.8). *Let $A \in \mathcal{F} \smallsetminus \{\varnothing\}$, and suppose that $s\colon A \to [0,\infty)$ is simple and $\mathcal{F}/\mathcal{B}^*$-measurable. Let $E \in \mathcal{F}_A$, where $\mathcal{F}_A$ is defined in the previous exercise. Consider two programs:*

Program 1 *Extend s from A to Ω as follows: Let $s^* \in \mathfrak{S}$ be such that $s^* = s$ on A and $s^* = 0$ on A^c.*

Program 2 *Do not extend s from A to Ω as in Program 1, but instead view s as a function defined everywhere relative to the measure space $(A, \mathcal{F}_A, \mu_A)$, where the notation is as in the previous exercise.*

These two programs are equivalent in the sense that $\mathcal{I}_E^{s}(s^*;\mu) = \mathcal{J}_E(s;\mu_A)$, *where* $\mathcal{I}_E^{s}$ *is this section's functional relative to the measure space* $(\Omega, \mathcal{F}, \mu)$, *and* $\mathcal{J}_E$ *is this section's functional relative to the measure space* $(A, \mathcal{F}_A, \mu_A)$.

Proof. Write $s = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i}$, where $\{A_i\}$ is disjoint and $\bigcup_{i=1}^{n} A_i = A$. Extend s to s^* as

$$s^* = \sum_{i=1}^{n} c_i \mathbb{1}_{A_i} + 0\mathbb{1}_{A^c}.$$

Then

$$\mathcal{I}_E^{s}(s^*;\mu) = \sum_{i=1}^{n} c_i \mu(A_i \cap E) + 0 \times \mu(A^c \cap E) = \sum_{i=1}^{n} c_i \mu(A_i \cap E).$$

Next, consider Program 2. We have

$$\mathcal{J}_E(s;\mu_A) = \sum_{i=1}^{n} c_i \mu_A(A_i \cap E) = \sum_{i=1}^{n} c_i \mu(A_i \cap E).$$

Thus, $\mathcal{I}_E^{s}(s^*;\mu) = \mathcal{J}_E(s;\mu_A)$. □

► Exercise 6.9 (6.1.9). *Quickly prove the following "almost everywhere" modification of (S3) and (S4). As usual,* $s, t \in \mathfrak{S}$ *and all sets are in* $\mathcal{F}$.

a. *If* $s = t$ μ-a.e. *on* E, *then* $\mathcal{I}_F^{s}(s) = \mathcal{I}_F^{s}(t)$ *for every* $F \subseteq E$.

b. *If* $s \leqslant t$ μ-a.e. *on* E, *then* $\mathcal{I}_F^{s}(s) \leqslant \mathcal{I}_F^{s}(t)$ *for every* $F \subseteq E$.

Proof. **(a)** Let $E_1 = \{\omega \in E : s(\omega) = t(\omega)\}$, and $E_2 = E \setminus E_1$. Then $\mu(E_2) = 0$. For an arbitrary $F \subseteq E$, define

$$F_1 = F \cap E_1, \quad \text{and} \quad F_2 = F \cap E_2.$$

Then $\mu(F_2) = 0$. Observe that $\mathcal{I}_{F_1}^{s}(s) = \mathcal{I}_{F_1}^{s}(t)$ by (S3), and $\mathcal{I}_{F_2}^{s}(s) = \mathcal{I}_{F_2}^{s}(t) = 0$ by (S2)(a). It follows from (S7) that

$$\mathcal{I}_F^{s}(s) = \mathcal{I}_{F_1}^{s}(s) + \mathcal{I}_{F_2}^{s}(s) = \mathcal{I}_{F_1}^{s}(s) = \mathcal{I}_{F_1}^{s}(t) = \mathcal{I}_{F_1}^{s}(t) + \mathcal{I}_{F_2}^{s}(t) = \mathcal{I}_F^{s}(t).$$

(b) Let $E_1 = \{\omega \in E : s(\omega) \leqslant t(\omega)\}$, and $E_2 = E \setminus E_1$; then $\mu(E_2) = 0$. For an arbitrary $F \subseteq E$, define $F_1 = F \cap E_1$ and $F_2 = F \cap E_2$. Then $\mathcal{I}_{F_1}^{s}(s) \leqslant \mathcal{I}_{F_1}^{s}(t)$ and $\mathcal{I}_{F_2}^{s}(s) = \mathcal{I}_{F_2}^{s}(t) = 0$. By (S7) we get the result. □

6.2 Stage Two: Nonnegative Functions

► Exercise 6.10 (6.2.1). *Prove*

a. (N3′) *If* $f = g$ μ-a.e. *on* E, *then* $\mathcal{I}_F^{n}(f) = \mathcal{I}_F^{n}(g)$ *for any* $F \subseteq E$.

b. (N4′) *If* $f \leqslant g$ μ-a.e. *on* E, *then* $\mathcal{I}_F^{n}(f) \leqslant \mathcal{I}_F^{n}(g)$ *for any* $F \subseteq E$.

c. (N7′) *If* $f, f_1, f_2, \ldots \in \mathfrak{N}$ *and* $f = \lim_n f_n$ μ-a.e. *on* E, *then* $\mathcal{I}_E^n(f) = \lim_n \mathcal{I}_E^n(f_n)$.

d. *If* $f \leqslant M$ μ-a.e. *on* E *for some* $M \in [0,\infty]$, *then* $\mathcal{I}_E^n(f) \leqslant M\mu(E)$.

PROOF. **(a)** Let $E_2 = \{\omega \in E : f(\omega) \neq g(\omega)\}$ and $E_1 = E \setminus E_2$. Let $F_1 = F \cap E_1$ and $F_2 = F \cap E_2$. Then $\mu(F_2) \leqslant \mu(E_2) = 0$, i.e., $\mu(F_2) = 0$. Thus,

$$\mathcal{I}_F^n(f) = \mathcal{I}_{F_1 \cup F_2}^n(f) = \mathcal{I}_{F_1}^n(f) + \mathcal{I}_{F_2}^n(f) = \mathcal{I}_{F_1}^n(f) = \mathcal{I}_{F_1}^n(g) = \mathcal{I}_F^n(g).$$

(b) Similar to (a) and so is omitted.

(c) Let $E_2 = \{\omega \in E : f(\omega) \neq \lim_n f_n(\omega)\}$ and $E_1 = E \setminus E_2$. Then $\mu(E_2) = 0$, and so

$$\mathcal{I}_E^n(f) = \mathcal{I}_{E_1}^n(f) + \mathcal{I}_{E_2}^n(f) = \mathcal{I}_{E_1}^n(f) = \lim_n \mathcal{I}_{E_1}^n(f_n) = \lim_n \mathcal{I}_E^n(f_n).$$

(d) Let $E_2 = \{\omega \in E : f(\omega) > M\}$ and $E_1 = E \setminus E_2$. Then $f \leqslant M$ on E_1 and $\mu(E_2) = 0$. Thus,

$$\mathcal{I}_E^n(f) = \mathcal{I}_{E_1}^n(f) \leqslant \mathcal{I}_{E_1}^n(M) = M\mu(E_1) = M\mu(E). \qquad \square$$

▶ EXERCISE 6.11 (6.2.2). *It was claimed in (N5) that* $\mathcal{I}_E^n(cf) = c\mathcal{I}_E^n(f)$ *for every* $c \in [0,\infty)$. *This result in fact holds in* $c = \infty$ *as well:* $\mathcal{I}_E^n(\infty f) = \infty\mathcal{I}_E^n(f)$. *Therefore, (N5) holds for all* $c \in [0,\infty]$. *Similarly, we may allow the numbers* $c_1, \ldots, c_n$ *to be in* $[0,\infty]$ *in the statement (N8).*

PROOF. If $f = 0$ μ-a.e. on E, then $\infty f = 0$ μ-a.e. on E and so $\mathcal{I}_E^n(\infty f) = \infty\mathcal{I}_E^n(f) = 0$. So assume that there exists $F \in \mathscr{F}$ with$F \subseteq E$ and $\mu(F) > 0$ such that $f > 0$ on F. Then $\infty f = \infty$ on F. Thus, $\mathcal{I}_E^n(\infty f) \geqslant \mathcal{I}_F^n(\infty) = \infty$, and $\infty\mathcal{I}_E^n(f) \geqslant \infty\mathcal{I}_F^n(f) = \infty$; that is, $\mathcal{I}_E^n(\infty f) = \infty\mathcal{I}_E^n(f) = \infty$. $\square$

▶ EXERCISE 6.12 (6.2.3). *This exercise concerns Fatou's Lemma.*

a. *Let* $\{A_n\}$ *denote a sequence of* $\mathscr{F}$*-sets. Show that* $\mu(\liminf A_n) \leqslant \liminf \mu(A_n)$ *by using Fatou's Lemma applied to the sequence of indicator functions* $\{\mathbb{1}_{A_n}\}_{n=1}^{\infty}$.

b. *Consider* $(\mathbb{R}, \mathscr{B}, \lambda)$. *If* $s_n = n^2\mathbb{1}_{[0,1/n]}$ *for each* $n \in \mathbb{N}$, *then* $\mathcal{I}^n(\liminf s_n) = 0$ *while* $\liminf \mathcal{I}^n(s_n) = \infty$,*hence strict inequality may hold in Fatou's Lemma.*

c. *In (b), with* $E \in \mathscr{B}$, *the sequence* $\mathbb{1}_E, 1 - \mathbb{1}_E, \mathbb{1}_E, 1 - \mathbb{1}_E, \ldots$ *provides another example where strict inequality holds in Fatou's Lemma.*

d. *It is not necessarily the case that* $\limsup \mathcal{I}_E^n(f_n) \leqslant \mathcal{I}_E^n(\limsup f_n)$ *if* $\mu(E) = \infty$. *However, if* $\mu(E) < \infty$, *the inequality holds, hence we have*

$$\mathcal{I}_E^n(\liminf f_n) \leqslant \liminf \mathcal{I}_E^n(f_n) \leqslant \limsup \mathcal{I}_E^n(f_n) \leqslant \mathcal{I}_E^n(\limsup f_n)$$

by putting everything together.

PROOF. **(a)** It is evident that $\{\mathbb{1}_{A_n}\} \subseteq \mathfrak{S} \subseteq \mathfrak{N}$ and $\liminf_n \mathbb{1}_{A_n} \in \mathfrak{N}$ since $\{A_n\} \subseteq \mathscr{F}$. For every $n \in \mathbb{N}$, let

$$g_n = \inf\{\mathbb{1}_{A_n}, \mathbb{1}_{A_{n+1}}, \ldots\}.$$

Then the sequence $\{g_n\}$ is nondecreasing and so $\lim_n g_n$ exists. Thus,

$$\lim_n g_n = \sup_n g_n = \sup_n \inf_{m \geqslant n} \mathbb{1}_{A_m} = \liminf_n \mathbb{1}_{A_n},$$

and $g_n \leqslant \mathbb{1}_{A_n}$ for all $n \in \mathbb{N}$. Also note that $\mathcal{I}^n(\mathbb{1}_{A_n}) = \mathcal{I}^s(\mathbb{1}_{A_n}) = \mu(A_n)$, and which implies that

$$\mathcal{I}^n(\lim_n g_n) = \mathcal{I}^n(\liminf_n \mathbb{1}_{A_n}) = \mathcal{I}^n(\mathbb{1}_{\liminf_n A_n}) = \mu\left(\liminf_n A_n\right),$$

where the second equality is from Exercise 2.14 (p. 36). Invoking Lebesgue's Monotone Convergence Theorem (MCT), we have

$$\begin{aligned}\mu\left(\liminf_n A_n\right) = \mathcal{I}^n(\lim_n g_n) = \lim_n \mathcal{I}^n(g_n) &= \liminf_n \mathcal{I}^n(g_n) \\ &\leqslant \liminf_n \mathcal{I}^n(\mathbb{1}_{A_n}) \\ &= \liminf_n \mu(A_n).\end{aligned}$$

(b) We first show that

$$\liminf_n s_n(\omega) = \begin{cases} 0 & \text{if } \omega \neq 0 \\ \infty & \text{if } \omega = 0. \end{cases}$$

Suppose that there exists $\omega \neq 0$ such that $\liminf_n s_n(\omega) = \alpha > 0$. Then for an arbitrary $\varepsilon \in (0, \alpha)$, there exists $N \in \mathbb{N}$ such that $s_n(\omega) > \varepsilon$ for all $n \geqslant N$. However, when n is large enough, $s_n(\omega) = 0$. A contradiction. Thus, $\liminf_n s_n = 0$ λ-a. e. on $\mathbb{R}$.

Therefore, $\mathcal{I}^n(\liminf_n s_n) = 0$. Nevertheless, $\mathcal{I}^n(s_n) = n^2\lambda[0, 1/n] = n$, and so $\liminf \mathcal{I}^n(s_n) = \lim_n n = \infty$.

(c) Write $1 - \mathbb{1}_E$ as $\mathbb{1}_{E^c}$. Then $\liminf_n\{\mathbb{1}_E, \mathbb{1}_{E^c}, \ldots\} = 0$ and so

$$\mathcal{I}^n(\liminf_n\{\mathbb{1}_E, \mathbb{1}_{E^c}, \ldots)\} = 0.$$

However, $\mathcal{I}^n(\mathbb{1}_E) = \lambda(E)$ and $\mathcal{I}^n(\mathbb{1}_{E^c}) = \lambda(E^c)$ imply that

$$\liminf_n \{\mathcal{I}^n(\mathbb{1}_E), \mathcal{I}^n(\mathbb{1}_{E^c}), \ldots\} = \min\left\{\lambda(E), \lambda\left(E^c\right)\right\}.$$

(d) We first extend Fatou's Lemma: If there exists $g \in L^1$ and $f_n \geqslant g$ on E, then

$$\mathcal{I}_E^n(\liminf_n f_n) \leqslant \liminf_n \mathcal{I}_E^n(f_n).$$

In this case, we have $f_n - g \geqslant 0$ on E and

$$\mathcal{I}_E^n(\liminf_n (f_n - g)) \leqslant \liminf_n \mathcal{I}_E^n(f_n - g)$$

by Fatou's Lemma. So

$$\mathcal{I}_E^n(\liminf_n f_n) - \mathcal{I}_E^n(g) \leqslant \liminf_n \mathcal{I}_E^n(f_n) - \mathcal{I}_E^n(g).$$

The result follows by cancelling $\mathcal{I}_E^n(g)$.

Now if $f_n \leqslant g$, then $-f_n \geqslant -g \in L^1$, and the extended Fatou's Lemma gives

$$\mathcal{I}_E^n(\liminf_n(-f_n)) \leqslant \liminf_n \mathcal{I}_E^n(-f_n),$$

so that

$$\mathcal{I}_E^n(-\liminf_n(-f_n)) \geqslant -\liminf_n \mathcal{I}_E^n(-f_n);$$

that is, $\mathcal{I}_E^n(\limsup_n f_n) \geqslant \limsup_n \mathcal{I}_E^n(f_n)$. □

► Exercise 6.13 (6.2.4). *Let Ω denote an arbitrary nonempty set, and fix attention upon a particular $\omega_0 \in \Omega$. Let $\mathcal{F} = 2^{\Omega}$, and define $\mu\colon \mathcal{F} \to \overline{\mathbb{R}}$ by writing $\mu(A) = 1$ if $\omega_0 \in A$ and $\mu(A) = 0$ if $\omega_0 \notin A$.*

a. *$(\Omega, \mathcal{F}, \mu)$ is a measure space.*

b. *Every $f\colon \Omega \to \mathbb{R}$ is $\mathcal{F}/\mathcal{B}^*$-measurable.*

c. *Let $E \in \mathcal{F}$ and $f \in \mathfrak{N}$. Then $\mathcal{I}_E^n(f) = f(\omega_0)\mathbb{1}_E(\omega_0)$.*

Proof. **(a)** It is evident that $\mathcal{F}$ is a σ-field, so it suffices to show that μ is a measure on $\mathcal{F}$. It is clear that $\mu(A) \geqslant 0$ for all $A \in \mathcal{F}$ and $\mu(\varnothing) = 0$. To see μ is countably additive, let $\{A_n\}_{n=1}^{\infty}$ be a disjoint sequence of $\mathcal{F}$-sets. If $\omega_0 \notin \bigcup_{n=1}^{\infty} A_n$, then $\omega_0 \notin A_n$ for all $n \in \mathbb{N}$; hence

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 0 = \sum_{n=1}^{\infty} \mu(A_n).$$

Otherwise, if there exists A_n such that $\omega_0 \in A_0$, then $\omega_0 \in \bigcup_{n=1}^{\infty} A_n$ and so

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 = \sum_{n=1}^{\infty} \mu(A_n).$$

(b) Every $f\colon \Omega \to \mathbb{R}$ is trivially $\mathcal{F}/\mathcal{B}^*$-measurable since $\mathcal{F} = 2^{\Omega}$: for every $B \in \mathcal{B}^*$, we have $f^{-1}(B) \in \mathcal{F}$.

(c) If $\omega_0 \notin E$, then $\mu(E) = \mathbb{1}_E(\omega_0) = 0$; thus, $\mathcal{I}_E^n(f) = f(\omega_0)\mathbb{1}_E(\omega_0) = 0$. If $\omega_0 \in E$, then $\mu(E) = \mathbb{1}_E(\omega_0) = 1$. Hence,

$$\mathcal{I}_E^n(f) = \mathcal{I}_{\{\omega_0\}}^n(f) + \mathcal{I}_{E \smallsetminus \{\omega_0\}}^n(f) = \mathcal{I}_{\{\omega_0\}}^n(f) = f(\omega_0). \qquad \square$$

► EXERCISE 6.14 (6.2.5). *Let Ω denote an uncountable set, and let $\mathcal{F} = \{A \subseteq \Omega : A$ is amc or A^c is amc$\}$. Define $\mu : \mathcal{F} \to \overline{\mathbb{R}}$ for all $A \in \mathcal{F}$ by writing $\mu(A) = 0$ if A is amc and $\mu(A) = 1$ if A^c is amc.*

a. *$(\Omega, \mathcal{F}, \mu)$ is a measure space.*

b. *$f : \Omega \to \overline{\mathbb{R}}$ is $\mathcal{F}/\mathcal{B}^*$-measurable iff there is $c \in \overline{\mathbb{R}}$ [depending on f] with $f = c$ μ-a. e. on Ω.*

c. *Let $E \in \mathcal{F}$ and $f \in \mathfrak{N}$, then $\mathcal{I}_E^n(f) = c \cdot \mu(E)$, where c is the constant such that $f = c$ μ-a. e. on Ω [as given in (b)].*

PROOF. **(a)** (i) $\Omega \in \mathcal{F}$ since $\Omega^c = \varnothing$ is amc. (ii) If $A \in \mathcal{F}$, then either A is amc or A^c is amc. If A is amc, then $A^c \in \mathcal{F}$ since $(A^c)^c = A$ is amc; if A^c is amc, then $A^c \in \mathcal{F}$. (iii) Let $\{A_n\} \subseteq \mathcal{F}$. Then either each A_n is amc or at least one A_n^c is amc. In the first case, $\bigcup A_n \in \mathcal{F}$ since countable unions of amc sets is itself amc; in the second case, let us assume that A_1^c is amc. We have $(\bigcup A_n)^c = \bigcap A_n^c \subseteq A_1^c$, so $(\bigcup A_n)^c$ is amc. It follows that $\bigcup A_n \in \mathcal{F}$ as well.

(b) First suppose that $f = c$ μ-a. e. for some $c \in \overline{\mathbb{R}}$; that is, there exists $N \in \mathcal{F}$ with $\mu(N) = 0$ such that $f(\omega) = c$ for all $\omega \in \Omega \smallsetminus N$. By definition, N is amc; thus, every subset of N is in $\mathcal{F}$. With this observation, we see that f is $\mathcal{F}/\mathcal{B}^*$-measurable.

Conversely, suppose that f is $\mathcal{F}/\mathcal{B}^*$-measurable. Let

$$C = \{t \in \overline{\mathbb{R}} : f^{-1}[-\infty, t] \text{ is amc}\}.$$

Note that $f^{-1}(-\infty) \in \mathcal{F}$. If $[f^{-1}(-\infty)]^c = f^{-1}(-\infty, \infty]$ is amc, then $f = -\infty$ μ-a. e.. So we suppose that $-\infty \in C$. Also, if $-\infty < b < a$ with $a \in C$, then $b \in C$ since $f^{-1}[-\infty, b] \subseteq f^{-1}[-\infty, a]$. Thus, $C = \{-\infty\}$ or C is some type of unbounded interval containing $-\infty$.

- If $C = \{-\infty\}$, then for any $t > -\infty$, the set $f^{-1}[-\infty, t]$ is not amc, and so is not in $\mathcal{F}$; hence $(f^{-1}[-\infty, t])^c = f^{-1}(t, \infty]$ is amc and is in $\mathcal{F}$ since f is $\mathcal{F}/\mathcal{B}^*$-measurable. Hence, $\mu(\{\omega \in \Omega : f(\omega) \neq -\infty\}) = 0$; that is, $f = -\infty$ μ-a. e..

- If C is an interval containing $-\infty$, let $c = \sup C$, so that $-\infty < c \leqslant \infty$. Let $t_1 \leqslant t_2 \leqslant \cdots$ be such that $t_n \to c$ and $t_n < c$ for each $n \in \mathbb{N}$. Then $f^{-1}[-\infty, t_n]$ is amc for each $n \in \mathbb{N}$. Thus,

$$\bigcup_n f^{-1}[-\infty, t_n] = f^{-1}\left(\bigcup [-\infty, t_n]\right) = f^{-1}[-\infty, c)$$

  is amc, i.e., $c \in C$. If $c = \infty$, then $\Omega = f^{-1}[-\infty, \infty]$ is amc; but Ω is uncountable so we get a contradiction. Hence, $-\infty < c < \infty$. Now, for every $d > c$, the set $f^{-1}[-\infty, d]$ is not amc; hence, $f^{-1}(d, \infty]$ is amc. Let $t_1 \geqslant t_2 \geqslant \cdots$ be such that $t_n \to c$ and $t_n > c$ for each $n \in \mathbb{N}$. Then $f^{-1}(t_n, \infty]$ is amc for each

$n \in \mathbb{N}$, and so $f^{-1}(c, \infty]$ is amc. In sum, the sets $f^{-1}[-\infty, c)$ and $f^{-1}(c, \infty]$ are both amc; thus,

$$\mu(\{\omega \in \Omega : f(\omega) \neq c\}) = \mu(f^{-1}[-\infty, c) \cup f^{-1}(c, \infty]) = 0;$$

that is, $f = c$ μ-a. e..

(c) Let $N \in \mathcal{F}$ be the set such that $\mu(N) = 0$ and $f(\omega) = c$ on $\Omega \smallsetminus N$. Let $E_1 = E \cap N$ and $E_2 = E \smallsetminus E_1$. Then $\mu(E_1) = 0$ and

$$\mathcal{I}_E^n(f) = \mathcal{I}_{E_1}^n(c) = c\mu(E_1) = c\mu(E). \qquad \square$$

► EXERCISE 6.15 (6.2.6). *Let Ω denote an arbitrary nonempty set, let $A \subseteq \Omega$, and let $f : \Omega \to [0, \infty]$ be given. Write*

$$\sum_{\omega \in A}^{*} f(\omega) = \sup \left\{ \sum_{\omega \in F} f(\omega) : F \subseteq A,\ F \text{ finite} \right\}.$$

a. *Suppose $A = \{\omega_1, \ldots, \omega_n\}$. Then $\sum_{\omega \in A}^{*} f(\omega) = \sum_{i=1}^{n} f(\omega_i)$, hence the definition above is consistent with what we're used to in the finite case.*

b. *Suppose $A = \{\omega_1, \omega_2, \ldots\}$ [a countable set]. Then $\sum_{\omega \in A}^{*} f(\omega) = \sum_{i=1}^{\infty} f(\omega_i)$.*

PROOF. **(a)** It is easy to see that $\sum_{i=1}^{n} f(\omega) \leqslant \sum_{\omega \in A}^{*} f(\omega)$ since $A \subseteq A$ and A is finite here. For the converse inclusion, observe that every $F \subseteq A$ is finite; thus, $\sum_{\omega \in F} f(\omega) \leqslant \sum_{\omega \in A} f(\omega)$ since $f(\omega) \geqslant 0$. We thus have $\sum_{\omega \in A}^{*} f(\omega) \leqslant \sum_{\omega \in A} f(\omega)$.

(b) Consider a sequence $\{A_n\}_{n=1}^{\infty}$ with $A_n = \{\omega_1, \ldots, \omega_n\}$ for all $n \in \mathbb{N}$. Then $A_n \uparrow A$. Also observe that for any finite set $F \subseteq A$, there exists A_n containing F. Thus,

$$\sum_{\omega \in A}^{*} f(\omega) = \sup \left\{ \sum_{\omega \in A_n} f(\omega) \right\} = \lim_n \sum_{\omega \in A_n} f(\omega) = \sum_{i=1}^{\infty} f(\omega_i). \qquad \square$$

► EXERCISE 6.16 (6.2.7). *Let Ω denote a nonempty set, let $\mathcal{F} = 2^{\Omega}$, and let $\mu : \mathcal{F} \to \overline{\mathbb{R}}$ be such that $\mu(A) =$ the number of points in A when A is finite, and $\mu(A) = \infty$ otherwise.*

a. *$(\Omega, \mathcal{F}, \mu)$ is a measure space. The measure μ is called the* counting measure *since μ "counts" the number of points in each $\mathcal{F}$-set.*

b. *Every $f : \Omega \to \overline{\mathbb{R}}$ is $\mathcal{F}/\mathcal{B}^*$-measurable.*

c. *Given any $E \in \mathcal{F}$ and $f \in \mathfrak{N}$, we have $\mathcal{I}_E^n(f) = \sum_{\omega \in E}^{*} f(\omega)$.*

PROOF. **(a)** and (b) are straightforward, so I just do (c). If $f \in \mathfrak{S}$ and $f > 0$ only on a finite subset of E, then $\mathcal{I}_E^n(f) = \mathcal{I}_E^s(f) = \sum_{\omega \in E}^{*} f(\omega)$.

