2. EQUILIBRIUM

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- Let's agree that throughout the notes $u_i(s_1, s_2)$, $i = 1, 2$ will denote Player i's payoff if Player 1's strategy is s_1 and Player 2's strategy is s_2.
- Remember that a **Nash equilibrium** is a pair of players' actions that are best responses to each other.
- Remember that if either player chooses a **mixed strategy** $x \cdot \textit{left} + (1-x) \cdot \textit{right}$, then they must be *indifferent* between playing "*left*" or "*right*".

Exercise 1.

Pure Strategy Nash Equilibrium

Step 1 If Player 2 chooses L, then Player 1's best response is to choose T whereas

$$u_1(T, L) = \underline{\hspace{3cm}} > \underline{\hspace{3cm}} = u_1(B, L).$$

Step 2 If Player 2 chooses R, then Player 1's best response is to choose T or B since

$$u_1(T, R) = \underline{\hspace{3cm}} = \underline{\hspace{3cm}} = u_1(B, R).$$

Step 3 If Player 1 chooses T, then Player 2's best response is to choose L whereas

$$u_2(T, L) = \underline{\hspace{3cm}} > \underline{\hspace{3cm}} = u_2(T, R).$$

Step 4 If Player 1 chooses B, then Player 2's best response is to choose L or R since

$$u_2(B, L) = \underline{\hspace{3cm}} = \underline{\hspace{3cm}} = u_2(B, R).$$

Step 5 Find the cells with two boxes. The pure strategy Nash equilibria in this game are

$$\underline{\hspace{4cm}} \quad \text{and} \quad \underline{\hspace{4cm}}.$$

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			Player 2	
			L $[y]$	R $[1-y]$
Player 1	T $[x]$		$\boxed{1}$, $\boxed{1}$	$\boxed{0}$, 0
	B $[1-x]$		0, $\boxed{0}$	$\boxed{0}$, $\boxed{0}$

A-1

Mixed Strategy Nash Equilibrium

Let Player 1's mixed strategy be $(x, 1-x)$ [Player 1 plays T with probability ____________ and plays B with probability ____________], and Player 2's mixed strategy be $\big(y, 1-y\big)$ [Player 2 plays L with probability ____________ and plays R with probability ____________].

Step 1

- Given Player 2's mixed strategy $(y, 1-y)$, Player 1 gets

$$\begin{cases} \underline{\hspace{6cm}}, & \text{if he plays } T \\ \underline{\hspace{6cm}}, & \text{if he plays } B \end{cases}$$

- Player 1 must be *indifferent* between T or B; hence

$$\underline{\hspace{9cm}}$$

Interpretation: $y = 0$ means that Player 2 chooses R deterministically; in this case, Player 1 is indifferent between playing T or B [always gets 0].

- Therefore, Player 2's mixed strategy is

$$\big(y, 1-y\big) = \underline{\hspace{4.5cm}}.$$

Step 2 Because this game is symmetric, we know immediately that

$$x = \underline{\hspace{4.5cm}}.$$

Step 3 Hence, the mixed NE in this game is

$$\underline{\hspace{4.5cm}}.$$

Exercise 2.

> ***Definition 0.1*** (Strongly Dominant Strategy)**.** In a two-player game the payoffs to a player from choosing a strongly dominant strategy are higher than those from choosing any other strategy in response to any strategy the other player chooses.

Step 1 In this game, L is Player 2's strongly dominant strategy because his payoff from choosing L is always higher than his payoff from choosing R [see Matrix B-1]:

$$
\begin{aligned}
u_2(T, \mathbf{L}) &= \underline{\hspace{3cm}} > \underline{\hspace{3cm}} = u_2(T, \mathbf{R}), \\
u_2(B, \mathbf{L}) &= \underline{\hspace{3cm}} > \underline{\hspace{3cm}} = u_2(B, \mathbf{R}).
\end{aligned}
$$

Player 2

Player 1	L	R
T	1, 1	0, 0
B	1, 1	0, 0

B-1

Step 2 So if Player 2 is *rational* he will never choose R. Since Player 2 will never choose R we can delete the column corresponding to his choice of R. This produces Matrix B-2.