Now let $f \in \mathfrak{N}$ and let $F \subseteq E$ be finite. Then $f\mathbb{1}_F \in \mathfrak{S}$ and $f > 0$ only on a finite subset of F; hence

$$\mathcal{I}_E^{n}(f) \geqslant \mathcal{I}_E^{n}(f\mathbb{1}_F) = \mathcal{I}_F^{n}(f) = \sum_{\omega\in F}^{*} f(\omega) = \sum_{\omega\in F} f(\omega).$$

Since F is an arbitrary finite subset of E, we have $\mathcal{I}_E^{n}(f) \geqslant \sum_{\omega\in E}^{*} f(\omega)$. This gives one inequality.

If $\sum_{\omega\in E}^{*} f(\omega) = \infty$, the previous inequality forces the result, so we may assume that $\sum_{\omega\in E}^{*} f(\omega) < \infty$. For any $s \in \mathfrak{S}_f$ we have

$$\sum_{\omega\in E}^{*} s(\omega) \leqslant \sum_{\omega\in E}^{*} f(\omega) < \infty;$$

from this deduce that $s > 0$ only on a finite subset of E; hence, we may find finite $F \subseteq E$ with $s = 0$ on $E \smallsetminus F$. Then

$$\mathcal{I}_E^{s}(s) = \sum_{\omega\in E}^{*} s(\omega) = \sum_{\omega\in F}^{*} s(\omega) \leqslant \sum_{\omega\in F}^{*} f(\omega) \leqslant \sum_{\omega\in E}^{*} f(\omega).$$

Since the above holds for any $s \in \mathfrak{S}_f$, it follows that

$$\mathcal{I}_E^{n}(f) = \sup_{s\in\mathfrak{S}_f} \mathcal{I}_E^{s}(s) \leqslant \sum_{\omega\in E}^{*} f(\omega). \qquad \square$$

► EXERCISE 6.17 (6.2.12). *Let Ω denote a nonempty set, and let $\mathcal{F}$ denote a σ-field on Ω. For each $n \in \mathbb{N}$, let μ_n denote a measure with domain $\mathcal{F}$. For each $n \in \mathbb{N}$, let $\sum_{i=1}^{n} \mu_i$ denote the measure that assigns the value $\sum_{i=1}^{n} \mu_i(A)$ to each $A \in \mathcal{F}$.*

a. *Let $s \in \mathfrak{S}$ and $n \in \mathbb{N}$. Then $\mathcal{I}_E^{s}(s; \sum_{i=1}^{n} \mu_i) = \sum_{i=1}^{n} \mathcal{I}_E^{s}(s; \mu_i)$.*

b. *Let $f \in \mathfrak{N}$ and $n \in \mathbb{N}$. Then $\mathcal{I}_E^{n}(f; \sum_{i=1}^{n} \mu_i) = \sum_{i=1}^{n} \mathcal{I}_E^{n}(f; \mu_i)$.*

PROOF. **(a)** Let $s = \sum_{j=1}^{m} c_j \mathbb{1}_{A_j} \in \mathfrak{S}$ and $n \in \mathbb{N}$. Then

$$\begin{aligned}\mathcal{I}_E^{s}(s; \sum_{i=1}^{n} \mu_i) = \sum_{j=1}^{m} c_j \cdot \left[\sum_{i=1}^{n} \mu_i\left(A_j \cap E\right)\right] &= \sum_{i=1}^{n}\sum_{j=1}^{m} c_i \cdot \mu_i\left(A_j \cap E\right) \\ &= \sum_{i=1}^{n} \mathcal{I}_E^{s}(s; \mu_i).\end{aligned}$$

(b) Let $f \in \mathfrak{N}$. By (MF18), there exists a nondecreasing sequence $\{s_k\}_{k=1}^{\infty} \subset \mathfrak{S}$ such that $s_m \to f$. Then by Lebesgue's Monotone Convergence Theorem,

$$
\begin{aligned}
\mathcal{I}_E^n(f;\sum_{i=1}^n \mu_i) = \mathcal{I}_E^n(\lim_m s_m;\sum_{i=1}^n) = \mathcal{I}_E^{s}(\lim_m s_m;\sum_{i=1}^n \mu_i) &= \lim_m \mathcal{I}_E^{s}(s_m;\sum_{i=1}^n \mu_i) \\
&= \lim_m \sum_{i=1}^n \mathcal{I}_E^{s}(s_m;\mu_i) \\
&= \sum_{i=1}^n \lim_m \mathcal{I}_E^{s}(s_m;\mu_i) \\
&= \sum_{i=1}^n \mathcal{I}_E^n(f;\mu_i). \qquad \square
\end{aligned}
$$

► EXERCISE 6.18 (6.2.14). *Keep $(\Omega, \mathcal{F}, \mu)$ general. Suppose that $\mathcal{F}_0 \subseteq \mathcal{F}$ is a σ-field on Ω, and let μ_0 denote the restriction of μ to μ_0. If a nonnegative $\mathcal{F}/\mathcal{B}^*$-measurable function f defined on Ω also happens to be $\mathcal{F}_0/\mathcal{B}^*$-measurable in addition, then $\mathcal{I}_E^n(f;\mu) = \mathcal{I}_E^n(f;\mu_0)$ for every $E \in \mathcal{F}_0$.*

PROOF. First consider the case of $f = \mathbb{1}_E$ where $E \in \mathcal{F}_0$. Then $\mathcal{I}_E^n(f;\mu) = \mu(E) = \mu_0(E) = \mathcal{I}_E^n(f;\mu_0)$. Next, let $f = \sum_{i=1}^n c_i \mathbb{1}_{A_i} \in \mathfrak{S}$. Then

$$\mathcal{I}_E^n(f;\mu) = \sum_{i=1}^n c_i \cdot \mathcal{I}_E^n(\mathbb{1}_{A_i};\mu) = \sum_{i=1}^n c_i \cdot \mathcal{I}_E^n(\mathbb{1}_{A_i};\mu_0) = \mathcal{I}_E^n(f;\mu_0).$$

Finally, let $f \in \mathfrak{N}$. Then there exists a nondecreasing sequence $\{s_n\}_{n=1}^\infty \subset \mathfrak{S}$ such that $s_n \to f$. Hence,

$$\mathcal{I}_E^n(f;\mu) = \mathcal{I}_E^n(\lim_n s_n;\mu) = \lim_n \mathcal{I}_E^n(s_n;\mu) = \lim_n \mathcal{I}_E^n(s_n;\mu_0) = \mathcal{I}_E^n(f;\mu_0). \qquad \square$$

► EXERCISE 6.19 (6.2.16, N17). *Let f denote a nonnegative $\mathcal{F}'/\mathcal{B}^*$-measurable function. We have the two equalities*

$$\mathcal{I}^n(f \circ T;\mu) = \mathcal{I}^n(f;\mu \circ T^{-1}) \tag{6.9}$$

and

$$\mathcal{I}_{T^{-1}(A')}^n(f \circ T;\mu) = \mathcal{I}_{A'}^n(f;\mu \circ T^{-1}) \quad \forall\ A' \in \mathcal{F}'. \tag{6.10}$$

PROOF. We have

$$
\begin{tikzcd}
(\Omega, \mathcal{F}, \mu) \arrow[r, "T"] \arrow[dr, "f \circ T"'] & (\Omega', \mathcal{F}', \mu \circ T^{-1}) \arrow[d, "f"] \\
& (\overline{\mathbb{R}}, \mathcal{B}^*, \overline{\lambda})
\end{tikzcd}
$$

- Let $f = \mathbb{1}_{A'}$, where $A' \in \mathcal{F}'$. Then $f \circ T = \mathbb{1}_{T^{-1}(A')}$, and

$$
\begin{aligned}
\mathcal{I}^n(\mathbb{1}_{A'} \circ T;\mu) = \mathcal{I}^n(\mathbb{1}_{T^{-1}(A')};\mu) = \mu(T^{-1}(A')) &= (\mu \circ T^{-1})(A') \\
&= \mathcal{I}^n(\mathbb{1}_{A'};\mu \circ T^{-1}).
\end{aligned}
$$

- Let $f = \sum_{i=1}^{n} c_i \mathbb{1}_{A'_i} \in \mathfrak{S}(\Omega', \mathscr{F}', \mu \circ T^{-1})$. Then $f \circ T = \sum_{i=1}^{n} c_i \mathbb{1}_{T^{-1}(A'_i)}$. It follows from (N8) that

$$\begin{aligned}
\mathcal{I}^n(f \circ T; \mu) = \mathcal{I}^n(\sum_{i=1}^{n} c_i \mathbb{1}_{T^{-1}(A'_i)}; \mu) &= \sum_{i=1}^{n} c_i \mathcal{I}^n(\mathbb{1}_{T^{-1}(A'_i)}; \mu) \\
&= \sum_{i=1}^{n} c_i \cdot \mathcal{I}^n(\mathbb{1}_{A'_i}; \mu \circ T^{-1}) \\
&= \mathcal{I}^n(\sum_{i=1}^{n} c_i \mathbb{1}_{A'_i}; \mu \circ T^{-1}) \\
&= \mathcal{I}^n(f; \mu \circ T^{-1}).
\end{aligned}$$

- Let $f \in \mathfrak{N}(\Omega', \mathscr{F}', \mu \circ T^{-1})$. By (MF18), there exists a nondecreasing sequence $\{s_n\} \subseteq \mathfrak{S}(\Omega', \mathscr{F}', \mu \circ T^{-1})$ such that $s_n \to f$. Thus, $\{s_n \circ T\} \subseteq \mathfrak{S}(\Omega', \mathscr{F}', \mu \circ T^{-1})$ is a nondecreasing sequence and $s_n \circ T \to f \circ T$. It follows from MCT that

$$\begin{aligned}
\mathcal{I}^n(f \circ T; \mu) = \mathcal{I}^n(\lim_n s_n \circ T; \mu) &= \lim_n \mathcal{I}^n(s_n \circ T; \mu) \\
&= \lim_n \mathcal{I}^n(s_n; \mu \circ T^{-1}) \\
&= \mathcal{I}^n(\lim_n s_n; \mu \circ T^{-1}) \\
&= \mathcal{I}^n(f; \mu \circ T^{-1}).
\end{aligned}$$

Replace f by $f \mathbb{1}_{A'}$. It suffices to show that

$$(f \mathbb{1}_{A'}) \circ T = (f \circ T) \mathbb{1}_{T^{-1}(A')}.$$

Note that $\mathbb{1}_{A'}$ is defined on Ω', while $\mathbb{1}_{T^{-1}(A')}$ is defined on Ω. For an arbitrary $\omega \in \Omega$, we have

$$\begin{aligned}
[(f \mathbb{1}_{A'}) \circ T](\omega) = f(T(\omega)) \cdot \mathbb{1}_{A'}(T(\omega)) &= (f \circ T)(\omega) \cdot \mathbb{1}_{T^{-1}(A')}(\omega) \\
&= [(f \circ T) \mathbb{1}_{T^{-1}(A')}](\omega). \qquad \square
\end{aligned}$$

6.3 Stage Three: General Measurable Functions

► Exercise 6.20 (6.3.1). *Let $E \in \mathscr{F}$ be such that $\mu(E) < \infty$, and let $f \in \mathfrak{M}$ be such that $f = 0$ on E^c and $m \leqslant f \leqslant M$ on E, where $m, M \in \mathbb{R}$. Then $\mathcal{I}^g(f)$ exists and is finite. Furthermore, we have $m\mu(E) \leqslant \mathcal{I}^g(f) \leqslant M\mu(E)$.*

Proof. Let $M' = \max\{|M|, |m|\}$. Then $f^+, f^- \leqslant M'$ on E. It follows that

$$\mathcal{I}_E^n(f^-), \mathcal{I}_E^n(f^+) \leqslant \mathcal{I}_E^n(M') = M'\mu(E) < \infty;$$

hence, $\mathcal{I}_E^g(f)$ eaif. The second claim follows (G6) since that $m, M \in \mathfrak{M}(E)$. $\square$

► EXERCISE 6.21 (6.3.2). *Let $E \in \mathcal{F}$, $\varepsilon > 0$ and let $f \in \mathfrak{M}$ be such that $\mathcal{I}_E^g(f)$ exists and is finite. There exists a subset $F \subseteq E$ such that $\mu(F) < \infty$ and $|\mathcal{I}_E^g(f) - \mathcal{I}_F^g(f)| < \varepsilon$.*

PROOF. First let $f \in \mathfrak{N}$. We first show that for all $x > 0$,

$$\mu(\{\omega \in E : f(\omega) > x\}) < \infty.$$

Suppose there exists $x > 0$ such that $\mu(\{\omega \in E : f(\omega) > x\}) = \infty$. Then

$$\mathcal{I}_E^n(f) \geqslant \mathcal{I}_{\{\omega \in E: f(\omega) > x\}}^n(x\mathbb{1}_E) = x\mu(\{\omega \in E : f(\omega) > x\}) = \infty.$$

A contradiction. For each $n \in \mathbb{N}$, let

$$E_n = \{\omega \in E : f(\omega) \geqslant 1/n\}.$$

Then for each $n \in \mathbb{N}$, we have $E_n \uparrow \{\omega \in E : f(\omega) \geqslant 0\} = E$ and $\mu(E_n) < \infty$. It follows that

$$\mathcal{I}_E^g(f) = \mathcal{I}_E^n(f) = \mathcal{I}_{\lim_n E_n}^n(f) = \lim_n \mathcal{I}_{E_n}^n(f) = \lim_n \mathcal{I}_{E_n}^g(f).$$

Thus, there exists E_N such that $|\mathcal{I}_E^g(f) - \mathcal{I}_{E_N}^g(f)| < \varepsilon$. Let $F = E_N$ and we are done.

Next let $f \in \mathfrak{M}$. Since $\mathcal{I}_E^g(f) < \infty$, we know that $\mathcal{I}_E^g(f^+) < \infty$ and $\mathcal{I}_E^g(f^-) < \infty$. By the previous argument, there exist $F', F'' \subseteq E$ such that

$$|\mathcal{I}_E^g(f^+) - \mathcal{I}_{F'}^g(f^+)| < \varepsilon/2 \quad \text{and} \quad |\mathcal{I}_E^g(f^-) - \mathcal{I}_{F''}^g(f^-)| < \varepsilon/2.$$

Let $F = F' \cup F'$. We get

$$\begin{aligned}
|\mathcal{I}_E^g(f) - \mathcal{I}_F^g(f)| &= |\mathcal{I}_E^g(f^+) - \mathcal{I}_E^g(f^-) - \mathcal{I}_F^g(f^+) + \mathcal{I}_F^g(f^-)| \\
&\leqslant |\mathcal{I}_E^g(f^+) - \mathcal{I}_F^g(f^+)| + |\mathcal{I}_E^g(f^-) - \mathcal{I}_F^g(f^-)| \\
&\leqslant |\mathcal{I}_E^g(f^+) - \mathcal{I}_{F'}^g(f^+)| + |\mathcal{I}_E^g(f^-) - \mathcal{I}_{F''}^g(f^-)| \\
&< \varepsilon.
\end{aligned}$$

□

► EXERCISE 6.22 (6.3.3). *f is such that $\mathcal{I}_E^g(f)$ exists and is finite iff for any $\varepsilon > 0$ there are functions g and h in $\mathfrak{M}$ such that $h \leqslant f \leqslant g$ on E and $\mathcal{I}_E^g(g - h) < \varepsilon$.*

PROOF. Suppose first that $\mathcal{I}_E^g(f)$ eaif. Then both $\mathcal{I}_E^n(f^+)$ and $\mathcal{I}_E^n(f^-)$ are finite. Given $\varepsilon > 0$, let $c = \varepsilon/4\mu(E)$. Let

$$h = f - c \quad \text{and} \quad g = f + c.$$

Then

$$\mathcal{I}_E^g(g - h) = \mathcal{I}_E^g(\varepsilon/2\mu(E)) = \varepsilon/2 < \varepsilon.$$

Now suppose that for every $\varepsilon > 0$ there exist $g, h \in \mathfrak{M}$ such that $h \leqslant f \leqslant g$ on E and $\mathcal{I}_E^g(g - h) < \varepsilon$. Since $h \leqslant g$, we have $g - h \in \mathfrak{N}$, and so $\mathcal{I}_E^g(g - h) = \mathcal{I}_E^n(g - h) = 0$. Then $g - h = 0$ μ-a. e. on E, and so g and h are finite and $g = h$

μ-a. e. on E, and so f is finite μ-a. e. on E. This proves that $\mathcal{I}_E^g(f)$ eaif on E. ☐

► EXERCISE 6.23 (6.3.4). *Let $f, f_1, f_2, \ldots$ denote a sequence of nonnegative functions in $\mathfrak{M}$. For each $n \in \mathbb{N}$ and $E \in \mathcal{F}$, define $\nu_n(E) = \mathcal{I}_E^g(f_n; \mu)$ and $\nu(E) = \mathcal{I}_E^g(f; \mu)$. Furthermore, assume that $\nu(\Omega), \nu_1(\Omega), \nu_2(\Omega), \ldots$ are finite and $f_n \to f$ μ-a. e. on Ω. Then*

$$\sup\{|\nu(E) - \nu_n(E)| : E \in \mathcal{F}\} \leqslant \mathcal{I}^g(|f_n - f|; \mu) \to 0 \text{ as } n \to \infty.$$

PROOF. Since $f, f_1, f_2, \ldots \in \mathfrak{N}$ and $f_n \xrightarrow{\text{a.e.}} f$ on Ω, we get

$$\begin{aligned}\sup_{E \in \mathcal{F}} |\nu(E) - \nu_n(E)| &= \sup_{E \in \mathcal{F}} \left|\mathcal{I}_E^g(f; \mu) - \mathcal{I}_E^g(f_n; \mu)\right| \\ &= \sup_{E \in \mathcal{F}} \left|\mathcal{I}_E^g(f - f_n; \mu)\right| \\ &\leqslant \sup_{E \in \mathcal{F}} \mathcal{I}_E^g(|f - f_n|; \mu) \\ &\leqslant \mathcal{I}^g(|f_n - f|; \mu) \\ &\to 0.\end{aligned}$$

☐

► EXERCISE 6.24 (6.3.5). *Let $(\Omega, \mathcal{F}, \mu) = (\mathbb{R}^k, \mathcal{B}^k, \lambda_k)$ with $E \in \mathcal{F}$, and let $f \in \mathfrak{M}$ be such that $\mathcal{I}^g(f)$ exists and is finite.*

a. *Suppose that $\{f_n\}$ is a sequence of functions in $\mathfrak{M}$ such that*

$$f_n(\boldsymbol{x}) = \begin{cases} f(\boldsymbol{x}) & \text{if } \boldsymbol{x} \text{ is such that } |f(\boldsymbol{x})| \leqslant n \text{ and } \|\boldsymbol{x}\| \leqslant n \\ 0 & \text{otherwise.} \end{cases}$$

Then $\lim_n \mathcal{I}_E^g(f_n)$ exists and equals $\mathcal{I}_E^g(\lim_n f_n)$.

b. *Reset everything in (a), and this time let $f_n(\boldsymbol{x}) = f(\boldsymbol{x}) \exp(-\|\boldsymbol{x}\|^2/n)$ for each $\boldsymbol{x} \in \mathbb{R}^k$ and $n \in \mathbb{N}$. Then $\lim_n \mathcal{I}_E^g(f_n) = \mathcal{I}_E^g(f)$.*

PROOF. **(a)** Since $\mathcal{I}^g(f)$ eaif, $\mathcal{I}^g(|f|)$ is finite by (G7). It is clear that $|f_n| \leqslant |f|$ and $f_n \to f$ (see Figure 6.1), all the claims follow from DCT.

(b) Observe that $f_n \to f$ (see Figure 6.2). ☐

► EXERCISE 6.25 (6.3.6). *Suppose that $\mu(\Omega) < \infty$, $\{f_n\}$ is a sequence of functions in $\mathfrak{M}$ such that there exists $M \in \mathbb{R}$ with $|f_n| \leqslant M$ for each $n \in \mathbb{N}$, and let $f \in \mathfrak{M}$ be such that $f_n \to f$ uniformly on E. Then $\lim_n \mathcal{I}_E^g(f_n) = \mathcal{I}_E^g(f)$.*

PROOF. $\mu(\Omega) < \infty$ and $M \in \mathbb{R}_+$ implies that $\mathcal{I}_\Omega^g(M) = M\mu(\Omega) < \infty$. Then for each $E \in \mathcal{F}$, we have $\mathcal{I}_E^g(M) < \infty$. The claim then follows from the DCT since uniform convergence implies pointwise convergence: $\lim_n f_n = f$. ☐

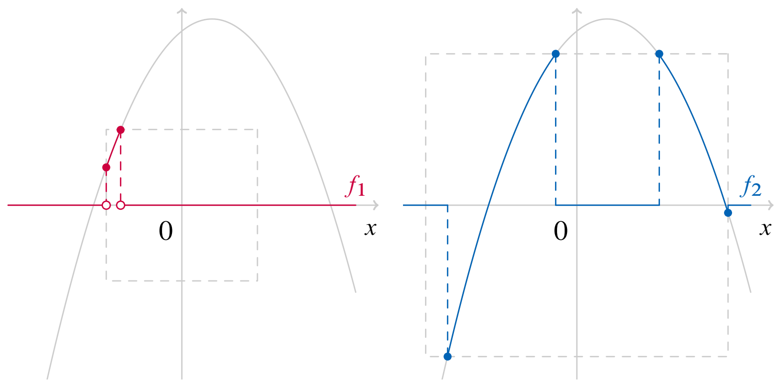

FIGURE 6.1. f_1 and f_2.

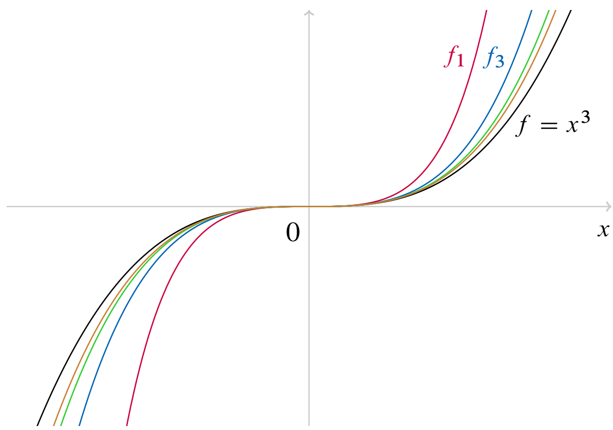

FIGURE 6.2. $f_n \to f$

▶ EXERCISE 6.26 (6.3.7). *Let* $\{f_n\}_{n=1}^{\infty}$ *denote a nondecreasing sequence of functions in* $\mathfrak{M}$ *such that* $\mathcal{I}_E^g(f_n)$ *exists and is finite for each* $n \in \mathbb{N}$ *and* $\sup_{n\in\mathbb{N}} \mathcal{I}_E^g(f_n) < \infty$. *Then* $\mathcal{I}_E^g(\lim_n f_n)$ *exists, is finite, and equals* $\lim_n \mathcal{I}_E^g(f_n)$. *This is one form of* Beppo Levi's Theorem.

PROOF. Let $g_n = f_n - f_1$ for all $n \in \mathbb{N}$. Then $\{g_n\}_{n=1}^{\infty} \subset \mathfrak{N}$, is nondecreasing, $g_n \uparrow \lim_n f_n - f_1$, and $\lim_n \mathcal{I}_E^g(g_n) = \lim_n \mathcal{I}_E^g(f_n - f_1) = \lim_n \mathcal{I}_E^g(f_n) - \mathcal{I}_E^g(f_1) = \sup_n \mathcal{I}_E^g(f_n) - \mathcal{I}_E^g(f_1) < \infty$. Then by MCT, $\mathcal{I}_E^g(\lim_n g_n) = \lim_n \mathcal{I}_E^g(g_n)$. Since

$$\mathcal{I}_E^g(\lim_n g_n) = \mathcal{I}_E^g(\lim_n f_n - f_1) = \mathcal{I}_E^g(\lim_n f_n) - \mathcal{I}_E^g(f_1),$$
$$\lim_n \mathcal{I}_E^g(g_n) = \lim_n \mathcal{I}_E^g(f_n - f_1) = \lim_n \mathcal{I}_E^g(f_n) - \mathcal{I}_E^g(f_1),$$

and $\mathcal{I}_E^g(f_1) < \infty$, we get the Beppo Levi's Theorem. □

▶ EXERCISE 6.27 (6.3.8). *Let* $\{f_n\}$, $\{g_n\}$, *and* $\{h_n\}$ *denote sequences of functions in* $\mathfrak{M}$ *such that* $f_n \xrightarrow{\text{a.e.}} f$, $g_n \xrightarrow{\text{a.e.}} g$, *and* $h_n \xrightarrow{\text{a.e.}} h$ *for some functions* $f, g, h \in \mathfrak{M}$.

Suppose for any $p \in \{g, g_1, g_2, \ldots, h, h_1, h_2, \ldots\}$ *that* $\mathcal{I}_E^g(p)$ *exists and is finite. Furthermore, suppose that* $\lim_n \mathcal{I}_E^g(g_n) = \mathcal{I}_E^g(g)$ *and* $\lim_n \mathcal{I}_E^g(h_n) = \mathcal{I}_E^g(h)$*. Also, assume that* $g_n \leqslant f_n \leqslant h_n$ *for every* $n \in \mathbb{N}$*.*

a. $\mathcal{I}_E^g(f_n)$ *exists and is finite for all* $n \in \mathbb{N}$, $\mathcal{I}_E^g(f)$ *exists and is finite, and* $\lim_n \mathcal{I}_E^g(f_n) = \mathcal{I}_E^g(f)$.

b. *DCT may be obtained from (a).*

PROOF. **(a)** Since $f_n \leqslant h_n$ and $\mathcal{I}_E^g(h_n) < \infty$, we have $\mathcal{I}_E^g(f_n) \leqslant \mathcal{I}_E^g(h_n) < \infty$ for all $n \in \mathbb{N}$; that is, $\mathcal{I}_E^g(f_n)$ exists and is finite for all $n \in \mathbb{N}$.

Since $f_n \leqslant h_n$ for all n, we get $\lim_n f_n \leqslant \lim_n h_n$, i.e., $f \leqslant h$; since $\mathcal{I}_E^g(h) < \infty$, we have $\mathcal{I}_E^g(f)$ exists and is finite.

Since $f_n \leqslant h_n$ for all n, we have $h_n - f_n \geqslant 0$ μ-a. e.. Fatou's Lemma yields

$$
\begin{aligned}
\mathcal{I}_E^g(h) - \mathcal{I}_E^g(f) = \mathcal{I}_E^g(h - f) &= \mathcal{I}_E^g(\lim_n (h_n - f_n)) \\
&= \mathcal{I}_E^g(\liminf_n (h_n - f_n)) \\
&\leqslant \liminf_n \mathcal{I}_E^g(h_n - f_n) \\
&= \liminf_n \left(\mathcal{I}_E^g(h_n) - \mathcal{I}_E^g(f_n)\right) \\
&= \mathcal{I}_E^g(h) + \liminf_n \left(-\mathcal{I}_E^g(f_n)\right);
\end{aligned}
$$

that is, $\mathcal{I}_E^g(f) \geqslant \limsup \mathcal{I}_E^g(f_n)$.

Finally, observe that $g_n \leqslant f_n$ yields $f_n - g_n \geqslant 0$ μ-a. e.. Applying Fatou's Lemma once again, we obtain

$$
\begin{aligned}
\mathcal{I}_E^g(f) - \mathcal{I}_E^g(g) = \mathcal{I}_E^g(f - g) &= \mathcal{I}_E^g(\lim_n (f_n - g_n)) \\
&= \mathcal{I}_E^g(\liminf_n (f_n - g_n)) \\
&\leqslant \liminf_n \mathcal{I}_E^g(f_n - g_n) \\
&= \liminf_n \mathcal{I}_E^g(f_n) - \mathcal{I}_E^g(g);
\end{aligned}
$$

that is, $\mathcal{I}_E^g(f_n) \leqslant \liminf_n \mathcal{I}_E^g(f_n)$. We thus get $\lim_n \mathcal{I}_E^g(f_n) = \mathcal{I}_E^g(f)$.

(b) Observe that if $|f| \leqslant g$, then $-g \leqslant f \leqslant g$. By (a) we get DCT. □

► EXERCISE 6.28 (6.3.12). *Suppose that* $\mathcal{I}^g_{\bigcup_{n=1}^\infty E_n}(f)$ *exists and is finite, where* $\{E_n\}_{n=1}^\infty$ *is a disjoint sequence of* $\mathcal{F}$*-sets. Then* $\mathcal{I}^g_{\bigcup_{n=1}^\infty E_n}(f) = \sum_{n=1}^\infty \mathcal{I}^g_{E_n}(f)$*, and the convergence of the series is absolute.*

PROOF. Since $\mathcal{I}^g_{\bigcup_{n=1}^\infty E_n}(f)$ eaif, (G1-b) implies that each of $\mathcal{I}^g_{E_n}(f)$ eaif. Therefore, $\mathcal{I}^g_{\bigcup_{n=1}^\infty E_n}(f^+), \mathcal{I}^g_{\bigcup_{n=1}^\infty E_n}(f^-), \mathcal{I}^g_{E_n}(f^+), \mathcal{I}^g_{E_n}(f^-) < \infty$. We have

$$
\begin{aligned}
\mathcal{I}^{g}_{\bigcup_{n=1}^{\infty} E_n}(f) &= \mathcal{I}^{g}_{\bigcup_{n=1}^{\infty} E_n}(f^+) - \mathcal{I}^{g}_{\bigcup_{n=1}^{\infty} E_n}(f^-) \\
&= \sum_{n=1}^{\infty} \mathcal{I}^{n}_{E_n}(f^+) - \sum_{n=1}^{\infty} \mathcal{I}^{n}_{E_n}(f^-) \\
&= \sum_{n=1}^{\infty} \left[\mathcal{I}^{n}_{E_n}(f^+) - \mathcal{I}^{n}_{E_n}(f^-) \right] \\
&= \sum_{n=1}^{\infty} \mathcal{I}^{g}_{E_n}(f).
\end{aligned}
$$

We now show that $\sum_{n=1}^{\infty} |\mathcal{I}^{g}_{E_n}(f)|$ converges. Since $\mathcal{I}^{g}_{E_n}(f)$ exists (and is finite), by (G7) we have $|\mathcal{I}^{g}_{E_n}(f)| \leqslant \mathcal{I}^{g}_{E_n}(|f|)$ for any $n \in \mathbb{N}$; since $\mathcal{I}^{g}_{E_n}(f)$ eaif for any $n \in \mathbb{N}$, we know that $\mathcal{I}^{g}_{E_n}(|f|)$ eaif for any $n \in \mathbb{N}$. Therefore,

$$
\sum_{n=1}^{\infty} \left| \mathcal{I}^{g}_{E_n}(f) \right| \leqslant \sum_{n=1}^{\infty} \mathcal{I}^{g}_{E_n}(|f|) = \mathcal{I}^{g}_{\bigcup_{n=1}^{\infty} E_n}(|f|) < \infty
$$

since $\mathcal{I}^{g}_{\bigcup_{n=1}^{\infty} E_n}(f)$ eaif. □

► EXERCISE 6.29 (6.3.14). *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of elements of $\mathfrak{N}$ converging to some $f \in \mathfrak{N}$. Furthermore, assume that there is $0 \leqslant M < \infty$ such that $\mathcal{I}^{g}(f_n) \leqslant M$ for each $n \in \mathbb{N}$. Then $\mathcal{I}^{g}(f)$ exists, is finite, and is no more than M.*

PROOF. Let $f_n \to f$. Then $\lim_n f_n = \liminf_n f_n$. By Fatou's Lemma,

$$
\mathcal{I}^{g}(f) = \mathcal{I}^{g}(\liminf_n f_n) \leqslant \liminf_n \mathcal{I}^{g}(f_n) \leqslant \limsup_n \mathcal{I}^{g}(f_n) \leqslant M. \qquad \square
$$

6.4 Stage Four: Almost Everywhere Defined Functions

► EXERCISE 6.30 (6.4.1). **(L10)** *If $f \in L^1(E)$ and $|g| \leqslant f$ μ-a. e. on E, then $g \in L^1(E)$. Also, any f that is bounded μ-a. e. on a set E with $\mu(E) < \infty$ and is zero μ-a. e. on E^c is in $L^1(E)$.*

(L16) *We have the following, where $A' \in \mathcal{F}'$:*

a. *If $\varphi \geqslant 0$, then $\int_{f^{-1}(A')} \varphi \circ f \, \mathrm{d}\mu = \int_{A'} \varphi \, \mathrm{d}\left(\mu \circ f^{-1}\right)$.*

b. *For general φ, $\int_{f^{-1}(A')} \varphi \circ f \, \mathrm{d}\mu$ exists and is finite iff $\int_{A'} \varphi \, \mathrm{d}\left(\mu \circ f^{-1}\right)$ exists and is finite, and in this case equality obtains.*

PROOF. We first prove (L10). $f \in L^1(E) \iff \mathcal{I}^{g}_{E}(f^*) < \infty$; since $|g| \leqslant f$ μ-a. e., we have $g^{*+} \leqslant f^*$ μ-a. e. and $g^{*-} \leqslant f^*$ μ-a. e.. Then the conclusion follows (G9). □

7

INTEGRALS RELATIVE TO LEBESGUE MEASURE

7.1 Semicontinuity

► Exercise 7.1 (7.1.1). **(SC4b)** *If $f(\boldsymbol{x}) = -\infty$, then f is USC at $\boldsymbol{x}$ iff $\lim_{\boldsymbol{y}\to\boldsymbol{x}} f(\boldsymbol{y}) = -\infty$.*

(SC7b) *$\overline{f}$ is USC, and is the minimal USC function $\geqslant f$.*

(SC9) *Let A denote a generic nonempty index set. For each $\alpha \in A$, suppose that f_α is a function from $\mathbb{R}^k$ into $\overline{\mathbb{R}}$. We have the following:*

a. *If f_α is LSC for each $\alpha \in A$, then $\sup_{\alpha\in A} f_\alpha$ is LSC.*

b. *If f_α is USC for each $\alpha \in A$, then $\inf_{\alpha\in A} f_\alpha$ is USC.*

Proof. **(SC4b)** Assume f is USC at $\boldsymbol{x}$. Pick $t > f(\boldsymbol{x}) = -\infty$. Then there is $\delta > 0$ such that $f(\boldsymbol{y}) < t$ for each $\boldsymbol{y} \in B(\boldsymbol{x}, \delta)$. Since t is arbitrary, we have $\lim_{\boldsymbol{y}\to\boldsymbol{x}} f(\boldsymbol{y}) = -\infty$. Conversely, assume that $\lim_{\boldsymbol{y}\to\boldsymbol{x}} f(\boldsymbol{y}) = -\infty$ and pick any $t > f(\boldsymbol{x}) = -\infty$. Then there is $\delta > 0$ such that $f(\boldsymbol{y}) < t$ for each $\boldsymbol{y} \in B(\boldsymbol{x}, \delta)$. Since t is generic, f is USC at $\boldsymbol{x}$ by definition.

(SC7b) We show that $\inf_{\delta>0}\sup_{\boldsymbol{y}\in B(\boldsymbol{x},\delta)} \overline{f}(\boldsymbol{y}) \leqslant \overline{f}(\boldsymbol{x})$ for each $\boldsymbol{x}$; then $\overline{f}$ is USC by (SC6). Suppose there is $\boldsymbol{x}$ so that the preceding inequality fails, then there exists t such that $\inf_{\delta>0}\sup_{\boldsymbol{y}\in B(\boldsymbol{x},\delta)} \overline{f}(\boldsymbol{y}) > t > \overline{f}(\boldsymbol{x})$. It follows that $\sup_{\boldsymbol{y}\in B(\boldsymbol{x},\delta)} \overline{f}(\boldsymbol{y}) > t$ for any $\delta > 0$, and therefore, there exists $\boldsymbol{y} \in B(\boldsymbol{x}, \delta)$ so that $\overline{f}(\boldsymbol{y}) > t$ for any $B(\boldsymbol{x}, \delta)$. Now consider an open ball of $\boldsymbol{y}$, $B(\boldsymbol{y}, r) \subseteq B(\boldsymbol{x}, \delta)$. We have

$$\overline{f}(\boldsymbol{y}) = \inf_{\delta'>0}\sup_{\boldsymbol{z}\in B(\boldsymbol{y},\delta')} f(\boldsymbol{z}) \leqslant \sup_{\boldsymbol{z}\in B(\boldsymbol{y},r)} f(\boldsymbol{z}) \leqslant \sup_{\boldsymbol{z}\in B(\boldsymbol{x},\delta)} f(\boldsymbol{z}),$$

that is, for any $B(\boldsymbol{x}, \delta)$, we have $\sup_{\boldsymbol{z}\in B(\boldsymbol{x},\delta)} f(\boldsymbol{z}) \geqslant \overline{f}(\boldsymbol{y}) > t$. But this implies that

$$\overline{f}(\boldsymbol{x}) = \inf_{\delta>0}\sup_{\boldsymbol{z}\in B(\boldsymbol{x},\delta)} f(\boldsymbol{z}) > t.$$

A contradiction.

With this outcome, we can show that $\underline{f}$ is LSC. Take any $\boldsymbol{x} \in \mathfrak{D}_f$. Then

$$\overline{-f}(\boldsymbol{x}) = \inf_{\delta>0} \sup_{\boldsymbol{y}\in B(\boldsymbol{x},\delta)} \big(-f\big)(\boldsymbol{y}) = -\sup_{\delta>0} \inf_{\boldsymbol{y}\in B(\boldsymbol{x},\delta)} f(\boldsymbol{y}) = -\underline{f}(\boldsymbol{x}).$$

Since $\overline{-f}$ is USC, $-(\overline{-f})$ is LSC, that is, $\underline{f} = -(-\underline{f}) = -(\overline{-f})$ is LSC.

(SC9a) Take any $\boldsymbol{x} \in \overline{\mathbb{R}}^k$ and $t < \sup_{\alpha\in A} f_\alpha(\boldsymbol{x})$. Then there exists $\alpha' \in A$ such that $f_{\alpha'}(\boldsymbol{x}) > t$; since $f_{\alpha'}$ is LSC, there is $B(\boldsymbol{x},\delta)$ such that $f_{\alpha'}(\boldsymbol{y}) > t$ for all $\boldsymbol{y} \in B(\boldsymbol{x},\delta)$. Since $\sup_{\alpha\in A} f_\alpha(\boldsymbol{y}) \geqslant f_{\alpha'}(\boldsymbol{y})$, we know that $\sup_{\alpha\in A} f_\alpha$ is LSC.

(SC9b) f_α is USC implies that $-f_\alpha$ is LSC; then $\sup_{\alpha\in A}(-f_\alpha) = -\inf_{\alpha\in A} f_\alpha$ is LSC. Hence, $\inf_{\alpha\in A} f_\alpha$ is USC.

□

► EXERCISE 7.2 (7.1.2). *Let $E \subset \overline{\mathbb{R}}^k$.*

a. *E is open iff $\mathbb{1}_E$ is LSC.*

b. *E is closed iff $\mathbb{1}_E$ is USC.*

c. *We have $\underline{\mathbb{1}_E} = \mathbb{1}_{E^\circ}$ and $\overline{\mathbb{1}_E} = \mathbb{1}_{\overline{E}}$.*

PROOF. □

8

THE L^p SPACES

8.1 L^p Space: The Case $1 \leqslant p < +\infty$

► Exercise 8.1 (8.1.1). *Pick $\mathfrak{A}, \mathfrak{B} \in \mathfrak{L}^p$. Then $\mathfrak{A} + \mathfrak{B} \in \mathfrak{L}^p$ and $\|\mathfrak{A} + \mathfrak{B}\|_p \leqslant \|\mathfrak{A}\|_p + \|\mathfrak{B}\|_p$.*

Proof. Let $h \in \mathfrak{A} + \mathfrak{B}$, so that $h = f + g$ for some $f \in \mathfrak{A}$ and $g \in \mathfrak{B}$. Now

$$\int |h|^p = \left(\int |f + g|^p\right)^{p/p} \leqslant \left[\left(\int |f|^p\right)^{1/p} + \left(\int |g|^p\right)^{1/p}\right]^p < +\infty$$

by Minkowski's Inequality, and hence $\mathfrak{A} + \mathfrak{B} \in \mathfrak{L}^p$. The above display also implies that

$$\|\mathfrak{A} + \mathfrak{B}\|_p^p \leqslant \left(\|\mathfrak{A}\|_p + \|\mathfrak{B}\|_p\right)^p,$$

i.e., $\|\mathfrak{A} + \mathfrak{B}\|_p \leqslant \|\mathfrak{A}\|_p + \|\mathfrak{B}\|_p$. □

► Exercise 8.2 (8.1.2). *Prove the Cauchy-Schwarz inequality*

$$\left|\sum_{k=1}^{n} a_k b_k\right| \leqslant \sqrt{\sum_{k=1}^{k} a_k^2} \sqrt{\sum_{k=1}^{n} b_k^2}.$$

Proof. Let $p = p' = 2$; then Hölder's Inequality becomes

$$\left|\int fg\right| \leqslant \|f\|_2 \cdot \|g\|_2. \tag{8.1}$$

Let $\Omega = \{\omega_1, \dots, \omega_n\}$, $\mathscr{F} = 2^\Omega$, μ be the counting measure, $f(\omega_i) = a_i$, and $g(\omega_i) = b_i$. Then $|\int fg| = |\sum_{\omega_i \in \Omega} f(\omega_i)\, g(\omega_i)| = |\sum_{k=1}^n a_k b_k|$,

$$\|f\|_2 = \left(\int |f|^2\right)^{1/2} = \left(\sum_{\omega_i \in \Omega} |f(\omega_i)|^2\right)^{1/2} = \sqrt{\sum_{k=1}^{n} a_k^2},$$

and similarly for $\|g\|_2$. Put these into (8.1) and we get the Cauchy-Schwarz inequality. See Shirali and Vasudeva (2006, Theorem 1.1.4) for a direct proof. □

► EXERCISE 8.3 (8.1.3). *Let* $1 < p_1, \dots, p_n < +\infty$ *be such that* $1/p_1 + \cdots + 1/p_n = 1$*, and pick functions* $f_1 \in L^{p_1}, \dots, f_n \in L^{p_n}$*. We wish to generalize Hölder's Inequality by showing that* $\prod_{i=1}^n f_i \in L^1$ *and* $|\int \prod_{i=1}^n f_i| \leqslant \prod_{i=1}^n \|f_i\|_{p_i}$.

a. *Show first that* $a_1 \cdots a_n \leqslant a_1^{p_1}/p_1 + \cdots + a_n^{p_n}/p_n$ *by generalizing the calculus result given in the section. [Here* $0 \leqslant a_1, \dots, a_n < +\infty$*.]*

b. *If* $\|f_1\|_{p_1} = 0$ *or ... or* $\|f_n\|_{p_n} = 0$*, the claim is trivial.*

c. *Use (a) to prove the claim when* $\|f_1\|_{p_1} = \cdots = \|f_n\|_{p_n} = 1$.

d. *Prove the claim when* $\|f_1\|_{p_1}, \dots, \|f_n\|_{p_n}$ *are positive.*

PROOF. **(a)** This is the *arithmetic mean-geometric mean inequality*, or AM-GM inequality, for short. Since ln is concave, we have

$$\sum_{i=1}^n \frac{1}{p_i} \ln x_i \leqslant \ln\left(\sum_{i=1}^n \frac{1}{p_i} x_i\right),$$

i.e.,

$$\ln\left(\prod_{i=1}^n x_i^{1/p_i}\right) \leqslant \ln\left(\sum_{i=1}^n \frac{x_i}{p_i}\right) \iff \prod_{i=1}^n x_i^{1/p_i} \leqslant \sum_{i=1}^n \frac{x_i}{p_i}.$$

Let $x_i^{1/p_i} = a_i$, then $x_i = a_i^{p_i}$ and we have the desired result.

(b) Let $\|f_i\|_{p_i} = 0$; then $f_i = 0$ μ-a.e. on Ω. But then $\prod_{i=1}^n f_i = 0$ μ-a.e. on Ω, hence $\prod_{i=1}^n f_i \in L^1$ and the desired inequality in this case is actually the trivial equation $0 = 0$.

(c) If $\|f_1\|_{p_1} = \cdots = \|f_n\|_{p_n} = 1$, observe that

$$\left|\prod_{i=1}^n f_i(\omega)\right| = \prod_{i=1}^n |f_i(\omega)| \leqslant \sum_{i=1}^n \frac{1}{p_i} |f_i(\omega)|^{p_i} \quad \forall\, \omega \in \Omega,$$

by the Am-GM inequality. Therefore,

$$\int \left|\prod_{i=1}^n f_i\right| \leqslant \sum_{i=1}^n \frac{1}{p_i} \int |f_i|^{p_i} = \sum_{i=1}^n \frac{1}{p_i} \|f_i\|_{p_i} = 1.$$

This shows that $\prod_{i=1}^n f_i \in L^1$.

(d) Define $f_i^* = f_i / \|f_i\|_{p_i}$ for any $i = 1, \dots, n$. We have

$$\left(\int |f_i^*|^{p_i}\right)^{1/p_i} = \left(\int |f_i/\|f_i\|_{p_i}|^{p_i}\right)^{1/p_i} = \left(\frac{1}{\|f_i\|_{p_i}^{p_i}} \int |f_i|^{p_i}\right)^{1/p_i} = 1,$$

which shows that $f_i^* \in L^{p_i}$ and $\|f_i^*\|_{p_i} = 1$. By (c), $\prod_{i=1}^n f_i^* \in L^1$ and $\int |\prod_{i=1}^n f_i^*| \leqslant 1$. Since $\prod_{i=1}^n f_i = \left(\prod_{i=1}^n \|f_i\|_{p_i}\right)\left(\prod_{i=1}^n f_i^*\right)$, we have

$$\int \left|\prod_{i=1}^n f_i\right| = \left(\prod_{i=1}^n \|f_i\|_{p_i}\right) \int \left|\prod_{i=1}^n f_i^*\right| \leqslant \prod_{i=1}^n \|f_i\|_{p_i} < +\infty,$$

giving $\prod_{i=1}^n f_i \in L^1$, and

$$\left|\int \prod_{i=1}^n f_i\right| \leqslant \int \left|\prod_{i=1}^n f_i\right| = \left\|\prod_{i=1}^n f_i\right\|_1 \leqslant \prod_{i=1}^n \|f_i\|_{p_i}. \qquad \square$$

► EXERCISE 8.4 (8.1.4). *We have equality in Hölder's Inequality iff there are nonnegative numbers A and B, not both zero, with $A|f|^p = B|g|^{p'}$ μ-a. e. on Ω.*

PROOF. We have equality in Hölder's Inequality iff

$$\frac{|f|}{\|f\|_p} \cdot \frac{|g|}{\|g\|_{p'}} = \frac{1}{p}\frac{|f|^p}{\|f\|_p^p} + \frac{1}{p'}\frac{|g|^{p'}}{\|g\|_{p'}^{p'}} \quad \mu\text{-a. e. on } \Omega,$$

which holds iff the AM-GM holds equality, that is,

$$\frac{|f|^p}{\|f\|_p^p} = \frac{|g|^{p'}}{\|g\|_{p'}^{p'}} \quad \mu\text{-a.e. on } \Omega. \qquad \square$$

► EXERCISE 8.5 (8.1.5). *Given $f \in L^p$ $\left[1 < p < +\infty\right]$, there is $g \in L^{p'}$ with $\|g\|_{p'} = 1$ and $\int fg = \|f\|_p$.*

PROOF. Let $g = (f/\|f\|_p)^{p-1}$. Then

$$\int |g|^{p'} = \int \left|\left(\frac{f}{\|f\|_p}\right)^{p-1}\right|^{p'} = \int \left|\frac{f}{\|f\|_p}\right|^p = \frac{1}{\|f\|_p^p}\int |f|^p = \frac{\|f\|_p^p}{\|f\|_p^p} = 1,$$

i.e., $\|g\|_{p'} = 1$. We also have

$$\int |fg| = \int \frac{|f|^p}{\|f\|_p^{p-1}} = \|f\|_p. \qquad \square$$

► EXERCISE 8.6 (8.1.6). *We now explore conditions for equality in Minkowski's Inequality. Let $f, g \in L^p$.*

a. *When $p = 1$, $\|f + g\|_p = \|f\|_p + \|g\|_p$ iff there exists positive $\mathcal{F}/\mathcal{B}^*$-measurable $h > 0$ defined on Ω with $fh = g$ μ-a. e. on $\{\omega \in \Omega : f(\omega)g(\omega) \neq 0\}$.*

b. *For $1 < p < +\infty$, equality obtains iff there are nonnegative real numbers A and B, not both zero, such that $Af = Bg$ μ-a. e. on Ω.*

PROOF. **(a)** When $p = 1$, we have

$$\begin{aligned}\|f+g\|_1 = \|f\|_1 + \|g\|_1 &\iff \int |f+g| = \int |f| + \int |g| \\ &\iff \int \big(|f+g| - |f| - |g|\big) = 0 \\ &\iff |f+g| = |f| + |g| \quad \mu\text{-a.e.} \\ &\iff \exists \mathscr{F}/\mathscr{B}^*\text{-measurable } h > 0 \text{ defined on } \Omega \\ &\qquad \text{with } fh = g \ \mu\text{-a.e. on } [f(\omega)g(\omega) \neq 0].\end{aligned}$$

(b) When $1 < p < \infty$, we have

$$\begin{aligned}\|f+g\|_p^p &= \int |f+g|^p \\ &= \int |f+g| \cdot |f+g|^{p-1} \\ &\overset{*}{\leqslant} \int |f| \cdot |f+g|^{p-1} + \int |g| \cdot |f+g|^{p-1} \\ &\overset{**}{\leqslant} \|f\|_p \cdot \||f+g|^{p-1}\|_{p'} + \|g\|_p \cdot \||f+g|^{p-1}\|_{p'} \\ &= \big(\|f\|_p + \|g\|_p\big) \|f+g\|_p^{p-1}.\end{aligned}$$

Hence, the Minkowski's Inequality holds with equality iff $(*)$ and $(**)$ hold with equality. The result follows from Exercise 8.4 that $(**)$ immediately. □

► EXERCISE 8.7 (8.1.7). *Let* $1 \leqslant p, q, r < +\infty$ *be such that* $1/r = 1/p + 1/q$. *Let* $f \in L^p$ *and* $g \in L^q$. *Then* $fg \in L^r$ *and* $\|fg\|_r \leqslant \|f\|_p \|g\|_q$.