Player 2

Player 1	L
T	1, 1
B	1, 1

B-2

Step 3 In the game in Matrix B-2, Player 1 is indifferent between T and B [he always gets 1]. Hence, Player 1's would like to put any probability x on T and probability $1 - x$ on B, i.e., Player 1's mixed strategies are

$$x \cdot T + (1 - x) \cdot B, \quad \text{for any } 0 \leqslant x \leqslant 1.$$

Step 4 Therefore, the set Nash equilibria in this game is

$$NE = \{x \cdot T + (1-x) \cdot B,\ L : 0 \leqslant x \leqslant 1\}.$$

Exercise 3.

■ **Pure Strategy Nash Equilibrium**

Player 2

		L	R
	T	10, 10	0, 0
Player 1	M	0, 0	10, 10
	B	4, 4	4, 4

C-1

■ **Mixed Strategy Nash Equilibrium**

> *Remark.* You can find the mixed NE's by the way introduced in my Notes 1. Here I introduce an alternative method. I think this method will save your time, especially in the exams.

Step 1 Note that for Player 1, B is dominated by the following mixed strategy

$$\left(\frac{1}{2}T + \frac{1}{2}M\right).$$

This is because [see Matrix C-2]

$$\begin{cases} u_1\left(\frac{1}{2}T + \frac{1}{2}M,\ L\right) = 10 \times \frac{1}{2} + 0 \times \frac{1}{2} = 5 > 4 = u_1(B, L), & \text{if Player 2 plays } L \\ u_1\left(\frac{1}{2}T + \frac{1}{2}M,\ R\right) = \underline{\hspace{6em}} > \underline{\hspace{2em}} = u_1(B, R), & \text{if Player 2 plays } R. \end{cases}$$

In words, for Player 1,

$$\begin{cases} \frac{1}{2}T + \frac{1}{2}M \text{ is better than } B, & \text{if Player 2 plays } L \\ \frac{1}{2}T + \frac{1}{2}M \text{ is better than } B, & \text{if Player 2 plays } R. \end{cases}$$

Player 2

Player 1		L	R
	T $[1/2]$	**10**, 10	**0**, 0
	M $[1/2]$	**0**, 0	**10**, 10
	B	4, 4	4, 4

C-2

Player 2

Player 1		L $[y]$	R $[1-y]$
	T $[x]$	10, 10	0, 0
	M $[1-x]$	0, 0	10, 10

C-3

Step 2 So if Player 1 is rational he will never play B [B is dominated by $\frac{1}{2}T+\frac{1}{2}M$]. Since Player 1 will never play B we can delete the row corresponding to his choice or B. This produces Matrix C-3.

Step 3 We can find the mixed NE in the game in Matrix C-3 by the way in Notes 1 [e.g., please refer the BOS]. You should complete it by yourself.

Exercise 4.

■ Pure Strategy Nash Equilibrium

Please find the pure NE's in this game [see Matrix D-1].

■ Mixed Strategy Nash Equilibrium

Remark. It is very hard to find the mixed Nash equilibria in this game at this stage. Do not feel upset if you cannot understand my solution now. Please notice the difference between Matrix C-1 and D-1.

Player 2

		L	R
Player 1	T	10, 10	0, 0
	M	0, 0	10, 10
	B	6, 6	6, 6

D-1

Player 2

		L $[z]$	R $[1-z]$
Player 1	T $[x]$	10, 10	0, 0
	M $[y]$	0, 0	10, 10
	B $[1-x-y]$	6, 6	6, 6

D-2

Step 1 Given Player 1' mixed strategy $\big(x, y, 1-x-y\big)$, if Player 2 would like to play a mixed strategy $(z, 1-z)$, it must be the case that Player 2 is indifferent between playing L or R, i.e.,

$$\underbrace{10\cdot x + 0\cdot y + 6\cdot(1-x-y)}_{\text{Player 2's expected payoff from playing } L} = \underbrace{0\cdot x + 10\cdot y + 6\cdot(1-x-y)}_{\text{Player 2's expected payoff from playing } R},$$

which solves for $x = y$. Thus, there are two cases: either $x = y \neq 0$ or $x = y = 0$ [not both].