PROOF. Let $p' = p/r$ and $q' = q/r$. Then $1/p' + 1/q' = r/p + r/q = 1$. Let $f^* = f^r$ and $g^* = g^r$. Then

$$\int |f^*|^{p'} = \int |f^r|^{p/r} = \int |f|^p < +\infty,$$

and

$$\int |g^*|^{q'} = \int |g^r|^{q/r} = \int |g|^q < +\infty,$$

i.e., $f^* \in L^{p'}$ and $g^* \in L^{q'}$. By the AM-GM inequality, for any $\omega \in \Omega$,

$$\big|f^*(\omega)g^*(\omega)\big| = |f^*(\omega)| \cdot |g^*(\omega)| \leqslant \frac{1}{p'}|f^*(\omega)|^{p'} + \frac{1}{q'}|g^*(\omega)|^{q'}.$$

Integrate the left and right sides of the above display, obtaining

$$\int |fg|^r = \int |f^*g^*| \leqslant \frac{1}{p'} \int |f^*|^{p'} + \frac{1}{q'} \int |g^*|^{q'} < +\infty,$$

that is, $fg \in L^r$ and $f^*g^* \in L^1$. Then by Hölder's Inequality, we have

$$\|f^*g^*\|_1 \leqslant \|f^*\|_{p'} \|g^*\|_{q'};$$

therefore,

$$\begin{aligned}\int |fg|^r = \int |f^* g^*| &\leqslant \left(\int |f^*|^{p'}\right)^{1/p'} \left(\int |g^*|^{q'}\right)^{1/q'} \\ &= \left[\left(\int |f|^p\right)^{1/p} \left(\int |g|^q\right)^{1/q}\right]^r \\ &= \left(\|f\|_p \, \|g\|_q\right)^r;\end{aligned}$$

that is, $\|fg\|_r \leqslant \|f\|_p \, \|g\|_q$. □

► EXERCISE 8.8 (8.1.8). *If* $1 \leqslant p < +\infty$, *Minkowski's Inequality gives* $\left|\|f\|_p - \|g\|_p\right| \leqslant \|f - g\|_p$ *for every* $f, g \in L^p$.

PROOF. Write $f = (f - g) + g$. We first show that $f - g \in L^p$ when $f, g \in L^p$.

$$\int |f - g|^p \leqslant \int |f|^p + \int |g|^p < +\infty.$$

Then by the Minkowski's Inequality, we have

$$\|f\|_p = \|(f - g) + g\|_p \leqslant \|f - g\|_p + \|g\|_p.$$

Rearrange the above display and we get the desired result. □

8.2 The Riesz-Fischer Theorem

► EXERCISE 8.9 (8.2.1). *Return to the formal definition of* L^p.

a. *Write out the formal definition of convergence in* L^p.

b. *State and prove the formal version of the Riesz-Fischer Theorem.*

PROOF. **(a)** Let $\mathfrak{F}, \mathfrak{F}_1, \mathfrak{F}_2, \ldots \in L^p$. $\{\mathfrak{F}_n\}_{n=1}^{\infty}$ converges to $\mathfrak{F}$ in L^p, $\mathfrak{F}_n \xrightarrow{L^p} \mathfrak{F}$, if and only if $\lim_n \|\mathfrak{F}_n - \mathfrak{F}\|_p = 0$.

(b) Straightforward. □

► EXERCISE 8.10 (8.2.2). *Let* (X, ρ) *denote a generic metric space. Let* $\mathcal{C}_b(X)$ *denote the collection of continuous real-valued bounded functions on* X. *For* $f \in \mathcal{C}_b(X)$, *write* $\|f\| = \sup_{x \in X} |f(x)|$, *the usual supremum norm. Then* $\mathcal{C}_b(X)$ *is a Banach space.*

PROOF. Let $\{f_n\}$ be a Cauchy sequence in $\mathcal{C}_b(X)$. Then for every $\varepsilon > 0$, there exists $N_\varepsilon \in \mathbb{N}$ such that for all $n, m \geqslant N_\varepsilon$, we have

$$\|f_n - f_m\| = \sup_{x \in X} |f_n(x) - f_m(x)| < \varepsilon.$$

Therefore, for every $x \in X$, we get $|f_n(x) - f_m(x)| < \varepsilon$ for all $n, m \geqslant N_\varepsilon$; that is, $\{f_k(x)\}$ is Cauchy in $\mathbb{R}$. The completeness of $\mathbb{R}$ yields

$$f_k(x) \to f(x),$$

for some $f(x) \in \mathbb{R}$. Now fix $n \geqslant N_\varepsilon$. Since $|\cdot|$ is continuous, we get

$$|f_n(x) - f(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)| \leqslant \varepsilon.$$

Hence, for every $n \geqslant N_\varepsilon$, we have

$$\|f - f_n\| = \sup_{x \in X} |f_n(x) - f(x)| \leqslant \varepsilon.$$

What has been just shown is that $\|f - f_n\| \to 0$ as $n \to \infty$. Note that this implies that $f_n \to f$ uniformly on X. Thus, f is continuous since every f_n is continuous. Also,

$$\|f\| \leqslant \|f - f_n\| + \|f_n\| < \infty.$$

Hence $f \in \mathscr{C}_b(X)$ and so $\mathscr{C}_b(X)$ is a Banach space. □

► Exercise 8.11 (8.2.3). *A function f on $\mathbb{R}^k$ is said to* vanish at infinity *iff $f(\boldsymbol{x}) \to 0$ as $\|\boldsymbol{x}\| \to \infty$. Show that the collection of continuous functions on $\mathbb{R}^k$ that vanish at infinity is a Banach space relative to the supremum norm given in the previous exercise.*

Proof. Let $\mathscr{C}_0(\mathbb{R}^k)$ denote the collection of continuous functions on $\mathbb{R}^k$ that vanish at infinity. We use an alternative definition (Rudin, 1986, Definition 3.16): A complex function f on a locally compact Hausdorff space X is said to *vanish at infinity* if to every $\varepsilon > 0$ there exists a compact set $K \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \notin K$.

Let $\{f_n\}$ be a Cauchy sequence in $\mathscr{C}_0(\mathbb{R}^k)$, i.e., assume that $\{f_n\}$ converges uniformly. Then its pointwise limit function f is continuous. Given $\varepsilon > 0$, there exists an n so that $\|f_n - f\| < \varepsilon/2$ and there is a compact set K so that $|f_n(x)| < \varepsilon/2$ outside K. Hence $|f(x)| < \varepsilon$ outside K, and we have proved that f vanishes at infinity. Thus $\mathscr{C}_0(\mathbb{R}^k)$ is complete. □

► Exercise 8.12 (8.2.4). *Let $\mathscr{C}_c(\mathbb{R}^k)$ denote the collection of continuous functions on $\mathbb{R}^k$ with compact support, and again consider the supremum norm. This collection is dense in the collection in the previous exercise, but it fails to be a Banach space.*

Proof. Refer Hewitt and Stromberg (1965, §7) and Rudin (1986, p. 69-71). The *support* of a (complex) function f on a topological space X is the closure of the set $\{x \in X : f(x) \neq 0\}$.

Given $f \in \mathscr{C}_0(\mathbb{R}^k)$ and $\varepsilon > 0$, there is a compact set K so that $|f(x)| < \varepsilon$ outside K. Urysohn's lemma (Rudin, 1986, 2.12) gives us a function $g \in \mathscr{C}_c(\mathbb{R}^k)$

such that $0 \leqslant g \leqslant 1$ and $g(x) = 1$ on K. Put $h = fg$. Then $h \in \mathscr{C}_c(\mathbb{R}^k)$ and $\|f - h\| < \varepsilon$. This proves that $\overline{\mathscr{C}_c(\mathbb{R}^k)} = \mathscr{C}_0(\mathbb{R}^k)$. □

► EXERCISE 8.13 (8.2.5). *A sequence $\{f_n\}$ in L^p may converge in pth mean to some $f \in L^p$ but at the same time fail to converge pointwise to f at* any *point in Ω. Therefore, convergence in L^p does not in general imply convergence μ*-a. e.

PROOF. Consider $([0,1], \mathscr{B}[0,1], \lambda)$. Consider the sequence

$$\mathbb{1}_{[0,1/2]}, \mathbb{1}_{[1/2,1]}, \mathbb{1}_{[0,1/4]}, \mathbb{1}_{[1/4,1/2]}, \mathbb{1}_{[1/2,3/4]}, \mathbb{1}_{[3/4,1]}, \mathbb{1}_{[0,1/8]}, \ldots.$$

Then $f_n \xrightarrow{L^p} 0$, but obviously $f(x) \not\to 0$ for all $x \in [0,1]$. □

8.3 L^p SPACE: THE CASE $0 < p < 1$

► EXERCISE 8.14 (8.3.1). *Let $f, g \in L^p$, where $0 < p < 1$. We know that $f + g \in L^p$ by the Minkowski-like Inequality result given earlier.*

a. *We have $(a+b)^p \leqslant a^p + b^p$ for every $0 < a, b < \infty$.*

b. *From (a), we have $\int |f+g|^p \leqslant \int |f|^p + \int |g|^p$.*

c. *If we write $\|f-g\|_p^p$ for the distance between f and g, then this distance function is truly a metric, if we identify functions equal μ*-a. e. *on Ω.*

d. *Writing $\|f-g\|_p$ for the distance between f and g does* not *define a metric on L^p.*

PROOF. **(a)** If $0 < a = b < \infty$, we have

$$(a+b)^p = 2^p a^p \leqslant 2a^p = a^p + b^p.$$

Next we assume that $0 < a < b < \infty$. Since $0 < p < 1$, the function x^p defined on $(0,\infty)$ is concave. Write b as a convex combination of a and $a+b$ as follows:

$$b = \frac{a}{b}a + \frac{b-a}{b}(a+b).$$

Then

$$b^p = \left(\frac{a}{b}a + \frac{b-a}{b}(a+b)\right)^p \leqslant \frac{a}{b}a^p + \frac{b-a}{b}(a+b)^p;$$

that is,

$$(a+b)^p \leqslant \frac{b^{p+1} - a^{p+1}}{b-a} \leqslant \frac{(b-a)(a^p + b^p)}{b-a} = a^p + b^p,$$

where the second inequality holds since

$$(b-a)(a^p + b^p) = b^{p+1} - a^{p+1} + ab\left(a^{p-1} - b^{p-1}\right) \geqslant b^{p+1} - a^{p+1}.$$

(b) It follows from (a) that

$$\int |f+g|^p \leqslant \int (|f|+|g|)^p \leqslant \int (|f|^p+|g|^p) = \int |f|^p + \int |g|^p.$$

(c) We use the informal definition. To see $\|f-g\|_p^p$ is a metric on L^p, we need to verify:

- $0 \leqslant \|f-g\|_p^p < \infty$ for every $f,g \in L^p$. It is true because by (b):

$$0 \leqslant \|f-g\|_p^p = \int |f-g|^p \leqslant \int |f|^p + \int |g|^p < \infty.$$

- $\|f-f\|_p^p = 0$ for each $f \in L^p$, and $\|f-g\|_p^p = 0$ forces $f = g$ μ-a. e. on Ω. The first claim is obvious, so we focus on the second one. If $\|f-g\|_p^p = \int |f-g|^p = 0$, then $|f-g|^p = 0$ μ-a. e., then $f = g$ μ-a. e.

- $\|f-g\|_p^p = \|g-f\|_p^p$ for every $f,g \in L^p$. This is evident.

- $\|f-h\|_p^p \leqslant \|f-g\|_p^p + \|g-h\|_p^p$ for every $f,g,h \in L^p$. It also follows from (b):

$$\begin{aligned}\|f-h\|_p^p = \int |f-h|^p &= \int |(f-g)+(g-h)|^p \\ &\leqslant \int (|f-g|+|g-h|)^p \\ &\leqslant \int |f-g|^p + \int |g-h|^p \\ &= \|f-g\|_p^p + \|g-h\|_p^p.\end{aligned}$$

Thus, $\|f-g\|_p^p$ is a metric on L^p when $0<p<1$.

(d) It follows from the Minkowski-like equality that the triangle inequality fails for $\|f-g\|_p$ when $0<p<1$. □

► EXERCISE 8.15 (8.3.2). *Consider the space $\Omega = (0,1)$ and let $0<p<1$. Write $\mathcal{B}$ for the Borel subsets of $(0,1)$, and write λ for Lebesgue measure restricted to $\mathcal{B}$. We will show that there exists no norm $\|\ \|$ on L^p such that $\lim_k \|f_k\|_p = 0$ forces $\lim_k \|f_k\| = 0$.*

a. *Suppose that such a norm $\|\ \|$ exists. Then there is $C \geqslant 0$ such that $\|f\| \leqslant C\|f\|_p$ for each $f \in L^p$.*

   Pick the minimal such C from (a).

b. *There is $0<c<1$ with $\int_0^c |f|^p = \int_c^1 |f|^p = \frac{1}{2}\int_0^1 |f|^p$.*

c. *Let $g = f\mathbb{1}_{(0,c]}$ and $h = f\mathbb{1}_{(c,1]}$, so that $f = g+h$. Then $\|g\|_p = \|h\|_p = 2^{-1/p}\|f\|_p$ and $\|f\| \leqslant \|g\|+\|h\| \leqslant C\|g\|_p + C\|h\|_p = C\times 2^{1-1/p}\|f\|_p$. Use the minimality of C to obtain $C \leqslant C2^{1-1/p}$, and deduce $C=0$.*

d. *Conclude that* $\|f\| = 0$ *for every* $f \in L^p$*, and hence the assumption in (a) entails a contradiction.*

PROOF. **(a)** Suppose that for every $C \geqslant 0$ there is $f \in L^p$ with $\|f\| > C\|f\|_p$. Then for every $k \in \mathbb{N}$ there is $f_k \in L^p$ with $\|f_k\| > k\|f_k\|_p$. Define $g_k = f_k/\|f_k\|$ for each $k \in \mathbb{N}$. Then $\|g_k\| = 1$ and for every $k \in \mathbb{N}$

$$1 = \|g_k\| > k\|g_k\|_p,$$

i.e., $\|g_k\| < 1/k$. Hence, $\lim_k \|g_k\|_p = 0$, so $\lim_k \|g_k\| = 0$ by the assumption that such a norm exists. But $\lim_k \|g_k\| = 1$.

(b) The function $\int_0^x |f|^p$ is continuous and increasing with respect to x. The claim follows immediately.

(c) It follows from (b) that $\|g\|_p^p = \|h\|_p^p = \frac{1}{2}\|f\|^p$, i.e.,

$$\|g\|_p = \|h\|_p = 2^{-1/p}\|f\|_p.$$

Since $\| \|$ is a norm, we have

$$\|f\| = \|g + h\| \leqslant \|g\| + \|h\| \leqslant C\|g\|_p + C\|h\|_p = C2^{1-1/p}\|f\|_p.$$

The minimality of C implies that $C \leqslant C2^{1-1/p}$. Hence, $C = 0$.

(d) By (a) and (c) we get $\|f\| \leqslant 0\|f\|_p = 0$; that is, $\|f\| = 0$ for all $f \in L^p$. But then $\| \|$ is not a norm. A contradiction. □

► EXERCISE 8.16 (8.3.3). *Let* $0 < p_0 < \infty$ *and let* $f \in L^{p_0}$ *be nonnegative. Let* $E_1 = \{\omega \in \Omega : f(\omega) = 0\}$, $E_2 = \{\omega \in \Omega : 0 < f(\omega) \leqslant 1\}$, *and* $E_3 = \{\omega \in \Omega : f(\omega) > 1\}$.

a. $\lim_{p\to 0^+} \int_{E_2} |f|^p = \mu(E_2)$.

b. $\lim_{p\to 0^+} \int_{E_3} |f|^p = \mu(E_3)$.

c. $\lim_{p\to 0^+} \int |f|^p = \mu(\{\omega \in \Omega : f(\omega) \neq 0\})$.

PROOF. **(a)** The function x^p decreasing with respect to p when $0 < x \leqslant 1$. By MCT we have

$$\lim_{p\to 0^+} \int_{E_2} |f|^p = \int_{E_2} \lim_{p\to 0^+} |f|^p = \int_{E_2} 1 = \mu(E_2).$$

(b) It follows from DCT.

(c) Let $\{p_n\}$ be a decreasing sequence converging to 0. Then

$$\begin{aligned}\lim_n \int |f|^p = \lim_n \left(\int_{E_2} |f|^p + \int_{E_3} |f|^p\right) &= \lim_n \int_{E_2} |f|^p + \lim_n \int_{E_3} |f|^p \\ &= \mu(E_2) + \mu(E_3) \\ &= \mu[f(\omega) \neq 0].\end{aligned}$$

□

► EXERCISE 8.17 (8.3.4). *Say $\mu(\Omega) = 1$, and let $f \in L^1$ be nonnegative. Write* $\log 0 = -\infty$.

a. $\int \log f \leqslant \log \int f$ *if* $\log f \in L^1$.

b. *If* $\log f \notin L^1$, *then* $\int (\log f)^+ < \infty$ *under the assumption* $f \in L^1$, *so it must be the case that* $\int (\log f)^- = \infty$. *Conclude that even if* $\log f \notin L^1$, $\int \log f$ *still exists and equals* $-\infty$, *giving the inequality in (a).*

c. $(f^r - 1)/r$ *decreases to* $\log f$ *as* $r \to 0^+$, *hence* $\lim_{r\to 0^+}(\int f^r - 1)/r = \int \log f$.

d. *Verify the inequalities*

$$\frac{1}{r}\left[\int f^r - 1\right] \geqslant \frac{1}{r}\log \int f^r \geqslant \frac{1}{r}\int \log f^r = \int \log f.$$

e. *Conclude that* $\lim_{r\to 0^+} \|f\|_r$ *exists and equals* $\exp(\int \log f)$. *If* $\log f \notin L^1$, *this is interpreted as* $\lim_{r\to 0^+} \|f\|_r = 0$.

PROOF. **(a)** If $\|f\|_1 = \int f = 0$, then $f = 0$ μ-a. e.. Hence, $\int \log f = \log \int f = -\infty$. Now assume that $\|f\|_1 > 0$. Since $\log x \leqslant x - 1$ when $0 \leqslant x < \infty$, we have

$$\int \log \frac{f}{\|f\|_1} \leqslant \int \left(\frac{f}{\|f\|_1} - 1\right) = \frac{\int f}{\|f\|_1} - \int 1 = 0;$$

hence,

$$\int \log f \leqslant \log \|f\|_1 = \log \int f.$$

(b) Observe that

$$(\log f)^+(\omega) = \begin{cases} 0 & \text{if } f(\omega) \in [0,1] \\ \log f(\omega) & \text{if } f(\omega) \in (1,\infty]. \end{cases}$$

Also, $\log f(\omega) < f(\omega) - 1$ when $\omega \in (1,\infty]$. Since $f \in L^1$, we have $\int f < \infty$. Thus,

$$\int (\log f)^+ = \int_{[f(\omega)>1]} \log f \leqslant \int_{[f(\omega)]>1} (f-1) < \infty.$$

Therefore, it must be the case that $\int (\log f)^- = \infty$, and so

$$\int \log f = \int (\log f)^+ - \int (\log f)^- = -\infty.$$

(c) Fix an arbitrary $\omega \in \Omega$. We have

$$\lim_{r\to 0^+} \frac{f(\omega)^r - 1}{r} = \log f(\omega);$$

hence, $(f^r - 1)/r \downarrow \log f$ as $r \to 0^+$, and consequently,

$$\left(f - \frac{f^r - 1}{r}\right) \uparrow \left(f - \log f\right) \quad \text{as } r \to 0^+.$$

It follows from the MCT that

$$\begin{aligned}\lim_{r\to 0+}\int\left(f-\frac{f^r-1}{r}\right) &= \int f - \lim_{r\to 0+}\int\frac{f^r-1}{r} \\ &= \int f - \int \lim_{r\to 0+}\frac{f^r-1}{r} \\ &= \int f - \int \log f.\end{aligned}$$

Since $\int f < \infty$ and $\mu(\Omega) = 1$, we get the desired result.

(d) The first inequality follows from the fact $f^r \geqslant 0$ and under this case $\log f^r \leqslant f^r - 1$. The second inequality follows from (a) and (b).

(e) □

8.4 L^p Space: The Case $p = +\infty$

► Exercise 8.18 (8.4.2). *Consider the σ-finite measure space $(\Omega, \mathcal{F}, \mu)$.*

a. $f \in L^\infty$ *iff there is a bounded $\mathcal{F}/\mathcal{B}^*$-measurable function g on Ω such that* $f = g$ μ-a. e. *on Ω.*

b. *If $f \in L^\infty$, then* $\|f\|_\infty = \inf\{\sup_{\omega\in\Omega}|g(\omega)| : g$ *is as in (a)*$\}$.

Proof. **(a)** First assume that there exists a bounded $\mathcal{F}/\mathcal{B}^*$-measurable function g on Ω such that $f = g$ μ-a. e.. Then there exists $M \geqslant 0$ such that $|g| \leqslant M$. Hence, $|f| \leqslant M$ μ-a. e.; that is, $\operatorname{ess\,sup} f \leqslant M$, and so $f \in L^\infty$.

Now suppose that $f \in L^\infty$. Define g on Ω by letting

$$g(\omega) = \begin{cases} f(\omega) & \text{if } |f(\omega)| \leqslant \operatorname{ess\,sup} f \\ 0 & \text{otherwise.}\end{cases}$$

This g is bounded, $\mathcal{F}/\mathcal{B}^*$-measurable, and $f = g$ μ-a. e..

(b) We first show that $\sup_{\omega\in\Omega}|g(\omega)| \geqslant \|f\|_\infty$ for all g as in (a). Suppose that $\sup_{\omega\in\Omega}|g(\omega)| < \|f\|_\infty$. Define

$$A := \left\{\omega \in \Omega : |f(\omega)| > \sup_{\omega\in\Omega}|g(\omega)|\right\}.$$

Then $\mu(A) > 0$; for otherwise $\|f\|_\infty \leqslant \sup_{\omega\in\Omega}|g(\omega)|$. But which means that $f > g$ on A and $\mu(A) > 0$. A contradiction. This shows that

$$\inf\left\{\sup_{\omega\in\Omega}|g(\omega)| : g \text{ as in (a)}\right\} \geqslant \|f\|_\infty.$$

We next show the reverse inclusion. Let $B := \{\omega \in \Omega : |f(\omega)| \leqslant \|f\|_\infty\}$; then $\mu(B^c) = 0$. Let $g = f\mathbb{1}_B$. Then g is bounded, $\mathcal{F}/\mathcal{B}^*$-measurable, and $f = g$ μ-a. e.. Furthermore,

$$\sup_{\omega\in\Omega} |(f\mathbb{1}_B)(\omega)| \leqslant \|f\|_\infty.$$

This prove that $\|f\|_\infty \in \{\sup_{\omega\in\Omega} : |g(\omega)| : g \text{ as in (a)}\}$, and the proof is completed. □

► EXERCISE 8.19 (8.4.4). *Quickly prove the $p = \infty$ version of Hölder's Inequality.*

PROOF. Let $f \in L^\infty$ and $g \in L^1$. By Claim 1 we get

$$|fg| \leqslant \operatorname{ess\,sup} fg \quad \mu\text{-a. e.}$$

Observe that

$$\operatorname{ess\,sup} fg = |g| \operatorname{ess\,sup} f = |g| \cdot \|f\|_\infty,$$

so we have

$$\int |fg| \leqslant \int |g| \cdot \|f\|_\infty = \|f\|_\infty \|g\|_1 < \infty.$$

Hence, $fg \in L^1$ and $\|fg\|_1 \leqslant \|f\|_\infty \|g\|_1$. □

► EXERCISE 8.20 (8.4.5). *Let $(\Omega, \mathcal{F}, \mu)$ be such that $\mu(\Omega) < \infty$, and let f denote a bounded $\mathcal{F}/\mathcal{B}^*$-measurable function on Ω.*

a. *For every $1 \leqslant p \leqslant \infty$ we have $f \in L^p$, hence $\|f\|_p$ exists.*

b. $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$.

PROOF. **(a)** Since f is bounded, there exists $0 \leqslant M \leqslant \infty$ such that $|f| \leqslant M$. Then $\operatorname{ess\,sup} f \leqslant M$; that is, $\|f\|_\infty$ exists. Now consider $1 \leqslant p < \infty$. We have

$$\int |f|^p \leqslant \int M^p = M^p \mu(\Omega) < \infty,$$

i.e., $\|f\|_p$ exists.

(b) If $f = 0$ μ-a. e., then $\|f\|_p = \|f\|_\infty = 0$ for all p, and the claim is trivial. So assume that $f \neq 0$ on a set of positive measure, so that $\|f\|_\infty > 0$. We first show that $\liminf_{p\to\infty} \|f\|_p \geqslant \|f\|_\infty$. Let $t \in (0, \|f\|_\infty)$. By the Chebyshev's Inequality (Exercise 8.24) we have

$$\|f\|_p \geqslant t\mu[|f(\omega)| \geqslant t]^{1/p}.$$

If $\mu[|f(\omega)| \geqslant t] = \infty$, the claim is trivial. If $\mu[|f(\omega)| \geqslant t] < \infty$, then

$$\liminf_{p\to\infty} \|f\|_p \geqslant \liminf_{p\to\infty} t\mu[|f(\omega)| \geqslant t]^{1/p} = \lim_{p\to\infty} t\mu[|f(\omega)| \geqslant t]^{1/p} = t.$$

Since $t \in (0, \|f\|_\infty)$ is arbitrary, we have $\lim_{p\to\infty} \|f\|_p \geqslant \|f\|_\infty$.

We next show that $\limsup_{p\to\infty} \|f\|_p \leqslant \|f\|_\infty$. It follows from (a) that $\|f\|_\infty$ exists, and so $|f| \leqslant \|f\|_\infty$ μ-a. e.. Then

$$\|f\|_p^p = \int |f|^p \leqslant \int \|f\|_\infty^p = \|f\|_\infty^p \mu(\Omega);$$

that is, $\|f\|_p \leqslant \|f\|_\infty \mu(\Omega)^{1/p}$. Then

$$\limsup_{p\to\infty} \|f\|_p \leqslant \|f\|_\infty.$$

Summarizing the findings, we have $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$. □

8.5 Containment Relations for L^p Spaces

► Exercise 8.21 (8.5.1). *Consider the measure space* $(\mathbb{R}, \mathscr{B}, \lambda)$. *Let* $1 \leqslant p < q < \infty$, *and let* r *be such that* $1/q < r < 1/p$.

a. *Define* f *on* $\mathbb{R}$ *by writing* $f(x) = x^{-r}\mathbb{1}_{(0,1)}(x)$ *for each* $x \in \mathbb{R}$. *Then* $f \in L^p$ *but* $f \notin L^q$. *Therefore, we do not in general have* $L^p \subseteq L^q$ *when* $p < q$.

b. *Let* $g(x) = x^{-r}\mathbb{1}_{(1,\infty)}(x)$ *for every* $x \in \mathbb{R}$. *Then* $g \in L^q$ *but* $g \notin L^p$. *Therefore, we do not in general have* $L^q \subseteq L^p$ *when* $p < q$.

Proof. **(a)** We have $-rq < -1 < -rp$. Hence,

$$\int |f|^p = \int_{(0,1)} x^{-rp} = \frac{x^{1-rp}}{1-rp}\bigg|_0^1 = \frac{1}{1-rp} < \infty,$$

$$\int |f|^q = \int_{(0,1)} x^{-rq} = \frac{x^{1-rq}}{1-rq}\bigg|_0^1 = \infty;$$

that is, $f \in L^p$, but $f \notin L^q$.

(b) We have

$$\int |g|^q = \int_{(1,\infty)} x^{-rq} = \frac{x^{1-rq}}{1-rq}\bigg|_1^\infty = \frac{1}{rq-1} < \infty,$$

$$\int |g|^p = \int_{(1,\infty)} x^{-rp} = \frac{x^{1-rp}}{1-rq}\bigg|_1^\infty = \infty;$$

hence, $g \in L^q$ but $g \notin L^p$. □

Remark 8.22 (Folland 1999, p.185). Thus we see two reasons why a function f may fail to be in L^p: either $|f|^p$ blows up too rapidly near some point, or it fails to decay sufficiently rapidly at infinity. In the first situation the behavior of $|f|^p$ becomes worse as p increases, while in the second it becomes better. In other words, if $p < q$, functions in L^p can be locally more singular than functions in L^q, whereas functions in L^q can be globally more spread out than functions in L^p. See Figure 8.1.

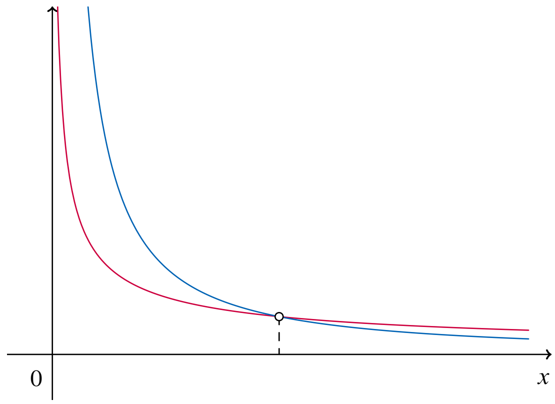

FIGURE 8.1. f^p and f^q.

▶ EXERCISE 8.23 (8.5.2). *Let $0 < p < r < \infty$. Then $L^p \cap L^\infty \subseteq L^r$, and for any $f \in L^p \cap L^\infty$ we have $\|f\|_r \leqslant \|f\|_p^{p/r} \|f\|_\infty^{1-p/r}$.*

PROOF. Let $A := \{\omega \in \Omega : |f(\omega)| \leqslant \|f\|_\infty\}$, so that $\mu(A^c) = 0$. Then

$$\int |f|^r = \int_A |f|^r = \int_A |f|^{r-p}|f|^p \leqslant \|f\|_\infty^{r-p} \int_A |f|^p = \|f\|_\infty^{r-p}\|f\|_p^p < \infty.$$

Hence, $f \in L^r$ and $\|f\|_r \leqslant \|f\|_p^{p/r}\|f\|_\infty^{1-p/r}$. □

▶ EXERCISE 8.24 (8.5.3). *For any $0 < p < \infty$ and $0 < M < \infty$ we have*

$$\left(\int |f|^p\right)^{1/p} \geqslant M\mu(\{\omega \in \Omega : |f(\omega)| \geqslant M\})^{1/p}.$$

PROOF. Let $E_M := \{\omega \in \Omega : |f(\omega)| \geqslant M\}$. Then

$$\|f\|_p^p = \int |f|^p \geqslant \int_{E_M} |f|^p \geqslant M^p \int_{E_M} 1 = M^p \mu(E_M);$$

that is, $\mu[|f(\omega)| \geqslant M] \leqslant (\|f\|_p/M)^p$. □

▶ EXERCISE 8.25 (8.5.4). *Let $0 < r < \infty$ and assume that $f \in L^r \cap L^\infty$, so that $f \in L^p$ for every $r < p < \infty$ by Exercise 8.23. We wish to show that $\lim_{p\to\infty}\|f\|_p = \|f\|_\infty$. Follow this outline:*

a. *Ignoring the trivial case where $f = 0$ μ-a.e. on Ω, let $f \neq 0$ on a set of positive measure, so that $\|f\|_\infty > 0$. Show that $\liminf_{p\to\infty}\|f\|_p \geqslant \|f\|_\infty$.*

b. *Show that $\limsup_{p\to\infty}\|f\|_p \leqslant \|f\|_\infty$.*

c. *Put (a) and (b) together to prove the claim.*

PROOF. **(a)** Pick an arbitrary $t \in (0, \|f\|_\infty)$. It follows from the Chebyshev's Inequality (Exercise 8.24) that

$$\|f\|_p \geqslant t \cdot \mu[|f(\omega)| \geqslant t]^{1/p}.$$

If $\mu[|f(\omega)| \geqslant t] = \infty$, then $\|f\|_p = \infty$ for all p, and the claim is trivial. If $\mu[|f(\omega)| \geqslant t] < \infty$, then $\lim_{p\to\infty} \mu[|f(\omega)| \geqslant t]^{1/p} = 1$ and so $\liminf_{p\to\infty} \|f\|_p \geqslant t$. Since $t \in (0, \|f\|_\infty)$ is chosen arbitrarily, we get

$$\liminf_{p\to\infty} \|f\|_p \geqslant \|f\|_\infty \tag{8.2}$$

(b) By Exercise 8.23 we have $\|f\|_p \leqslant \|f\|_r^{r/p} \|f\|_\infty^{1-r/p}$, for every $p \in (r, \infty)$. Hence,

$$\limsup_{p\to\infty} \|f\|_p \leqslant \|f\|_\infty. \tag{8.3}$$

(c) Combining (8.2) and (8.3) we get

$$\|f\|_\infty \leqslant \liminf_{p\to\infty} \|f\|_p \leqslant \limsup_{p\to\infty} \|f\|_p \leqslant \|f\|_\infty.$$

Hence, $\lim_{p\to\infty} \|f\|_p = \|f\|_\infty$. □

► EXERCISE 8.26 (8.5.5). *Let $\mu(\Omega) = 1$ and $1 \leqslant p \leqslant q \leqslant \infty$. Show for arbitrary f that*

$$\int |f| \leqslant \left(\int |f|^p\right)^{1/p} = \left(\int |f|^q\right)^{1/q} \leqslant \operatorname{ess\,sup} f,$$

so that $\|f\|_1 \leqslant \|f\|_p \leqslant \|f\|_q \leqslant \|f\|_\infty$.

PROOF. It follows from Claim 1 and the assumption that $\mu(\Omega) = 1$. □

8.6 Approximation

8.7 More Convergence Concepts

► EXERCISE 8.27 (8.7.1). *Prove the following simple claims.*

a. *Let $\{f_n\}_{n=1}^\infty$ denote a Cauchy sequence in L^p, where $0 < p < \infty$. Show that $\{f_n\}_{n=1}^\infty$ is a Cauchy sequence in measure: for every $\varepsilon > 0$ and $\delta > 0$ there is $N \in \mathbb{N}$ such that for every $n, m \geqslant N$ we have $\mu(\{\omega \in \Omega : |f_n(\omega) - f_m(\omega)| > \delta\}) < \varepsilon$.*

b. *Let $f, f_1, f_2, \ldots \in L^p$ and suppose that $f_n \xrightarrow{L^p} f$, where $0 < p < \infty$. If $g \in L^\infty$, then $fg, f_1g, f_2g, \ldots \in L^p$ and $f_ng \xrightarrow{L^p} fg$.*

PROOF. **(a)** Choose arbitrary $\varepsilon > 0$ and $\delta > 0$. It follows from Chebyshev's Inequality that

$$\mu[|f_n(\omega) - f_m(\omega)| \geqslant \delta] \leqslant \frac{\|f_n - f_m\|_p^p}{\delta^p}.$$

Let $\varepsilon' = \varepsilon^{1/p}\delta$. Then there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\|_p < \varepsilon'$ when $n, m \geqslant N$ since $\{f_n\}$ is Cauchy in L^p. Hence, when $n, m \geqslant N$ we get

$$\mu[|f_n(\omega) - f_m(\omega)| \geqslant \delta] < \frac{\varepsilon\delta^p}{\delta^p} = \varepsilon.$$

(b) We have

$$\|fg\|_p^p = \int |fg|^p = \int |f|^p|g|^p \leqslant \int |f|^p\|g\|_\infty^p = \|g\|_\infty^p \int |f|^p = \|f\|_p^p\|g\|_\infty^p < \infty;$$

that is, $fg \in L^p$. Similarly we can show that $f_n g \in L^p$ for all $n \in \mathbb{N}$. Finally,

$$\int |f_n g - fg|^p = \int |f_n - f|^p|g|^p = \|g\|_p \int |f_n - f|^p \to 0$$

since $f_n \xrightarrow{L^p} f$. □

► EXERCISE 8.28 (8.7.2). *While convergence in pth mean implies convergence in measure, it is not the case that convergence in measure implies convergence in $p$the mean.*

PROOF. Consider the probability space $([0,1], \mathscr{B}_{[0,1]}, \lambda)$, where λ is Lebesgue measure and set

$$f_n = 2^n \mathbb{1}_{(0,1/n)}.$$

Then

$$\lim_n \lambda(|f_n - 0| > \varepsilon) = \lim_n \lambda(0, 1/n) = 0.$$

However,

$$\int |f_n|^p = 2^{np}/n \to \infty.$$

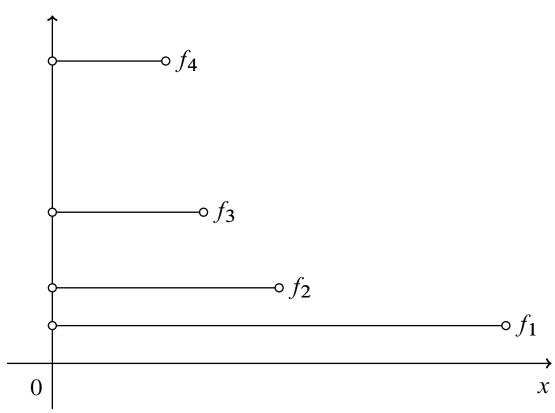

FIGURE 8.2. $f_n \xrightarrow{\mu} 0$, but $f_n \stackrel{L^p}{\not\longrightarrow} 0$.

Thus, convergence in measure does not imply L^p convergence. What can go wrong is that the nth function in the sequence can be huge on a very small set (see Figure 8.2). □

► EXERCISE 8.29 (8.7.3). *It is possible for a sequence $\{f_n\}_{n=1}^{\infty}$ in L^p to converge μ-a.e. to some $f \in L^p$ but not in pth mean. That is, convergence μ-a.e. does not force convergence in pth mean.*

PROOF. Consider the setting in the previous exercise again. □

► EXERCISE 8.30 (8.7.4). *It is possible for a sequence $\{f_n\}_{n=1}^{\infty}$ in L^p to converge in pth mean to zero, but $\{f_n\}_{n=1}^{\infty}$ converges at no point of Ω.*

PROOF. Consider $([0,1], \mathscr{B}_{[0,1]}, \lambda)$. Set

$$\begin{aligned} &f_1 = \mathbb{1}_{[0,1/2]}, && f_2 = \mathbb{1}_{[1/2,1]}, && \\ &f_3 = \mathbb{1}_{[0,1/3]}, && f_4 = \mathbb{1}_{[1/3,2/3]}, && f_5 = \mathbb{1}_{[2/3,1]}, \\ &f_6 = \mathbb{1}_{[0,1/4]}, && \cdots && \end{aligned}$$

For every $p > 0$,

$$\begin{aligned} &\int |f_1|^p = \int |f_2|^p = \frac{1}{2}, \\ &\int |f_3|^p = \int |f_4|^p = \int |f_5|^p = \frac{1}{3}, \\ &\int |f_6|^p = \frac{1}{4}. \end{aligned}$$

So $\int |f_n|^p \to 0$ and $f_n \xrightarrow{L^p} 0$. However, $\{f_n\}$ converges at no point. □

► EXERCISE 8.31 (8.7.5). *It is possible to have functions $f, f_1, f_2, \ldots \in L^{p_1} \cap L^{p_2}$ such that $f_n \xrightarrow{L^{p_1}} f$ but $f_n \not\xrightarrow{L^{p_2}} f$.*

PROOF. Consider $((0,\infty), \mathscr{B}_{(0,\infty)}, \lambda)$. Set

$$f_n = n^{-1}\mathbb{1}_{(n,2n)};$$

see Figure 8.3. Then

$$\int |f_n|^p = \frac{n}{n^p} = n^{1-p}.$$

The sequence $\{n^{1-p}\}$ converges if $p > 1$, and diverges if $p \leqslant 1$. Thus, $f_n \xrightarrow{L^p} 0$ when $1 < p < \infty$, but $\|f_n\|_1$ fails to converge to 0. □

8.8 Prelude to the Riesz Representation Theorem

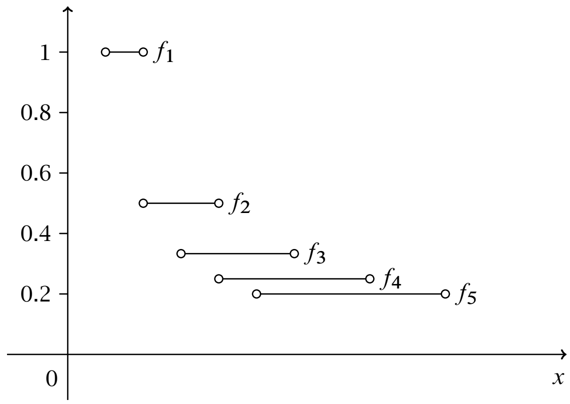

FIGURE 8.3.

9

THE RADON-NIKODYM THEOREM

9.1 The Radon-Nikodym Theorem, Part I

► Exercise 9.1 (9.1.1). *In the definition of an additive set function, show that the series $\sum_{n=1}^{\infty} \varphi(A_n)$ must converge absolutely.*

Proof. Observe that $\bigcup_{n=1}^{\infty} A_n = \bigcup_{k=1}^{\infty} A_{n_k}$ for every rearrangement $\{n_k\}_{k=1}^{\infty}$ of the positive integers, hence both $\sum_{n=1}^{\infty} \varphi(A_n)$ and $\sum_{k=1}^{\infty} \varphi(A_{n_k})$ should be defined and equal, that is, the series is unconditionally convergent. By the Riemann series theorem, it is absolutely convergent. □

► Exercise 9.2 (9.1.2). *In Claim 4, quickly verify that φ^- is a finite measure with support A^-.*

Proof. For all $A \in \mathscr{F}$ we have $A \cap A^- \subseteq A^-$, and the negativity of A^- with respect to φ implies that $\varphi^-(A) = -\varphi(A \cap A^-) \geqslant 0$. Therefore, φ^- is nonnegative. Next, $\varphi^-(\varnothing) = \varphi(\varnothing) = 0$. We now exhibit countable additivity for φ^-. Let $\{A_n\}_{n=1}^{\infty}$ denote a disjoint sequence of $\mathscr{F}$-sets. Then

$$\begin{aligned}\varphi^-\left(\bigcup_{n=1}^{\infty} A_n\right) = -\varphi\left(\left(\bigcup_{n=1}^{\infty} A_n\right) \cap A^-\right) &= -\varphi\left(\bigcup_{n=1}^{\infty} (A_n \cap A^-)\right)\\ &= -\sum_{n=1}^{\infty} \varphi\left(A_n \cap A^-\right)\\ &= \sum_{n=1}^{\infty} \left[-\varphi\left(A_n \cap A^-\right)\right]\\ &= \sum_{n=1}^{\infty} \varphi^-(A_n).\end{aligned}$$

This shows that φ^- is a measure. Since $\varphi^-(\Omega) = -\varphi(A^-) \in \mathbb{R}$, φ^- is a finite measure. To see that A^- is a support of φ^-, observe that $\varphi^-\left((A^-)^c\right) = -\varphi\left((A^-)^c \cap A^-\right) = -\varphi(\varnothing) = 0$. □

► EXERCISE 9.3 (9.1.3). *Suppose that* $\left(A^{+}, A^{-}\right)$ *and* $\left(B^{+}, B^{-}\right)$ *are Hahn decompositions with respect to an additive set function* φ. *Then* $\varphi\left(A^{+} \Delta B^{+}\right)=\varphi\left(A^{-} \Delta B^{-}\right)=0$.

PROOF. We first do the set operations:

$$
\begin{aligned}
A^{+} \Delta B^{+}=\left(A^{+} \cup B^{+}\right) \smallsetminus\left(A^{+} \cap B^{+}\right) & =\left(A^{+} \cup B^{+}\right) \smallsetminus\left(A^{-c} \cap B^{-c}\right) \\
& =\left(A^{+} \cup B^{+}\right) \smallsetminus\left(A^{-} \cup B^{-}\right)^{c} \\
& =\left(A^{+} \cup B^{+}\right) \cap\left(A^{-} \cup B^{-}\right) \\
& =\left(A^{+} \cap B^{-}\right) \cup\left(A^{-} \cap B^{+}\right),
\end{aligned}
$$

and $\left(A^{+} \cap B^{-}\right) \cap\left(A^{-} \cap B^{+}\right)=\varnothing$. Since $A^{+} \cap B^{-} \subseteq A^{+}$, we have $\varphi\left(A^{+} \cap B^{-}\right) \geqslant 0$; since $A^{+} \cap B^{-} \subseteq A^{+} \subseteq B^{-}$, we have $\varphi\left(A^{+} \cap B^{-} \subseteq A^{+}\right) \leqslant 0$; hence, $\varphi\left(A^{+} \cap B^{-} \subseteq A^{+}\right)=0$. Similarly, $\varphi\left(A^{-} \cap B^{+}\right)=0$, and so $\varphi\left(A^{+} \Delta B^{+}\right)=0$. Using this way, we can also show that $\varphi\left(A^{-} \Delta B^{-}\right)=0$. □

► EXERCISE 9.4 (9.1.4). *The Jordan decomposition of an additive set function* φ *is unique.*

PROOF. Let $\left(A^{+}, A^{-}\right)$ and $\left(B^{+}, B^{-}\right)$ denote Hahn decomposition of Ω with respect to φ. Let $\varphi_{A}^{+}(E)=\varphi\left(E \cap A^{+}\right)$ and $\varphi_{B}^{+}(E)=\varphi\left(E \cap B^{+}\right)$ for every $E \in \mathcal{F}$; define φ_{A}^{-} and φ_{B}^{-} similarly. Then $\varphi=\varphi_{A}^{+}-\varphi_{A}^{-}$ is the Jordan decomposition of φ relative to the Hahn decomposition $\left(A^{+}, A^{-}\right)$ and $\varphi=\varphi_{B}^{+}-\varphi_{B}^{-}$ is the Jordan decomposition of φ relative to the Hahn decomposition $\left(B^{+}, B^{-}\right)$. We now show that $\varphi_{A}^{+}=\varphi_{B}^{+}$ and $\varphi_{A}^{-}=\varphi_{B}^{-}$. For any $E \in \mathcal{F}$, we have

$$
\begin{aligned}
\varphi_{A}^{+}(E)=\varphi\left(E \cap A^{+}\right) & =\varphi\left(E \cap\left(B^{+} \cup B^{-}\right) \cap A^{+}\right) \\
& =\varphi\left(E \cap A^{+} \cap B^{+}\right)+\varphi\left(E \cap A^{+} \cap B^{-}\right) \\
& =\varphi\left(E \cap A^{+} \cap B^{+}\right),
\end{aligned}
$$

where $\varphi\left(E \cap A^{+} \cap B^{-}\right)=0$ since: (i) $E \cap A^{+} \cap B^{-} \subseteq A^{+}$ implies that $\varphi(E \cap A^{+} \cap B^{-}) \geqslant 0$; (ii) $E \cap A^{+} \cap B^{-} \subseteq B^{-}$ implies that $\varphi\left(E \cap A^{+} \cap B^{-}\right) \leqslant 0$. Similarly, we can show $\varphi_{B}^{+}=\varphi\left(E \cap A^{+} \cap B^{+}\right)=\varphi_{A}^{+}$ and $\varphi_{A}^{-}=\varphi_{B}^{-}$. □

► EXERCISE 9.5 (9.1.5). *This problem relates somewhat the notion of absolute continuity with the familiar* ε-δ *concepts.*

a. *Let* μ *and* ν *denote measures with common domain* $\mathcal{F}$ *and such that* ν *is finite. Then* $\nu \ll \mu$ *iff for every* $\varepsilon>0$ *there is* $\delta>0$ *such that* $\mu(A)<\delta$ *forces* $\nu(A)<\varepsilon$.

b. *The claim in (a) is not necessarily true if* ν *is infinite, since the condition* $\nu \ll \mu$ *does not imply the* ε-δ *condition.*

PROOF. **(a)** Suppose first that for every $\varepsilon > 0$ there is $\delta > 0$ such that $\mu(A) < \delta$ forces $\nu(A) < \varepsilon$. We desire to show that $\nu \ll \mu$. If $\mu(A) = 0$ and $\varepsilon > 0$ is given (and the corresponding δ is found), then $\mu(A) < \delta$, hence $\nu(A) < \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\nu(A) = 0$, whence $\nu \ll \mu$.

To show the other direction, suppose that there is $\varepsilon > 0$ such that for every $\delta > 0$ there is a set $A \in \mathscr{F}$ with $\mu(A) < \delta$ and $\nu(A) \geqslant \varepsilon$. In particular, there is $\varepsilon > 0$ such that there is a sequence $\{A_n\}_{n=1}^{\infty}$ of $\mathscr{F}$-sets with $\mu(A) < 1/n^2$ and $\nu(A) \geqslant \varepsilon$ for each $n \in \mathbb{N}$. Let $A = \overline{\lim} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m$. For every $n \in \mathbb{N}$ we have

$$\mu(A) \leqslant \mu\left(\bigcup_{m=n}^{\infty} A_m\right) \leqslant \sum_{m=n}^{\infty} \mu(A_m) < \sum_{m=n}^{\infty} \frac{1}{m^2},$$

so $\mu(A) = 0$. However, we also have

$$\nu(A) = \nu\left(\overline{\lim} A_n\right) \geqslant \overline{\lim}\, \nu(A_n) \geqslant \varepsilon > 0$$

by property (M10) in Section 2.2. This shows that there is $A \in \mathscr{F}$ such that $\mu(A) = 0$ and $\nu(A) > 0$, so $\nu \not\ll \mu$.