Step 2 We first consider the case of $\boldsymbol{x = y \neq 0}$. In this case, Player 1 must be indifferent between T and M; hence,

$$\underbrace{10\cdot z + 0\cdot(1-z)}_{u_1\left(T,\, z\cdot L+(1-z)\cdot R\right)} = \underbrace{0\cdot z + 10\cdot(1-z)}_{u_1\left(M,\, z\cdot L+(1-z)\cdot R\right)} \Rightarrow z = \frac{1}{2};$$

that is, if Player 2 plays $\frac{1}{2}L + \frac{1}{2}R$, Player 1 is indifferent between playing T or M. Therefore, given Player 2's mixed strategy $\frac{1}{2}L + \frac{1}{2}R$, if Player 1 plays T or

M, his expected payoff is

$$\begin{aligned}
u_1\left(T,\ \frac{1}{2}L+\frac{1}{2}R\right) = u_1\left(M,\ \frac{1}{2}L+\frac{1}{2}R\right) &= 10\times\frac{1}{2}+0\times\left(1-\frac{1}{2}\right) \\
&\overset{\text{or}}{=} 0\times\frac{1}{2}+10\times\left(1-\frac{1}{2}\right) \\
&= 5,
\end{aligned}$$

but if Player 1 plays B, his expected payoff is

$$\begin{aligned}
u_1\left(B,\ \frac{1}{2}L+\frac{1}{2}R\right) &= 6\times\frac{1}{2}+6\times\frac{1}{2} \\
&= 6 \\
&> 5.
\end{aligned}$$

Hence, if Player 2's mixed strategy is $\frac{1}{2}L+\frac{1}{2}R$, Player 1 will choose B certainly, that is, Player 1's best response is B, rather than putting any positive probability x on T and probability y on B, i.e., $x=y=0$. A contradiction.

Step 3 So it must be the case that $\boldsymbol{x=y=0}$. In this case, Player 1 will choose B deterministically, but which means that B is Player 1's best response given Player 2's mixed strategy is $(z,1-z)$, i.e.,

$$\boxed{\begin{cases} u_1\left(B,\ z\cdot L+(1-z)\cdot R\right) \geqslant u_1\left(T,\ z\cdot L+(1-z)\cdot R\right), & B \text{ is better than } T \\ u_1\left(B,\ z\cdot L+(1-z)\cdot R\right) \geqslant u_1\left(M,\ z\cdot L+(1-z)\cdot R\right), & B \text{ is better than } M \end{cases}}$$

that is,

$$\underbrace{z\cdot 6+(1-z)\cdot 6}_{u_1(B,\ z\cdot L+(1-z)\cdot R)} \geqslant \underbrace{z\cdot 10+(1-z)\cdot 0}_{u_1(T,\ z\cdot L+(1-z)\cdot R)} \Rightarrow \boxed{z\leqslant\frac{3}{5}}$$

$$\underbrace{z\cdot 6+(1-z)\cdot 6}_{u_1(B,\ z\cdot L+(1-z)\cdot R)} \geqslant \underbrace{z\cdot 0+(1-z)\cdot 10}_{u_1(M,\ z\cdot L+(1-z)\cdot R)} \Rightarrow \boxed{z\geqslant\frac{2}{5}}$$

Hence, Player 1 would like to play B certainly if and only if Player 2 plays L with probability z, plays R with probability $1-z$, and

$$\boxed{\frac{2}{5}\leqslant z\leqslant\frac{3}{5}.}$$

		Player 2	
		L $[z]$	R $[1-z]$
Player 1	**B**	6, **6**	6, **6**

D-2

Step 4 Finally, we need to check that Player 2 will really play $(z, 1-z)$ with $\frac{2}{5} \leqslant z \leqslant \frac{3}{5}$. If Player 1's strategy is B, then the game becomes Matrix D-2

In the game in Matrix D-2, Player 2 always gets 6, so he would like to put any probability z on L, and put $(1-z)$ on R, including

$$\boxed{\frac{2}{5} \leqslant z \leqslant \frac{3}{5}.}$$

Step 5 Therefore, the set of mixed Nash equilibria in this game is

$$\left\{\left(B,\ z \cdot L + (1-z) \cdot R\right) \,\middle|\, \frac{2}{5} \leqslant z \leqslant \frac{3}{5}\right\}.$$