(b) Let $\Omega = \mathbb{Z}$, let $\mathscr{F} = 2^{\Omega}$, let ν denote the counting measure, so that ν is infinite, and let μ be such that $\mu(\{n\}) = \frac{1}{n^2}$ for each $n \in \mathbb{Z}$, so that μ is finite. □

► EXERCISE 9.6 (9.1.6). *Let μ, ν, ν_1, and ν_2 denote measures, each having common domain $\mathscr{F}$.*

a. *If $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$, then $\nu_1 + \nu_2 \perp \mu$.*

b. *If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then $\nu_1 + \nu_2 \ll \mu$.*

c. *If $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$, then $\nu_1 \perp \nu_2$.*

d. *If $\nu \ll \mu$ and $\nu \perp \mu$, then $\nu = 0$.*

e. *If $\mu \perp \mu$, then $\mu = 0$.*

f. *If μ and ν are σ-finite with $\nu \ll \mu$, then $\nu(\{\omega \in \Omega : \frac{d\nu}{d\mu}(\omega) = 0\}) = 0$.*

PROOF. **(a)** Let $\nu_1 \perp \mu$ and $\nu_2 \perp \mu$. Then there exist $D_1 \in \mathscr{F}$ with $\nu_1(D_1) = \mu(D_1^c) = 0$, and $D_2 \in \mathscr{F}$ with $\nu_2(D_2) = \mu(D_2^c) = 0$. Let $D = D_1 \cap D_2$. We show that D supports μ and D^c supports $\nu_1 + \nu_2$. As for μ, we have

$$\mu(D^c) = \mu(D_1^c \cup D_2^c) \leqslant \mu(D_1^c) + \mu(D_2^c) = 0.$$

As for $\nu_1 + \nu_2$, we have

$$(\nu_1 + \nu_2)(D) = \nu_1(D_1 \cap D_2) + \nu_2(D_1 \cap D_2) \leqslant \mu_1(D_1) + \nu_2(D_2) = 0.$$

Therefore, $(\nu_1 + \nu_2)(D) = \mu(D^c) = 0$, that is, $\nu_1 + \nu_2 \perp \mu$.

(b) If $\mu(A) = 0$, then $(\nu_1 + \nu_2)(A) = \nu_1(A) + \nu_2(A) = 0$; hence, $\nu_1 + \nu_2 \ll \mu$.

(c) Since $\nu_2 \perp \mu$, there exists $D \in \mathcal{F}$ with $\nu_2(D) = \mu(D^c) = 0$; since $\nu_1 \ll \mu$, $\mu(D^c) = 0$ forces $\nu_1(D^c) = 0$. Therefore, there is $D \in \mathcal{F}$ with $\nu_1(D^c) = \nu_2(D) = 0$, that is, $\nu_1 \perp \nu_2$.

(d) Since $\nu \perp \mu$, there is $D \in \mathcal{F}$ with $\nu(D) = \mu(D^c) = 0$. For any $E \in \mathcal{F}$, we have

$$\nu(E) = \nu(E \cap D) + \nu\left(E \cap D^c\right) = 0 + 0 = 0,$$

where $\nu(E \cap D) = 0$ since $\nu(E \cap D) \leqslant \nu(D) = 0$, and $\nu(E \cap D^c) \leqslant \nu(D^c) = 0$ since $\mu(D^c) = 0$ and $\nu \ll \mu$.

(e) Let $\nu = \mu$ in (d) and we get the result.

(f) We have

$$\nu[\mathrm{d}\nu/\mathrm{d}\mu = 0] = \int_{[\mathrm{d}\nu/\mathrm{d}\mu=0]} \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu = 0. \qquad \square$$

► Exercise 9.7 (9.1.8). *Let $f \in L^1(\Omega, \mathcal{F}, \mu)$. Define $\nu(E) = \int_E f\,\mathrm{d}\mu$ for every $E \in \mathcal{F}$.*

a. *ν is an additive set function such that $\nu^+(E) = \int_E f^+\,\mathrm{d}\mu$ and $\nu^-(E) = \int_E f^-\,\mathrm{d}\mu$ for every $E \in \mathcal{F}$.*

b. *If $A^+ = \{\omega \in \Omega : f(\omega) > 0\}$ and $A^- = A^{+c}$, then (A^+, A^-) is a Hahn decomposition with respect to ν.*

Proof. **(b)** We first show (b). For every $E \in \mathcal{F}$ with $E \subseteq A^+$, we have

$$\nu(E) = \int_E f\,\mathrm{d}\mu = \int_{E\cap A^+} f\,\mathrm{d}\mu = \int_E f\mathbb{1}_{A^+}\,\mathrm{d}\mu = \int_E f^+\,\mathrm{d}\mu \geqslant 0,$$

and for every $E \in \mathcal{F}$ with $E \subseteq A^-$ we have

$$\nu(E) = \int_E f\,\mathrm{d}\mu = \int_{E\cap A^-} f\,\mathrm{d}\mu = \int_E f\mathbb{1}_{A^-}\,\mathrm{d}\mu = \int_E f^-\,\mathrm{d}\mu \leqslant 0.$$

Hence, (A^+, A^-) is a Hahn decomposition with respect to ν.

(a) It follows from (L6) (p. 251) that ν is an additive set function. Now by part (b) and the uniqueness of Hahn decomposition (Exercise 9.3), we get the desired result. $\square$

► Exercise 9.8 (9.1.9). *Let $\mathfrak{M}$ denote the collection of additive set functions φ with domain $\mathcal{F}$.*

a. *$\mathfrak{M}$ is a linear space over $\mathbb{R}$: for $a, b \in \mathbb{R}$ and $\varphi_1, \varphi_2 \in \mathfrak{M}$ we have $a\varphi_1 + b\varphi_2 \in \mathfrak{M}$.*

b. *Given $\varphi \in \mathfrak{M}$, define $\|\varphi\| = \varphi^+(\Omega) + \varphi^-(\Omega)$, where $\varphi = \varphi^+ - \varphi^-$ is the Jordan decomposition of φ. Then $\|\ \|$ is a norm on $\mathfrak{M}$.*

c. *Is $\mathfrak{M}$ a Banach space?*

PROOF. **(a)** It is clear that $a\varphi_1 + b\varphi_2 : \mathcal{F} \to \mathbb{R}$, and for a disjoint sequence $\{A_n\} \subseteq \mathcal{F}$ we have

$$\begin{aligned}(a\varphi_1 + b\varphi_2)\left(\bigcup_{n=1}^{\infty} A_n\right) &= a\varphi_1\left(\bigcup_{n=1}^{\infty} A_n\right) + b\varphi_2\left(\bigcup_{n=1}^{\infty} A_n\right) \\ &= \sum_{n=1}^{\infty} a\varphi_1(A_n) + \sum_{n=1}^{\infty} b\varphi_2(A_n) \\ &= \sum_{n=1}^{\infty}\left[a\varphi(A_n) + b\varphi_2(A_n)\right] \\ &= \sum_{n=1}^{\infty}(a\varphi_1 + b\varphi_2)(A_n).\end{aligned}$$

(b) Clearly, $\|\varphi\| \geqslant 0$ for all $\varphi \in \mathfrak{M}$, and $\|\mathfrak{o}\| = 0$, where $\mathfrak{o}(E) = 0$ for all $E \in \mathcal{F}$. Now if $\|\varphi\| = 0$, then $\varphi^+(\Omega) + \varphi^-(\Omega) = 0$ implies that $\varphi^+(\Omega) = \varphi^-(\Omega) = 0$. Since φ^+ and φ^- are finite measures on $\mathcal{F}$ (by Claim 4, p. 373), for every $E \in \mathcal{F}$ we have $\varphi^+(E) \leqslant \varphi^+(\Omega) = 0$ and $\varphi^-(E) \leqslant \varphi^-(\Omega) = 0$; that is,

$$\varphi(E) = \varphi^+(E) - \varphi^-(E) = 0.$$

We finally show that the triangle inequality. Let $\varphi_1, \varphi_2 \in \mathfrak{M}$. Then

$$\begin{aligned}\|\varphi_1 + \varphi_2\| &= (\varphi_1 + \varphi_2)^+(\Omega) + (\varphi_1 + \varphi_2)^-(\Omega) \\ &\leqslant \varphi_1^+(\Omega) + \varphi_2^+(\Omega) + \varphi_1^-(\Omega) + \varphi^-(\Omega) \\ &= \|\varphi_1\| + \|\varphi_2\|.\end{aligned}$$

This proves that $(\mathfrak{M}, \|\ \|)$ is a normed space. □

► EXERCISE 9.9 (9.1.10). *Let (A^+, A^-) denote a Hahn decomposition of the additive set function φ, and let $\varphi = \varphi^+ - \varphi^-$ denote the Jordan decomposition. We have*

$$\begin{aligned}\varphi^+(A) &= \sup\{\varphi(E) : E \in \mathcal{F}, E \subseteq A\}, \\ \varphi^-(A) &= -\inf\{\varphi(E) : E \in \mathcal{F}, E \subseteq A\},\end{aligned}$$

for every $A \in \mathcal{F}$.

PROOF. Let $A, E \in \mathcal{F}$ with $E \subseteq A$. Then

$$\varphi(E) = \varphi(E \cap A^+) + \varphi(E \cap A^-) = \varphi^+(E) - \varphi^-(E) \leqslant \varphi^+(E) \leqslant \varphi^+(A).$$

Thus, $\varphi^+(A)$ is an upper bound of $\{\varphi(E) : E \in \mathcal{F}, E \subseteq A\}$. We next show that $\varphi^+(A)$ is actually in the former set: let $E = A \cap A^+$. Then $E \in \mathcal{F}$, $E \subseteq A$, and

$$\varphi(E) = \varphi(A \cap A^+) = \varphi^+(A).$$

We then have

$$\begin{aligned}\varphi^-(A) = \varphi^+(A) - \varphi(A) &= \sup\{\varphi(E) : E \in \mathcal{F}, E \subseteq A\} - \varphi(A) \\ &= \sup\{\varphi(E) - \varphi(A) : E \in \mathcal{F}, E \subseteq A\} \\ &= \sup\{-\varphi(A \smallsetminus E) : E \in \mathcal{F}, E \subseteq A\} \\ &= \sup\{-\varphi(F) : F \in \mathcal{F}, F \subseteq A\} \\ &= -\inf\{\varphi(F) : F \in \mathcal{F}, F \subseteq A\}. \qquad \square\end{aligned}$$

► EXERCISE 9.10 (9.1.12). *Let μ, ν, ν_1, ν_2 and ρ denote σ-finite measures having domain $\mathcal{F}$. We have the following claims.*

a. *If $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, then $\mathrm{d}(\nu_1 \pm \nu_2)/\mathrm{d}\mu = \mathrm{d}\nu_1/\mathrm{d}\mu \pm \mathrm{d}\mu_2/\mathrm{d}\mu$ μ-a.s. on Ω.*

b. *If $\nu \ll \mu$ and $\mu \ll \rho$, then $\nu \ll \rho$ and $\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\rho}$ μ-a.e. on Ω.*

c. *If $\nu \ll \mu$ and $\mu \ll \nu$, then $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \mathbb{1}_{[\mathrm{d}\mu/\mathrm{d}\nu > 0]} \times \frac{1}{\mathrm{d}\mu/\mathrm{d}\nu}$ μ-a.e. on Ω.*

d. *Let $\mu \ll \rho$ and $\nu \ll \rho$. Then $\nu \ll \mu$ if and only if $\rho[\frac{\mathrm{d}\nu}{\mathrm{d}\rho} > 0, \frac{\mathrm{d}\mu}{\mathrm{d}\rho} > 0] = 0$, in which case we have*

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \mathbb{1}_{[\mathrm{d}\mu/\mathrm{d}\rho > 0]} \times \frac{\mathrm{d}\nu/\mathrm{d}\rho}{\mathrm{d}\mu/\mathrm{d}\rho} \quad \mu\text{-a.s. on } \Omega.$$

PROOF. **(a)** Since $\nu_1 \ll \mu$ and $\nu_2 \ll \mu$, we get $\nu_1 \pm \nu_2 \ll \mu$, and that for every $E \in \mathcal{F}$,

$$\nu_1(E) = \int_E \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu}\,\mathrm{d}\mu, \quad \nu_2(E) = \int_E \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}\,\mathrm{d}\mu,$$

and

$$(\nu_1 \pm \nu_2)(E) = \int_E \frac{\mathrm{d}(\nu_1 \pm \nu_2)}{\mathrm{d}\mu}\,\mathrm{d}\mu.$$

Clearly, $(\nu_1 \pm \nu_2)(E) = \nu_1(E) \pm \nu_2(E)$. Hence,

$$\frac{\mathrm{d}(\nu_1 \pm \nu_2)}{\mathrm{d}\mu} = \frac{\mathrm{d}\nu_1}{\mathrm{d}\mu} \pm \frac{\mathrm{d}\nu_2}{\mathrm{d}\mu}.$$

(b) Let $\nu \ll \mu$ and $\mu \ll \rho$. Take an arbitrary $E \in \mathcal{F}$ so that $\rho(E) = 0$; then $\mu(E) = 0$; then $\nu(E) = 0$ and so $\nu \ll \rho$. Next, for every $E \in \mathcal{F}$, we have

$$\nu(E) = \int_E \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\mathrm{d}\mu, \quad \mu(E) = \int_E \frac{\mathrm{d}\mu}{\mathrm{d}\rho}\mathrm{d}\rho, \quad \text{and} \quad \nu(E) = \int_E \frac{\mathrm{d}\nu}{\mathrm{d}\rho}\mathrm{d}\rho.$$

It follows from (L14) (p. 259) that

$$\nu(E) = \int_E \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\,\mathrm{d}\mu = \int_E \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\rho}\,\mathrm{d}\rho;$$

that is, $\frac{\mathrm{d}\nu}{\mathrm{d}\rho} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\rho}$.

(c) It follows from (b) that

$$\frac{\mathrm{d}\nu}{\mathrm{d}\nu} = \frac{\mathrm{d}\nu}{\mathrm{d}\mu}\frac{\mathrm{d}\mu}{\mathrm{d}\nu}.$$

Therefore,

$$\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \mathbb{1}_{[\mathrm{d}\mu/\mathrm{d}\nu>0]} \times \frac{1}{\mathrm{d}\mu/\mathrm{d}\nu}.$$

(d) □

► EXERCISE 9.11 (9.1.15). *Suppose that μ and ν are σ-finite measures on $\mathscr{F}$. The Lebesgue decomposition of ν with respect to μ is unique. That is, if $\nu = \nu_{\mathrm{ac}} + \nu_{\mathrm{s}}$ where ν_{ac} and ν_{s} are σ-finite measures with $\nu_{\mathrm{ac}} \ll \mu$ and $\nu_{\mathrm{s}} \perp \mu$, and if in addition $\nu = \nu'_{\mathrm{ac}} + \nu'_{\mathrm{s}}$ where ν'_{ac} and ν'_{s} are σ-finite measures with $\nu'_{\mathrm{ac}} \ll \mu$ and $\nu'_{\mathrm{s}} \perp \mu$, then $\nu_{\mathrm{ac}} = \nu'_{\mathrm{ac}}$ and $\nu_{\mathrm{s}} = \nu'_{\mathrm{s}}$.*

PROOF. Since $\nu_{\mathrm{s}} \perp \mu$, there exists $A \in \mathscr{F}$ such that A supports ν_{s} and A^c supports μ; that is,

$$\nu_{\mathrm{s}}(A^c) = \mu(A) = 0.$$

Since $\nu'_{\mathrm{s}} \perp \mu$, there exists $B \in \mathscr{F}$ such that B supports ν'_{s} and B^c supports μ; that is,

$$\nu'_{\mathrm{s}}(B^c) = \mu(B) = 0.$$

Since $\nu_{\mathrm{s}}(A^c \cap B^c) \leqslant \nu_{\mathrm{s}}(A^c) = 0$, and $\nu'_{\mathrm{s}}(A^c \cap B^c) \leqslant \nu'_{\mathrm{s}}(B^c) = 0$, we have that $A \cup B$ supports both ν_{s} and ν'_{s}. Since $\mu(A \cup B) \leqslant \mu(A) + \mu(B) = 0$, we have that $(A \cup B)^c$ supports μ. Let $S := A \cup B$, so that

$$\mu(S) = \nu_{\mathrm{s}}(S^c) = \nu'_{\mathrm{s}}(S^c) = 0$$

We now show that $\nu_{\mathrm{ac}} = \nu'_{\mathrm{ac}}$. For every $E \in \mathscr{F}$, we have

$$\begin{aligned}
\nu_{\mathrm{ac}}(E) = \nu_{\mathrm{ac}}(E \cap S^c) + \nu_{\mathrm{ac}}(E \cap S) &= \nu_{\mathrm{ac}}(E \cap S^c) && [\nu_{\mathrm{ac}} \ll \mu] \\
&= \nu_{\mathrm{ac}}(E \cap S^c) + \nu_{\mathrm{s}}(E \cap S^c) \\
&= \nu(E \cap S^c) \\
&= \nu'_{\mathrm{ac}}(E \cap S^c) + \nu'_{\mathrm{s}}(E \cap S^c) \\
&= \nu'_{\mathrm{ac}}(E \cap S^c) \\
&= \nu'_{\mathrm{ac}}(E \cap S^c) + \nu'_{\mathrm{ac}}(E \cap S) \\
&= \nu'_{\mathrm{ac}}(E).
\end{aligned}$$

Hence, $\nu_{\mathrm{ac}} = \nu'_{\mathrm{ac}}$, and so $\nu_{\mathrm{s}} = \nu'_{\mathrm{s}}$. □

10

PRODUCTS OF TWO MEASURE SPACES

10.1 Product Measures

Remark 10.1.

$$
\begin{aligned}
&(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)\,,\\
&(A \cup B) \times (C \cup D) \neq (A \times C) \cup (B \times D)\,,\\
&A \times (B \cap C) = (A \times B) \cap (A \times C)\,,\\
&A \times (B \cup C) = (A \times B) \cup (A \times C)\,.
\end{aligned}
$$

► Exercise 10.2 (10.1.1). *Let Ω_1 denote an uncountable set, and let $\mathcal{F}_1$ denote the σ-field of subsets of Ω_1 that are at most countable or have at most countable complements. Let Ω_2 and $\mathcal{F}_2$ be identical to Ω_1 and $\mathcal{F}_1$, respectively. Let $D = \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 : \omega_1 = \omega_2\}$. We have $D_{\omega_1} \in \mathcal{F}_2$ and $D^{\omega_2} \in \mathcal{F}_1$ for every $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$, but $D \notin \mathcal{F}_1 \otimes \mathcal{F}_2$.*

Proof. For every ω_1, we have $D_{\omega_1} = \{\omega_2\}$ with $\omega_2 = \omega_1$. For every $\omega_2 \in \Omega_2$, we have $D^{\omega_2} = \{\omega_1\}$ with $\omega_1 = \omega_2$. Hence, $D_{\omega_1} \in \mathcal{F}_2$ and $D^{\omega_2} \in \mathcal{F}_1$. □

► Exercise 10.3 (10.1.2). *Let $A \subseteq \Omega_1$ and $B \subseteq \Omega_2$.*

a. *Suppose that $A \times B \neq \varnothing$. Then $A \times B \in \mathcal{F}_1 \otimes \mathcal{F}_2$ iff $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$.*

b. *Suppose that $A \times B = \varnothing$. Then obviously $A \times B \in \mathcal{F}_1 \otimes \mathcal{F}_2$, but it is not always the case that $A \in \mathcal{F}_1$ and $B \in \mathcal{F}_2$.*

Proof. **(a)** The *if* part is evident since $\mathcal{F}_1 \times \mathcal{F}_2 \subseteq \mathcal{F}_1 \otimes \mathcal{F}_2$. Now take an arbitrary $\omega_1 \in A$ (such a point exists because $A \times B \neq \varnothing$ implies that $A \neq \varnothing$). Then $(A \times B)_{\omega_1} = B \in \mathcal{F}_2$. Similarly we show that $A \in \mathcal{F}_1$.

(b) If there exists $A \notin \mathcal{F}_1$, then we have $A \times \varnothing = \varnothing \in \mathcal{F}_1 \otimes \mathcal{F}_2$. But obviously $A \notin \mathcal{F}_1$. □

► Exercise 10.4 (10.1.3). *Prove the following set-theoretical facts.*

a. *Let $A_1 \times B_1$ and $A_2 \times B_2$ both be nonempty. Then $A_1 \times B_1 \subseteq A_2 \times B_2$ iff $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$.*

b. *Let $A_1 \times B_1$ and $A_2 \times B_2$ both be nonempty. Then $A_1 \times B_1 = A_2 \times B_2$ iff $A_1 = A_2$ and $B_1 = B_2$.*

c. *Let $A \times B$, $A_1 \times B_1$, and $A_2 \times B_2$ be nonempty. Then $A \times B$ is the disjoint union of $A_1 \times B_1$ and $A_2 \times B_2$ iff either (i) A is the disjoint union of A_1 and A_2 and $B = B_1 = B_2$ or (ii) $A = A_1 = A_2$ and B is the disjoint union of B_1 and B_2.*

d. *The "only if" parts of (a) and (b) do not necessarily hold for empty Cartesian products. What about (c)?*

Proof. **(a)** The *if* part is automatic, so we only do the *only if* part. Suppose that $A_1 \times B_1 \subseteq A_2 \times B_2$. If, say, $A_1 \nsubseteq A_2$, then there exists $\omega_1' \in A_1 \smallsetminus A_2$. Take an arbitrary $\omega_2 \in B_1$. Then $(\omega_1', \omega_2) \in A_1 \times B_1$ but $(\omega_1', \omega_2) \notin A_2 \times B_2$. A contradiction.

(b) Using the fact that $A_1 \times B_1 = A_2 \times B_2$ iff $A_1 \times B_1 \subseteq A_2 \times B_2$ and $A_2 \times B_2 \subseteq A_1 \times B_1$, and the result in (a), we get the desired outcome.

(c) Straightforward. □

► Exercise 10.5 (10.1.5). *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ denote σ-fields on Ω_1 and Ω_2, respectively. It may not be the case that $\mathcal{F}_1 \times \mathcal{F}_2$ is a σ-field on $\Omega_1 \times \Omega_2$.*

Proof. Consider $(\mathbb{R}^k, \mathcal{B}^k, \lambda_k)$ and $(\mathbb{R}^m, \mathcal{B}^m, \lambda_m)$. Then $\mathcal{B}^k \times \mathcal{B}^m \subset \mathcal{B}^k \otimes \mathcal{B}^m$. □

► Exercise 10.6 (10.1.6). *Prove Claims 2(b) and 3(b) by mimicking the proofs of Claims 2(a) and 3(a).*

Proof. **(2(b))** We show that if $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, then $E^{\omega_2} \in \mathcal{F}_1$ for every $\omega_2 \in \Omega_2$. Define

$$\mathcal{D} = \{E \in \mathcal{F}_1 \otimes \mathcal{F}_2 : E^{\omega_2} \in \mathcal{F}_1 \text{ for every } \omega_2 \in \Omega_2\}.$$

First observe that $\Omega_1 \times \Omega_2 \in \mathcal{F}_1 \times \mathcal{F}_2 \subseteq \mathcal{F}_1 \otimes \mathcal{F}_2$ and $(\Omega_1 \times \Omega_2)^{\omega_2} = \Omega_1 \in \mathcal{F}_1$ for every $\omega_2 \in \Omega_2$. Therefore, $\Omega_1 \times \Omega_2 \in \mathcal{D}$. Next, if $E \in \mathcal{D}$, then we have $E^c \in \mathcal{F}_1 \otimes \mathcal{F}_2$ and $(E^c)^{\omega_2} = (E^{\omega_2})^c \in \mathcal{F}_1$ for every $\omega_2 \in \Omega_2$, so that $E^c \in \mathcal{D}$. Next, if $\{E_n\}$ is a sequence of $\mathcal{D}$-sets, then $\bigcup E_n \in \mathcal{F}_1 \otimes \mathcal{F}_2$ and $(\bigcup E_n)^{\omega_2} = \bigcup (E_n)^{\omega_2} \in \mathcal{F}_1$ for every $\omega_2 \in \Omega_2$, whence $\bigcup E_n \in \mathcal{D}$. Therefore, $\mathcal{D}$ is a σ-field on $\Omega_1 \times \Omega_2$ and $\mathcal{D} \subseteq \mathcal{F}_1 \otimes \mathcal{F}_2$. We desire to strengthen this inclusion to an equality. To do this, let $E_1 \in \mathcal{F}_1$ and $E_2 \in \mathcal{F}_2$. Then for every $\omega_2 \in \Omega_2$ we have

$$(E_1 \times E_2)^{\omega_2} = \begin{cases} E_1 & \text{if } \omega_2 \in E_2 \\ \varnothing & \text{if } \omega_2 \notin E_2 \end{cases} \in \mathcal{F}_1.$$

This shows that $\mathcal{F}_1 \times \mathcal{F}_2 \subseteq \mathcal{D}$. Since $\mathcal{D}$ is a σ-filed, we have $\mathcal{F}_1 \otimes \mathcal{F}_2 \subseteq \mathcal{D}$. This yields $\mathcal{D} = \mathcal{F}_1 \otimes \mathcal{F}_2$.

(3(b)) We show that if $f\colon \Omega_1 \times \Omega_2 \to \overline{\mathbb{R}}$ be $\mathcal{F}_1 \otimes \mathcal{F}_2/\mathcal{B}^*$-measurable, then f^{ω_2} is $\mathcal{F}_1/\mathcal{B}^*$-measurable for every $\omega_2 \in \Omega_2$. To do this, first consider the case of $f = \mathbb{1}_E$, where $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$. Next, pick $\omega_1 \in \Omega_1$ and $\omega_2 \in \Omega_2$. We have

$$(\mathbb{1}_E)^{\omega_2}(\omega_1) = 1 \iff \mathbb{1}_E(\omega_1, \omega_2) = 1 \iff (\omega_1, \omega_2) \in E \iff \omega_1 \in E^{\omega_2} \iff \mathbb{1}_{E^{\omega_2}}(\omega_1) = 1,$$

and hence

$$(\mathbb{1}_E)^{\omega_2} = \mathbb{1}_{E^{\omega_2}}.$$

By Claim 2(b), $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$ forces $E^{\omega_2} \in \mathcal{F}_1$. Therefore, if $f = \mathbb{1}_E$ where $E \in \mathcal{F}_1 \otimes \mathcal{F}_2$, we have that f^{ω_2} is the indicator function $\mathbb{1}_{E^{\omega_2}}$ of the $\mathcal{F}_1$-set E^{ω_2}, and hence is $\mathcal{F}_1/\mathcal{B}^*$-measurable. This proves 3(b) when f is an indicator function on $\Omega_1 \times \Omega_2$ of a set in $\mathcal{F}_1 \otimes \mathcal{F}_2$.

Next, let $f = \sum_{i=1}^n c_i \mathbb{1}_{E_i}$, where $E_1, \ldots, E_n \in \mathcal{F}_1 \otimes \mathcal{F}_2$ are disjoint with union $\Omega_1 \times \Omega_2$ and $c_1, \ldots, c_n \in \mathbb{R}$, so that f is an $\mathcal{F}_1 \otimes \mathcal{F}_2/\mathcal{B}^*$-measurable simple function on $\Omega_1 \times \Omega_2$. For every $\omega_2 \in \Omega_2$, observe that $f^{\omega_2} = \sum_{i=1}^n c_i \mathbb{1}_{(E_i)^{\omega_2}}$, a finite linear combination of the indicator functions $(\mathbb{1}_{E_1})^{\omega_2}, \ldots, (\mathbb{1}_{E_n})^{\omega_2}$, and each of these is $\mathcal{F}_1/\mathcal{B}^*$-measurable by the previous paragraph. It follows that f^{ω_2} is $\mathcal{F}_1/\mathcal{B}^*$-measurable for every $\omega_2 \in \Omega_2$. This proves the result when f is an $\mathcal{F}_1 \otimes \mathcal{F}_2/\mathcal{B}^*$-measurable simple function on $\Omega_1 \times \Omega_2$.

Next, suppose that f is a nonnegative $\mathcal{F}_1 \otimes \mathcal{F}_2/\mathcal{B}^*$-measurable function on $\Omega_1 \times \Omega_2$. There exists a nondecreasing sequence $\{s_n\}$ of nonnegative finite-valued $\mathcal{F}_1 \otimes \mathcal{F}_2/\mathcal{B}^*$-measurable simple functions on $\Omega_1 \times \Omega_2$ with $\lim_n s_n = f$. By the previous paragraph, we have that $(s_n)^{\omega_2}$ is $\mathcal{F}_1/\mathcal{B}^*$-measurable for every $\omega_2 \in \Omega_2$ and $n \in \mathbb{N}$. Since

$$f^{\omega_2} = \left(\lim_n s_n\right)^{\omega_2} = \lim_n (s_n)^{\omega_2}$$

for every $\omega_2 \in \Omega_2$, we have that f^{ω_2} is the limit of a sequence of $\mathcal{F}_1/\mathcal{B}^*$-measurable functions and hence is itself $\mathcal{F}_1/\mathcal{B}^*$-measurable. This proves the result when f is a nonnegative $\mathcal{F}_1 \otimes \mathcal{F}_2/\mathcal{B}^*$-measurable function on $\Omega_1 \times \Omega_2$.

Finally, if f is a general $\mathcal{F}_1 \otimes \mathcal{F}_2/\mathcal{B}^*$-measurable function on $\Omega_1 \times \Omega_2$, then the functions f^+ and f^-, both being nonnegative $\mathcal{F}_1 \otimes \mathcal{F}_2/\mathcal{B}^*$-measurable functions on $\Omega_1 \times \Omega_2$, are such that $(f^+)^{\omega_2}$ and $(f^-)^{\omega_2}$ are $\mathcal{F}_1/\mathcal{B}^*$-measurable for every $\omega_2 \in \Omega_2$. Observing that

$$f^{\omega_2} = (f^+ - f^-)^{\omega_2} = (f^+)^{\omega_2} - (f^-)^{\omega_2}$$

for every $\omega_2 \in \Omega_2$, we see for every $\omega_2 \in \Omega_2$ that f^{ω_2} is the difference of two $\mathcal{F}_1/\mathcal{B}^*$-measurable functions on $\Omega_1 \times \Omega_2$ and hence is $\mathcal{F}_1/\mathcal{B}^*$-measurable. This completes the proof. □

► EXERCISE 10.7 (10.1.7). *The product of* $(\mathbb{R}^k, \mathcal{B}^k, \lambda_k)$ *and* $(\mathbb{R}^m, \mathcal{B}^m, \lambda_m)$ *is*

$$(\mathbb{R}^{k+m}, \mathcal{B}^{k+m}, \lambda_{k+m}).$$

In other words, $\mathscr{B}^k \otimes \mathscr{B}^m = \mathscr{B}^{k+m}$ *and* $\lambda_k \otimes \lambda_m = \lambda_{k+m}$*.*

Proof. We first show that $\mathscr{B}^k \otimes \mathscr{B}^m = \mathscr{B}^{k+m}$ visa showing that $\mathscr{B}^k \times \mathscr{B}^m \subset \mathscr{B}^{k+m}$ (proper subset). Consider the projection $\pi_k : \mathbb{R}^{k+m} \to \mathbb{R}^k$. Let $\mathscr{O}^k$ and $\mathscr{O}^{k+m}$ be the set of open sets of $\mathbb{R}^k$ and $\mathbb{R}^{k+m}$, respectively. Endowed with Tychonoff' topology, π_k is continuous. Hence,

$$\pi_k^{-1}(\mathscr{B}^k) = \pi_k^{-1}(\sigma(\mathscr{O}^k)) = \sigma(\pi_k^{-1}(\mathscr{O}^k)) \subseteq \sigma(\mathscr{O}^{k+m}) = \mathscr{B}^{k+m}.$$

Similarly, we have $\pi_m^{-1}(\mathscr{B}^m) \subseteq \mathscr{B}^{k+m}$. Therefore,

$$\mathscr{B}^k \times \mathscr{B}^m = \pi_k^{-1}(\mathscr{B}^k) \cap \pi_m^{-1}(\mathscr{B}^m) \subseteq \mathscr{B}^{k+m}.$$

To see that the containment is strict, observe that the open unit ball D in $\mathscr{B}^{k+m}$ cannot be written as $A_1 \times A_2$ with $A_1 \subseteq \mathbb{R}^k$ and $A_2 \subseteq \mathbb{R}^m$, let along with $A_1 \in \mathscr{B}^k$ and $A_2 \in \mathscr{B}^m$. From the above argument, we have

$$\mathscr{B}^k \otimes \mathscr{B}^m = \sigma(\mathscr{B}^k \times \mathscr{B}^m) \subseteq \mathscr{B}^{k+m}.$$

Define $\mathscr{A}_1$ = intervals of the form $(-\infty, x]$. We have $\mathscr{A}_1^{k+m} = \mathscr{A}_1^k \times \mathscr{A}_1^m \subseteq \mathscr{B}^k \times \mathscr{B}^m$; hence,

$$\mathscr{B}^{k+m} = \sigma(\mathscr{A}_1^{k+m}) \subseteq \sigma(\mathscr{B}^k \times \mathscr{B}^m) = \mathscr{B}^k \otimes \mathscr{B}^m.$$

It follows from Claim 4 of Section 4.2 that $\lambda_{k+m}(A \times B) = \lambda_k(A)\lambda_m(B)$ for every $A \in \mathscr{B}^k$ and $B \in \mathscr{B}^m$. Since $(\mathbb{R}^k, \mathscr{B}^k, \lambda_k)$ and $(\mathbb{R}^m, \mathscr{B}^m, \lambda_m)$ are σ-finite, by Claim 6 we have $\lambda_{k+m} = \lambda_k \otimes \lambda_m$. □

► Exercise 10.8 (10.1.8). *Let* $\Omega_1 = \Omega_2 = [0,1]$*. Let* $\mathscr{F}_1 = \mathscr{F}_2$ *denote the Borel subsets of* $[0,1]$*. Let* μ_1 *denote Lebesgue measure restricted to* $\mathscr{F}_1$*, and let* μ_2 *denote the counting measure on* $[0,1]$*. Let* $E = \{(\omega_1, \omega_2) \in \Omega_1 \times \Omega_2 : \omega_1 = \omega_2\}$*.*

a. $E \in \mathscr{F}_1 \otimes \mathscr{F}_2$.

b. $\int_{\Omega_1} \mu_2(E_{\omega_1})\,\mathrm{d}\mu_1(\omega_1) = 1$.

c. $\int_{\Omega_2} \mu_1(E^{\omega_2})\,\mathrm{d}\mu_2(\omega_2) = 0$.

Proof. **(a)** We prove $E \in \mathscr{F}_1 \otimes \mathscr{F}_2$ by showing that E is closed in $[0,1] \times [0,1]$. It is true because $[0,1]$ is Hausdorff (see Willard, 2004, Theorem 13.7).

(b) Since μ_2 is a counting measure, we have

$$\int_{\Omega_1} \mu_2(E_{\omega_1})\,\mathrm{d}\mu_1(\omega_1) = \int_{\Omega_1} \mu_2(\omega_2)\,\mathrm{d}\mu_1(\omega_1) = \int_{\Omega_1} 1\,\mathrm{d}\mu_1(\omega_1) = 1.$$

(c) We have

$$\int_{\Omega_2} \mu_1(E^{\omega_2})\,\mathrm{d}\mu_2(\omega_2) = \int_{\Omega_2} \mu_1(\omega_1)\,\mathrm{d}\mu_2(\omega_2) = \int_{\Omega_2} 0\,\mathrm{d}\mu_2(\omega_2) = 0.$$ □

► Exercise 10.9 (10.1.10, Cavalieri's Principle). *If* $E, F \in \mathscr{F}_1 \otimes \mathscr{F}_2$ *are such that* $\mu_2(E_{\omega_1}) = \mu_2(F_{\omega_1})$ *for every* $\omega_1 \in \Omega_1$*, then* $\mu_1 \otimes \mu_2(E) = \mu_1 \otimes \mu_2(F)$*.*

PROOF. We have

$$\mu_1 \otimes \mu_2(E) = \int_{\Omega_1} \mu_2(E_{\omega_1})\,\mathrm{d}\mu_1 = \int_{\Omega_1} \mu_2(F_{\omega_1})\,\mathrm{d}\mu_1 = \mu_1 \otimes \mu_2(F). \qquad \square$$

10.2 The Fubini Theorems

11

ARBITRARY PRODUCTS OF MEASURE SPACES

11.1 Notation and Conventions

▶ Exercise 11.1 (11.1.1). *Let Ω_1 denote a nonempty set, and let $\mathcal{A}$ denote a nonempty collection of subsets of Ω_1. Let Ω_2 denote a nonempty set, and let $\mathcal{B}$ denote a nonempty collection of subsets of Ω_2.*

a. *Let $B \subseteq \Omega_2$ be nonempty. Then $\sigma_{\Omega_1 \times B}(\mathcal{A} \times \{B\}) = \sigma(\mathcal{A}) \times \{B\}$.*

b. $\sigma(\mathcal{A} \times \mathcal{B}) = \sigma(\sigma(\mathcal{A}) \times \sigma(\mathcal{B}))$.

Proof. **(a)** Since $\mathcal{A} \times \{B\} \subseteq \sigma(\mathcal{A}) \times \{B\}$, and $\sigma(\mathcal{A}) \times \{B\}$ is a σ-field on $\Omega_1 \times B$, we get

$$\sigma_{\Omega_1 \times B}(\mathcal{A} \times \{B\}) \subseteq \sigma(\mathcal{A}) \times \{B\}.$$

To see the converse inclusion, define

$$\mathcal{C} := \left\{A \in \sigma(\mathcal{A}) : A \times B \in \sigma_{\Omega \times B}(\mathcal{A} \times \{B\})\right\}.$$

If $A \in \mathcal{A}$, then $A \times B \in \mathcal{A} \times \{B\} \subseteq \sigma_{\Omega_1 \times B}(\mathcal{A} \times \{B\})$, so $A \in \mathcal{C}$; thus $\mathcal{A} \subseteq \mathcal{C}$. We then show that $\mathcal{C}$ is a σ-field. (i) $\Omega_1 \in \mathcal{C}$. (ii) If $A \in \mathcal{C}$, then $(A \times B)^c = A^c \times B \in \sigma_{\Omega_1 \times B}(\mathcal{A} \times \{B\})$, i.e., $A^c \in \mathcal{C}$. (iii) If $\{A_n\}_{n=1}^{\infty} \subseteq \mathcal{C}$, then $(\bigcup A_n) \times B = \bigcup (A_n \times B)$, i.e., $\bigcup A_n \in \mathcal{C}$. Therefore, $\sigma(\mathcal{A}) = \mathcal{C}$.

(b) Since $\mathcal{A} \times \mathcal{B} \subseteq \sigma(\mathcal{A}) \times \sigma(\mathcal{B})$, we have

$$\sigma(\mathcal{A} \times \mathcal{B}) \subseteq \sigma(\sigma(\mathcal{A}) \times \sigma(\mathcal{B})).$$

Next, for every $B \in \sigma(\mathcal{B})$ we have $\sigma(\mathcal{A}) \times \{B\} = \sigma_{\Omega_1 \times B}(\mathcal{A} \times \{B\}) \subseteq \sigma(\mathcal{A} \times \mathcal{B})$ by (a). Therefore,

$$\bigcup_{B \in \sigma(\mathcal{B})} \left[\sigma(\mathcal{A}) \times \{B\}\right] \subseteq \sigma(\mathcal{A} \times \mathcal{B});$$

that is, $\sigma(\mathcal{A}) \times \sigma(\mathcal{B}) \subseteq \sigma(\mathcal{A} \times \mathcal{B})$. But then $\sigma(\sigma(\mathcal{A}) \times \sigma(\mathcal{B})) \subseteq \sigma(\mathcal{A} \times \mathcal{B})$. □

▶ Exercise 11.2 (11.1.2). *Prove the claim in the Identification Lemma for the case where P is a two-element set, which case is really the only one that we use.*

PROOF. By the assumption, $P = \{1, 2\}$. Then

$$\mathcal{N}_0 = \mathcal{F}_{D_1} \times \mathcal{F}_{D_2}.$$

Let $\mathcal{A}_1$ denote the collection of sets of the form $\bigtimes_{i \in D_1} A_i$, where $A_i \in \mathcal{F}_i$ for each $i \in D_1$ and at most finitely many A_i's differ from Ω_i. Then $\mathcal{F}_{D_1} = \sigma(\mathcal{A}_1)$. Let $\mathcal{A}_2$ denote the collection of sets of the form $\bigtimes_{i \in D_2} A_i$, where $A_i \in \mathcal{F}_i$ for each $i \in D_2$ and at most finitely many A_i's differ from Ω_i. Then $\mathcal{F}_{D_2} = \sigma(\mathcal{A}_2)$, and

$$\mathcal{N}_1 = \mathcal{A}_1 \times \mathcal{A}_2.$$

Therefore, $\sigma(\mathcal{N}_0) = \sigma(\mathcal{N}_1)$ iff

$$\sigma(\sigma(\mathcal{A}_1) \times \sigma(\mathcal{A}_2)) = \sigma(\mathcal{A}_1 \times \mathcal{A}_2).$$

The above equality follows from Exercise 11.1(b) immediately. □

► EXERCISE 11.3 (11.1.3). *Prove the Identification Lemma in full generality for the case where P is an arbitrary set.*

PROOF. By definition, $\mathcal{N}_0$ is the collection of sets $\bigtimes_{p \in P} A_{D_p}$, where $A_{D_p} \in \mathcal{F}_{D_p}$ for each $p \in P$ and $A_{D_p} \neq \Omega_{D_p}$ for at most finitely many $p \in P$. Further, $\mathcal{N}_1$ is the collection of sets of the form

$$\bigtimes_{p \in P} \left(\bigtimes_{i \in D_p} A_i \right),$$

where $A_i \neq \Omega_i$ for at most finitely many $i \in \bigcup_{p \in P} D_p$. For each $p \in P$, let $\mathcal{A}_{D_p}$ denote the collection of $\bigtimes_{i \in D_p} A_i$. We then have

$$\sigma(\mathcal{N}_0) = \sigma\left(\bigtimes_{p \in P} \mathcal{F}_{D_p} \right),$$

$$\sigma(\mathcal{N}_1) = \sigma\left(\bigtimes_{p \in P} \mathcal{A}_{D_p} \right).$$

Notice that $\mathcal{F}_{D_p} = \sigma(\mathcal{A}_{D_p})$ for every $p \in P$. Thus we need to show that

$$\sigma\left(\bigtimes_{p \in P} \sigma(\mathcal{A}_{D_p}) \right) = \sigma\left(\bigtimes_{p \in P} \mathcal{A}_{D_p} \right).$$

Generalizing the result in Exercise 11.1(b) yields the desired outcome. □

► EXERCISE 11.4 (11.1.4). *Show that $\mathcal{E}_F$ is a semiring on Ω_F.*

PROOF. Given a finite subset $F \subseteq I$, we define $\mathcal{E}_F$ by writing

$$\mathcal{E}_F = \left\{ \bigtimes_{i \in F} A_i : A_i \in \mathcal{F}_i \text{ for every } i \in F \right\}.$$

Clearly, $\varnothing \in \mathcal{E}_F$. Take two sets $B, C \in \mathcal{E}_F$ and write them as $B = \bigtimes_{i \in F} B_i$ and $C = \bigtimes_{i \in F} C_i$, where $B_i, C_i \in \mathcal{F}_i$ for each $i \in F$. Then

$$B \cap C = \left(\bigtimes_{i \in F} B_i \right) \cap \left(\bigtimes_{i \in F} C_i \right) = \bigtimes_{i \in F} (B_i \cap C_i) \in \mathcal{E}_F.$$

Finally, suppose that $\varnothing \neq B \subseteq C$ (otherwise the proof is trivial). Then $B_i \subseteq C_i$ for every $i \in F$. It is easy to see that $C \smallsetminus B$ is a finite disjoint union of $\mathcal{E}_F$-sets. ◻

► EXERCISE 11.5 (11.1.5). *Let $\mathcal{A}$ denote a semiring on Ω_1, and let $\mathcal{B}$ denote a semiring on Ω_2. Then $\mathcal{A} \times \mathcal{B}$ is a semiring on $\Omega_1 \times \Omega_2$.*

PROOF. It is evident that $\mathcal{A} \times \mathcal{B}$ contains $\varnothing$ and is a π-system. Now let $A_1 \times B_1 \subseteq A_2 \times B_2$, where $A_1, A_2 \in \mathcal{A}$ and $B_1, B_2 \in \mathcal{B}$. Then $A_1 \subseteq A_2$ and $B_1 \subseteq B_2$, and

$$(A_2 \times B_2) \smallsetminus (A_1 \times B_1) = [A_1 \times (B_2 \smallsetminus B_1)] \sqcup [(A_2 \smallsetminus A_1) \times B_2].$$

Observe that $B_2 \smallsetminus B_1$ may be written as a finite disjoint union $\bigsqcup_{i=1}^k D_i$ of $\mathcal{B}$-sets, and $A_2 \smallsetminus A_1$ may be written as a finite disjoint union $\bigsqcup_{j=1}^{\ell} C_j$. It follows that

$$\begin{aligned}(A_2 \times B_2) \smallsetminus (A_1 \times B_1) &= \left[A_1 \times \left(\bigsqcup_{i=1}^{k} D_i \right) \right] \sqcup \left[\left(\bigsqcup_{j=1}^{\ell} C_j \right) \times B_2 \right] \\ &= \left[\bigsqcup_{i=1}^{k} (A_1 \times D_i) \right] \sqcup \left[\bigsqcup_{j=1}^{\ell} (C_j \times B_2) \right].\end{aligned}$$

Hence $(A_2 \times B_2) \smallsetminus (A_1 \times B_1)$ is a finite disjoint union of sets in $\mathcal{A} \times \mathcal{B}$. ◻

11.2 CONSTRUCTION OF THE PRODUCT MEASURE

► EXERCISE 11.6 (11.2.1). *Refer to the proof of Claim 4.*

a. *Prove Subclaim 1.*

b. *Prove Subclaim 2.*

c. *Why can't we use the same type of proof as used to demonstrate the finite additivity of μ to show that μ as defined on $\mathcal{H}$ is countably additive?*

PROOF. **(a)** We show that if $A_{F_1} \in \mathcal{F}_{F_1}$, then there exists $C_{F_1 \cup F_2} \in \mathcal{F}_{F_1 \cup F_2}$ with the property that

$$\Phi_{F_1}^{-1}(A_{F_1} \times \Omega_{F_1^c}) = \Phi_{F_1 \cup F_2}^{-1}(C_{F_1 \cup F_2} \times \Omega_{(F_1 \cup F_2)^c}). \tag{11.1}$$

Define the collection

$$\mathcal{C} := \left\{A_{F_1} \in \mathcal{F}_{F_1} : \text{there is } C_{F_1 \cup F_2} \in \mathcal{F}_{F_1 \cup F_2} \text{ such that (11.1) holds}\right\}.$$

Then $\mathcal{C} \subseteq \mathcal{F}_{F_1}$.

We first show that $\mathcal{E}_{F_1} \subseteq \mathcal{C}$. Take an arbitrary $\bigtimes_{i \in F_1} A_i \in \mathcal{E}_{F_1}$. Then

$$\Phi_{F_1}^{-1}\left[\left(\bigtimes_{i \in F_1} A_i\right) \times \Omega_{F_1^c}\right] = \bigtimes_{i \in I} Q_i,$$

where $Q_i = A_i$ for each $i \in F_1$ and $Q_i = \Omega_i$ for each $i \in F_1^c$. Define the set

$$C_{F_1 \cup F_2} = \bigtimes_{i \in F_1 \cup F_2} R_i,$$

where $R_i = A_i$ for each $i \in F_1$ and $R_i = \Omega_i$ for each $i \in F_2$. We have $C_{F_1 \cup F_2} \in \mathcal{E}_{F_1 \cup F_2} \subseteq \mathcal{F}_{F_1 \cup F_2}$, and

$$\Phi_{F_1 \cup F_2}^{-1}\left[C_{F_1 \cup F_2} \times \Omega_{(F_1 \cup F_2)^c}\right] = \bigtimes_{i \in I} Q_i.$$

Comparing the last two displayed equations shows that $\bigtimes_{i \in F_1} A_i \in \mathcal{C}$. That is, we have $\mathcal{E}_{F_1} \subseteq \mathcal{C}$.

We now turn to showing that $\mathcal{C}$ is a σ-filed on Ω_{F_1}. We first show that $\Omega_{F_1} \in \mathcal{C}$. This is because

$$\Phi_{F_1}^{-1}(\Omega_{F_1} \times \Omega_{F_1^c}) = \Omega = \Phi_{F_1 \cup F_2}^{-1}(\Omega_{F_1 \cup F_2} \times \Omega_{(F_1 \cup F_2)^c}),$$

and $\Omega_{F_1 \cup F_2} \in \mathcal{F}_{F_1 \cup F_2}$. We now discuss closure under complementation. Suppose that $A_{F_1} \in \mathcal{C}$, and let $C_{F_1 \cup F_2} \in \mathcal{F}_{F_1 \cup F_2}$ be such that (11.1) holds. Then

$$\begin{aligned}
\Phi_{F_1}^{-1}(A_{F_1}^c \times \Omega_{F_1^c}) &= \Phi_{F_1}^{-1}\left[(\Omega_{F_1} \times \Omega_{F_1^c}) \setminus (A_{F_1} \times \Omega_{F_1^c})\right] \\
&= \Phi_{F_1}^{-1}(\Omega_{F_1} \times \Omega_{F_1^c}) \setminus \Phi_{F_1}^{-1}(A_{F_1} \times \Omega_{F_1^c}) \\
&= \Phi_{F_1 \cup F_2}^{-1}\left(\Omega_{F_1 \cup F_2} \times \Omega_{(F_1 \cup F_2)^c}\right) \setminus \Phi_{F_1 \cup F_2}^{-1}\left(C_{F_1 \cup F_2} \times \Omega_{(F_1 \cup F_2)^c}\right) \\
&= \Phi_{F_1 \cup F_2}^{-1}\left[\left(\Omega_{F_1 \cup F_2} \times \Omega_{(F_1 \cup F_2)^c}\right) \setminus \left(C_{F_1 \cup F_2} \times \Omega_{(F_1 \cup F_2)^c}\right)\right] \\
&= \Phi_{F_1 \cup F_2}^{-1}\left(C_{F_1 \cup F_2}^c \times \Omega_{(F_1 \cup F_2)^c}\right).
\end{aligned}$$

Since $C_{F_1 \cup F_2}^c \in \mathcal{F}_{F_1 \cup F_2}$, it follows that $A_{F_1}^c \in \mathcal{C}$.

We now show that $\mathcal{C}$ is closed under countable intersections. Let $\{A_{F_1}^{(n)}\} \subseteq \mathcal{C}$, and let $C_{F_1 \cup F_2}^{(n)} \in \mathcal{F}_{F_1 \cup F_2}$ denote the corresponding sets for $A_{F_1}^{(n)}$ that satisfies (11.1) for every $n \in \mathbb{N}$:

$$\Phi_{F_1}^{-1}\left(A_{F_1}^{(n)} \times \Omega_{F_1^c}\right) = \Phi_{F_1 \cup F_2}^{-1}\left(C_{F_1 \cup F_2}^{(n)} \times \Omega_{(F_1 \cup F_2)^c}\right).$$

We have

$$
\begin{aligned}
\Phi_{F_1}^{-1}\left[\left(\bigcap_{n=1}^{\infty} A_{F_1}^{(n)}\right)\times \Omega_{F_1^c}\right] &= \Phi_{F_1}^{-1}\left[\bigcap_{n=1}^{\infty}\left(A_{F_1}^{(n)}\times\Omega_{F_1^c}\right)\right]\\
&= \bigcap_{n=1}^{\infty}\Phi_{F_1}^{-1}\left(A_{F_1}^{(n)}\times\Omega_{F_1^c}\right)\\
&= \bigcap_{n=1}^{\infty}\Phi_{F_1\cup F_2}^{-1}\left(C_{F_1\cup F_2}^{(n)}\times\Omega_{(F_1\cup F_2)^c}\right)\\
&= \Phi_{F_1\cup F_2}^{-1}\left[\bigcap_{n=1}^{\infty}\left(C_{F_1\cup F_2}^{(n)}\times\Omega_{(F_1\cup F_2)^c}\right)\right]\\
&= \Phi_{F_1\cup F_2}^{-1}\left[\left(\bigcap_{n=1}^{\infty}C_{F_1\cup F_2}^{(n)}\right)\times\Omega_{(F_1\cup F_2)^c}\right].
\end{aligned}
$$

Since $\bigcap_{n=1}^{\infty} C_{F_1\cup F_2}^{(n)} \in \mathcal{F}_{F_1\cup F_2}$, it follows that $\bigcap_{n=1}^{\infty} A_{F_1}^{(n)} \in \mathcal{C}$. Therefore, $\mathcal{C}$ is a σ-filed on Ω_{F_1}, and $\mathcal{E}_{F_1} \subseteq \mathcal{C}$. Hence, $\mathcal{F}_{F_1} = \mathcal{C}$.

(b) We prove that if $A_{F_1} \in \mathcal{F}_{F_1}$, then

$$\Phi_{F_1}^{-1}(A_{F_1}\times\Omega_{F_1^c}) = \Phi_{F_3}^{-1}\left[\Phi_{F_1,F_3}^{-1}(A_{F_1}\times\Omega_{F_3\setminus F_1})\times\Omega_{F_3^c}\right]. \tag{11.2}$$

Let

$$\mathcal{C} = \left\{A_{F_1}\in\mathcal{F}_{F_1} : (11.2) \text{ holds for } A_{F_1}\right\}.$$

We first show that $\mathcal{E}_{F_1} \subseteq \mathcal{C}$. Pick $\bigtimes_{i\in F_1} A_i \in \mathcal{E}_{F_1}$. Then

$$\Phi_{F_1}^{-1}\left[\left(\bigtimes_{i\in F_1} A_i\right)\times\Omega_{F_1^c}\right] = \bigtimes_{i\in I} Q_i,$$

where $Q_i = A_i$ for each $i \in F_1$ and $Q_i = \Omega_i$ for each $i \in F_1^c$. Similarly, we have

$$\Phi_{F_1,F_3}^{-1}\left[\left(\bigtimes_{i\in F_1} A_i\right)\times\Omega_{F_3\setminus F_1}\right] = \bigtimes_{i\in F_3} R_i,$$

where $R_i = A_i$ for each $i \in F_1$ and $R_i = \Omega_i$ for each $i \in F_3 \setminus F_1$. Hence,

$$\Phi_{F_3}^{-1}\left[\Phi_{F_1,F_3}^{-1}\left(\left(\bigtimes_{i\in F_1} A_{F_1}\right)\times\Omega_{F_3\setminus F_1}\right)\times\Omega_{F_3^c}\right] = \Phi_{F_3}^{-1}\left(\bigtimes_{i\in F_3} R_i\right) = \bigtimes_{i\in I} S_i,$$

where

$$S_i = \begin{cases} R_i & \text{if } i\in F_3\\ \Omega_i & \text{if } i\in F_3^c\end{cases} = \begin{cases} A_i & \text{if } i\in F_1\\ \Omega_i & \text{if } i\in F_3\setminus F_1\\ \Omega_i & \text{if } i\in F_3^c\end{cases} = \begin{cases} A_i & \text{if } i\in F_1\\ \Omega_i & \text{if } i\in F_1^c.\end{cases}$$

Therefore, $\bigtimes_{i\in F_1} A_i \in \mathcal{C}$; that is, $\mathcal{E}_{F_1} \subseteq \mathcal{C}$.

We next show that $\mathcal{C}$ is a σ-field on Ω_{F_1}. It is clear that

$$\Phi_{F_1}^{-1}(\Omega_{F_1} \times \Omega_{F_1^c}) = \Omega = \Phi_{F_3}^{-1}\left[\Phi_{F_1,F_3}^{-1}(\Omega_{F_1} \times \Omega_{F_3\setminus F_1}) \times \Omega_{F_3^c}\right],$$

so $\Omega_{F_1} \in \mathcal{C}$. Now suppose that $A_{F_1} \in \mathcal{C}$. Then

$$\begin{aligned}
\Phi_{F_1}^{-1}\left(A_{F_1}^c \times \Omega_{F_1^c}\right) &= \Phi_{F_1}^{-1}\left[\left(\Omega_{F_1} \times \Omega_{F_1^c}\right) \setminus \left(A_{F_1} \times \Omega_{F_1^c}\right)\right] \\
&= \Phi_{F_1}^{-1}\left(\Omega_{F_1} \times \Omega_{F_1^c}\right) \setminus \Phi_{F_1}^{-1}\left(A_{F_1} \times \Omega_{F_1^c}\right) \\
&= \Omega \setminus \Phi_{F_3}^{-1}\left[\Phi_{F_1,F_3}^{-1}\left(A_{F_1} \times \Omega_{F_3\setminus F_1}\right) \times \Omega_{F_3^c}\right] \\
&= \Phi_{F_3}^{-1}\left[\left(\Omega_{F_3} \times \Omega_{F_3^c}\right) \setminus \left(\Phi_{F_1,F_3}^{-1}\left(A_{F_1} \times \Omega_{F_3\setminus F_1}\right) \times \Omega_{F_3^c}\right)\right] \\
&= \Phi_{F_3}^{-1}\left[\left(\Omega_{F_3} \setminus \Phi_{F_1,F_3}^{-1}\left(A_{F_1} \times \Omega_{F_3\setminus F_1}\right)\right) \times \Omega_{F_3^c}\right] \\
&= \Phi_{F_3}^{-1}\left[\Phi_{F_1,F_3}^{-1}\left(A_{F_1}^c \times \Omega_{F_3\setminus F_1}\right) \times \Omega_{F_3^c}\right];
\end{aligned}$$

that is, $A_{F_1}^c \in \mathcal{C}$ whenever $A_{F_1} \in \mathcal{C}$. We finally show that $\mathcal{C}$ is closed under countable intersections. Take an arbitrary sequence $\{A_{F_1}^{(n)}\} \subseteq \mathcal{C}$. Then

$$\begin{aligned}
\Phi_{F_1}^{-1}\left[\left(\bigcap_{n=1}^{\infty} A_{F_1}^{(n)}\right) \times \Omega_{F_1^c}\right] &= \Phi_{F_1}^{-1}\left[\bigcap_{n=1}^{\infty}\left(A_{F_1}^{(n)} \times \Omega_{F_1^c}\right)\right] \\
&= \bigcap_{n=1}^{\infty} \Phi_{F_1}^{-1}\left(A_{F_1}^{(n)} \times \Omega_{F_1^c}\right) \\
&= \bigcap_{n=1}^{\infty} \Phi_{F_3}^{-1}\left[\Phi_{F_1,F_3}^{-1}\left(A_{F_1}^{(n)} \times \Omega_{F_3\setminus F_1}\right) \times \Omega_{F_3^c}\right] \\
&= \Phi_{F_3}^{-1}\left[\Phi_{F_1,F_3}^{-1}\left(\left(\bigcap_{n=1}^{\infty} A_{F_1}^{(n)}\right) \times \Omega_{F_3\setminus F_1}\right) \times \Omega_{F_3^c}\right].
\end{aligned}$$

Thus, $\mathcal{C}$ is a σ-field containing $\mathcal{E}_{F_1}$, and so $\mathcal{C} = \mathcal{F}_{F_1}$.

(c) Given a sequence $\{F_n\}$ of finite subsets of I, it is not necessarily the case that $\bigcup_{n=1}^{\infty} F_n$ is a finite subset of I. □

► EXERCISE 11.7 (11.2.2). *Prove equalities* $(*)$ *and* $(**)$ *given in the proof of Subclaim 3 of Claim 7.*

PROOF. **(∗)** Take $j \in \mathbb{N}$. Let $\Phi\colon \Omega_{\mathbb{N}_{m+j}} \to \Omega_{\mathbb{N}_m} \times \Omega_{\{m+1,\dots,m+j\}}$ be the bijection that associates each $(\omega_1, \dots, \omega_{m+j})$ with $((\omega_1, \dots, \omega_m), (\omega_{m+1}, \dots, \omega_{m+j}))$. We first prove

$$\Phi_{\mathbb{N}_m}^{-1}\left(A_{\mathbb{N}_m}\times\Omega_{\mathbb{N}_m^c}\right)=\Phi_{\mathbb{N}_{m+j}}^{-1}\left[\Phi^{-1}\left(A_{\mathbb{N}_m}\times\Omega_{\{m+1,\dots,m+j\}}\right)\times\Omega_{\mathbb{N}_{m+j}^c}\right]. \qquad (*)$$

Define

$$\mathcal{C}=\left\{A_{\mathbb{N}_m}\in\mathcal{F}_{\mathbb{N}_m}:(*)\text{ holds for }A_{\mathbb{N}_m}\right\}.$$

As usual, we show that $\mathcal{E}_{\mathbb{N}_m}\subseteq\mathcal{C}$ and $\mathcal{C}$ is a σ-filed on $\Omega_{\mathbb{N}_m}$. Let $A_{\mathbb{N}_m}=\bigtimes_{i=1}^m A_i$, where $A_i\in\mathcal{F}_i$ for each $i\in\{1,\dots,m\}$. Then

$$\Phi_{\mathbb{N}_m}^{-1}\left[\left(\bigtimes_{i=1}^m A_i\right)\times\Omega_{\mathbb{N}_m^c}\right]=A_1\times\cdots\times A_m\times\Omega_{m+1}\times\Omega_{m+2}\times\cdots,$$

and

$$\begin{aligned}
&\Phi_{\mathbb{N}_{m+j}}^{-1}\left[\Phi^{-1}\left[\left(\bigtimes_{i=1}^m A_i\right)\times\Omega_{\{m+1,\dots,m+j\}}\right]\times\Omega_{\mathbb{N}_{m+j}^c}\right]\\
&\qquad=\Phi_{\mathbb{N}_{m+j}}^{-1}\left[\left(A_1\times\cdots\times A_m\times\Omega_{m+1}\times\cdots\times\Omega_{m+j}\right)\times\Omega_{\mathbb{N}_{m+j}^c}\right]\\
&\qquad=A_1\times\cdots\times A_m\times\Omega_{m+1}\times\Omega_{m+2}\times\cdots.
\end{aligned}$$

Hence, $\mathcal{E}_{\mathbb{N}_m}\subseteq\mathcal{C}$.

We turn to show that $\mathcal{C}$ is a σ-field on $\Omega_{\mathbb{N}_m}$. It is evident that $\Omega_{\mathbb{N}_m}\in\mathcal{C}$ since

$$\Phi_{\mathbb{N}_m}^{-1}\left(\Omega_{\mathbb{N}_m}\times\Omega_{\mathbb{N}_m^c}\right)=\Omega=\Phi_{\mathbb{N}_{m+j}}^{-1}\left[\Phi^{-1}\left(\Omega_{\mathbb{N}_m}\times\Omega_{\{m+1,\dots,m+j\}}\right)\times\Omega_{\mathbb{N}_{m+j}^c}\right].$$

Now suppose that $A_{\mathbb{N}_m}\in\mathcal{C}$. Then

$$\begin{aligned}
\Phi_{\mathbb{N}_m}^{-1}\left(A_{\mathbb{N}_m}^c\times\Omega_{\mathbb{N}_m^c}\right)&=\Phi_{\mathbb{N}_m}^{-1}\left[\left(\Omega_{\mathbb{N}_m}\times\Omega_{\mathbb{N}_m^c}\right)\smallsetminus\left(A_{\mathbb{N}_m}\times\Omega_{\mathbb{N}_m^c}\right)\right]\\
&=\Omega\smallsetminus\left(A_1\times\cdots\times A_m\times\Omega_{m+1}\times\Omega_{m+2}\times\cdots\right),
\end{aligned}$$

and

$$\begin{aligned}
&\Phi_{\mathbb{N}_{m+j}}^{-1}\left[\Phi^{-1}\left(A_{\mathbb{N}_m}^c\times\Omega_{\{m+1,\dots,m+j\}}\right)\times\Omega_{\mathbb{N}_{m+j}^c}\right]\\
&\quad=\Phi_{\mathbb{N}_{m+j}}^{-1}\left[\left(\Omega_{\mathbb{N}_{m+j}}\smallsetminus\left(A_1\times\cdots\times A_m\times\Omega_{m+1}\times\cdots\times\Omega_{m+j}\right)\right)\times\Omega_{\mathbb{N}_{m+j}^c}\right]\\
&\quad=\Phi_{\mathbb{N}_{m+j}}^{-1}\Big[\left(\Omega_{\mathbb{N}_{m+j}}\times\Omega_{\mathbb{N}_{m+j}^c}\right)\\
&\qquad\smallsetminus\left(\left(A_1\times\cdots\times A_m\times\Omega_{m+j}\times\cdots\times\Omega_{m+j}\right)\times\Omega_{\mathbb{N}_{m+j}^c}\right)\Big]\\
&\quad=\Omega\smallsetminus\left(A_1\times\cdots\times A_m\times\Omega_{m+1}\times\Omega_{m+2}\times\cdots\right);
\end{aligned}$$

that is, $A_{\mathbb{N}_m}\in\mathcal{C}$ forces $A_{\mathbb{N}_m}^c\in\mathcal{C}$. To verify that $\mathcal{C}$ is closed under countable unions, take an arbitrary sequence $\{A_{\mathbb{N}_m}^{(n)}\}\subseteq\mathcal{C}$. We then have

$$\Phi_{\mathbb{N}_m}^{-1}\left[\left(\bigcup_{n=1}^{\infty} A_{\mathbb{N}_m}^{(n)}\right) \times \Omega_{\mathbb{N}_m^c}\right] = \bigcup_{n=1}^{\infty} \Phi_{\mathbb{N}_m}^{-1}\left(A_{\mathbb{N}_m}^{(n)} \times \Omega_{\mathbb{N}_m^c}\right),$$

and

$$\begin{aligned}
&\Phi_{\mathbb{N}_{m+j}}^{-1}\left[\Phi^{-1}\left(\left(\bigcup_{n=1}^{\infty} A_{\mathbb{N}_m}^{(n)}\right) \times \Omega_{\{m+1,\dots,m+j\}}\right) \times \Omega_{\mathbb{N}_{m+j}^c}\right] \\
&\quad = \Phi_{\mathbb{N}_{m+j}}^{-1}\left[\left(\bigcup_{n=1}^{\infty} \Phi^{-1}\left(A_{\mathbb{N}_m}^{(n)} \times \Omega_{\{m+1,\dots,m+j\}}\right)\right) \times \Omega_{\mathbb{N}_{m+j}^c}\right] \\
&\quad = \bigcup_{n=1}^{\infty} \Phi_{\mathbb{N}_{m+j}}^{-1}\left[\Phi^{-1}\left(A_{\mathbb{N}_m}^{(n)} \times \Omega_{\{m+1,\dots,m+j\}}\right) \times \Omega_{\mathbb{N}_{m+j}^c}\right] \\
&\quad = \bigcup_{n=1}^{\infty} \Phi_{\mathbb{N}_m}^{-1}\left(A_{\mathbb{N}_m}^{(n)} \times \Omega_{\mathbb{N}_m^c}\right) \\
&\quad = \Phi_{\mathbb{N}_m}^{-1}\left[\left(\bigcup_{n=1}^{\infty} A_{\mathbb{N}_m}^{(n)}\right) \times \Omega_{\mathbb{N}_m^c}\right].
\end{aligned}$$

Hence, $\bigcup_{n=1}^{\infty} A_{\mathbb{N}_m}^{(n)} \in \mathcal{C}$. Therefore, $\mathcal{C}$ is a σ-filed containing $\mathcal{E}_{\mathbb{N}_m}$, and so $\mathcal{C} = \mathcal{F}_{\mathbb{N}_m}$.

(∗∗) We now prove

$$\begin{aligned}
&\mathbb{1}_{\Phi^{-1}\left(A_{\mathbb{N}_m} \times \Omega_{\{m+1,\dots,m+j\}}\right)}\left(\xi_1, \dots, \xi_n, \omega_{n+1}, \dots, \omega_{m+j}\right) \\
&\qquad = \mathbb{1}_{A_{\mathbb{N}_m}}\left(\xi_1, \dots, \xi_n, \omega_{n+1}, \dots, \omega_m\right).
\end{aligned} \tag{$**$}$$

Define

$$\mathcal{D} = \left\{A_{\mathbb{N}_m} \in \mathcal{F}_{\mathbb{N}_m} : (**) \text{ holds for each } \omega_{n+1} \in \Omega_{n+1}, \dots, \omega_m \in \Omega_m\right\}.$$

Once again, we prove this claim by showing that $\mathcal{E}_{\mathbb{N}_m} \subseteq \mathcal{D}$ and $\mathcal{D}$ is a σ-filed on $\Omega_{\mathbb{N}_m}$.

Let $A_{\mathbb{N}_m} = \bigtimes_{i=1}^m A_i$, where $A_i \in \mathcal{F}_i$ for each $i \in \{1, \dots, m\}$. Then

$$
\begin{aligned}
&\mathbb{1}_{\Phi^{-1}\left[\left(\bigtimes_{i=1}^{m} A_i\right)\times\Omega_{\{m+1,\ldots,m+j\}}\right]}\left(\xi_1,\ldots,\xi_n,\omega_{n+1},\ldots,\omega_{m+j}\right)=1\\
&\iff \left(\xi_1,\ldots,\xi_n,\omega_{n+1},\ldots,\omega_{m+j}\right)\in\Phi^{-1}\left[\left(\bigtimes_{i=1}^{m} A_i\right)\times\Omega_{\{m+1,\ldots,m+j\}}\right]\\
&\iff \left(\xi_1,\ldots,\xi_n,\omega_{n+1},\ldots,\omega_{m+j}\right)\in A_1\times\cdots\times A_m\times\Omega_{m+1}\times\cdots\times\Omega_{m+j}\\
&\iff \left(\xi_1,\ldots,\xi_n,\omega_{m+1}\ldots,\omega_m\right)\in A_1\times\cdots A_m\\
&\iff \mathbb{1}_{A_1\times\cdots\times A_m}\left(\xi_1,\ldots,\xi_n,\omega_{m+1}\ldots,\omega_m\right)=1;
\end{aligned}
$$

that is, $\mathcal{E}_{\mathbb{N}_m}\subseteq\mathcal{D}$.

We next show that $\mathcal{D}$ is a σ-field on $\Omega_{\mathbb{N}_m}$. It is easy to see that

$$
\begin{aligned}
&\mathbb{1}_{\Phi^{-1}\left(\Omega_{\mathbb{N}_m}\times\Omega_{\{m+1,\ldots,m+j\}}\right)}\left(\xi_1,\ldots,\xi_n,\omega_{m+1},\ldots,\omega_{m+j}\right)\\
&\qquad\qquad=1=\mathbb{1}_{\Omega_{\mathbb{N}_m}}\left(\xi_1,\ldots,\xi_n,\omega_{m+1},\ldots,\omega_m\right),
\end{aligned}
$$

so $\Omega_{\mathbb{N}_m}\in\mathcal{D}$. Suppose that $A_{\mathbb{N}_m}\in\mathcal{D}$. Then

$$
\begin{aligned}
&\mathbb{1}_{\Phi^{-1}\left(A^c_{\mathbb{N}_m}\times\Omega_{\{m+1,\ldots,m+j\}}\right)}\left(\xi_1,\ldots,\xi_n,\omega_{m+1},\ldots,\omega_{m+j}\right)=1\\
&\iff \left(\xi_1,\ldots,\xi_n,\omega_{m+1},\ldots,\omega_{m+j}\right)\in\Phi^{-1}\left(A^c_{\mathbb{N}_m}\times\Omega_{\{m+1,\ldots,m+j\}}\right)\\
&\iff \left(\xi_1,\ldots,\xi_n,\omega_{m+1},\ldots,\omega_{m+j}\right)\in\Omega_{\mathbb{N}_{m+j}}\setminus\Phi^{-1}\left(A_{\mathbb{N}_m}\times\Omega_{\{m+1,\ldots,m+j\}}\right)\\
&\iff \mathbb{1}_{\Phi^{-1}\left(A_{\mathbb{N}_m}\times\Omega_{\{m+1,\ldots,m+j\}}\right)}\left(\xi_1,\ldots,\xi_n,\omega_{m+1},\ldots,\omega_{m+j}\right)=0\\
&\iff \mathbb{1}_{A_{\mathbb{N}_m}}\left(\xi_1,\ldots,\xi_n,\omega_{m+1},\ldots,\omega_m\right)=0\\
&\iff \mathbb{1}_{A^c_{\mathbb{N}_m}}\left(\xi_1,\ldots,\xi_n,\omega_{m+1},\ldots,\omega_m\right)=1.
\end{aligned}
$$

Hence, $A^c_{\mathbb{N}_m}\in\mathcal{D}$ whenever $A_{\mathbb{N}_m}\in\mathcal{D}$. Finally, we verify that $\mathcal{D}$ is closed under countable unions. Take an arbitrary sequence $\{A^{(n)}_{\mathbb{N}_m}\}\subseteq\mathcal{D}$. Observe that

$$
\begin{aligned}
\mathbb{1}_{\Phi^{-1}\left[\left(\bigcup_{n=1}^{\infty}A^{(n)}_{\mathbb{N}_m}\right)\times\Omega_{\{m+1,\ldots,m+j\}}\right]}&=\mathbb{1}_{\bigcup_{n=1}^{\infty}\Phi^{-1}\left(A^{(n)}_{\mathbb{N}_m}\times\Omega_{\{m+1,\ldots,m+j\}}\right)}\\
&=\sup_n \mathbb{1}_{\Phi^{-1}\left(A^{(n)}_{\mathbb{N}_m}\times\Omega_{\{m+1,\ldots,m+j\}}\right)},
\end{aligned}
$$

and

$$
\mathbb{1}_{\bigcup_{n=1}^{\infty}A^{(n)}_{\mathbb{N}_m}}=\sup_n \mathbb{1}_{A^{(n)}_{\mathbb{N}_m}}.
$$

Since

$$
\begin{aligned}
&\mathbb{1}_{\Phi^{-1}\left(A^{(n)}_{\mathbb{N}_m}\times\Omega_{\{m+1,\ldots,m+j\}}\right)}\left(\xi_1,\ldots,\xi_n,\omega_{m+1},\ldots,\omega_{m+j}\right)\\
&\qquad\qquad=\mathbb{1}_{A^{(n)}_{\mathbb{N}_m}}\left(\xi_1,\ldots,\xi_n,\omega_{m+1},\ldots,\omega_m\right)
\end{aligned}
$$

for every $n\in\mathbb{N}$, we get

$$\sup_n \mathbb{1}_{\Phi^{-1}\left(A_{\mathbb{N}_m}^{(n)} \times \Omega_{\{m+1,\dots,m+j\}}\right)} \left(\xi_1, \dots, \xi_n, \omega_{m+1}, \dots, \omega_{m+j}\right)$$
$$= \sup_n \mathbb{1}_{A_{\mathbb{N}_m}^{(n)}} \left(\xi_1, \dots, \xi_n, \omega_{m+1}, \dots, \omega_m\right);$$

that is, $\bigcup_{n=1}^{\infty} A_{\mathbb{N}_m}^{(n)} \in \mathcal{D}$. This proves that $\mathcal{D} = \mathcal{F}_{\mathbb{N}_m}$. $\square$

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