5. Zero-sum games

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Remember that in a zerosum game, $u_1(s_1, s_2) + u_2(s_1, s_2) = 0$, $\forall\, s_1, s_2$.

Exercise 1.

Step 1 Refer Matrix A, we know that for Player 1, his minmax pure strategy is G. This is because Player 1 guarantees

$$\begin{cases} 1, & \text{if } s_1 = F \text{ and } s_2 = E \\ 2, & \text{if } s_1 = G \text{ and } s_2 = D \\ 1, & \text{if } s_1 = H \text{ and } s_2 = D \\ 1, & \text{if } s_1 = I \text{ and } s_2 = A \text{ or } C \\ 0, & \text{if } s_1 = J \text{ and } s_2 = A. \end{cases}$$

Step 2 Similarly, you can check that Player 2's minmax pure strategy is E:

$$u_2(s_1, s_2) = \begin{cases} \_\_\_\_\_\_, & \text{if } (s_1, s_2) = (H, A) \\ \_\_\_\_\_\_, & \text{if } (s_1, s_2) = \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_, & \text{if } (s_1, s_2) = \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_, & \text{if } (s_1, s_2) = \_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_, & \text{if } (s_1, s_2) = \_\_\_\_\_\_\_\_\_\_. \end{cases}$$

Exercise 2. Refer Matrix A, and we find the unique pure Nash equilibrium is (G, E).

Exercise 3.

Remark. Find the set of Nash equilibria by yourself. Please refer Notes 1 in the event that you forget how to solve the game of Matching Pennies.

Let us denote Player i's payoff in Matrix C-1 as $\widetilde{u}_i(s_1, s_2)$, and his payoff in Matching Pennies as $u_i(s_1, s_2)$, where $i = 1, 2$, $s_1 \in \{T, B\}$, and $s_2 \in \{L, R\}$.

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	A	B	C	D	E
F	2, 3	6, **1**	7, **0**	7, 2	**1**, 2
G	7, 3	8, 2	4, 2	2, **0**	9, 4
H	2, **1**	9, 5	3, 2	**1**, 8	3, 5
I	**1**, 8	4, 3	**1**, **0**	5, 9	8, 2
J	**0**, 5	2, 6	2, 3	4, 6	7, 3

Matrix A

Player 2

Player 1	L	R
T	2, − 3	0, − 1
B	0, − 1	2, − 3

Matrix C-1

Player 2

Player 1	L	R
T	1, − 1	−1, 1
B	−1, 1	1, − 1

Matching Pennies

Step 1 Consider Player 1's affine transformation

$$\widetilde{u}_1(s_1, s_2) = a \cdot u_1(s_1, s_2) + b, \quad \text{where } a > 0.$$

[Note that by definition of an affine transformation, a must strictly greater than zero.] Thus, there are four equations — every equation corresponds to a pair of payoffs:

$$\begin{cases} 2 = 1 \cdot a + b \\ 0 = -1 \cdot a + b \\ 0 = -1 \cdot a + b \\ 2 = 1 \cdot a + b. \end{cases}$$

Note that the first equation is the same as the last one, and the second equation is the same as the third one, so we can write the above system of equations as follows:

$$\begin{cases} 2 = 1 \cdot a + b \\ 0 = -1 \cdot a + b. \end{cases}$$

Solve this system and we get

$$\begin{cases} a = 1 \\ b = 1. \end{cases}$$

Then we have

$$\widetilde{u}_1(s_1, s_2) = u_1(s_1, s_2) + 1, \quad \text{for any } s_1, s_2.$$

[See Matrix C-2]. Hence, we know that Player 1's payoff in Matrix C-1 is the affine transformation from the game of Matching Pennies, that is, his preference does not change.

	L	R
T	$2 = 1 + 1$	$0 = -1 + 1$
B	$0 = -1 + 1$	$2 = 1 + 1$

Matrix C-2

Step 2 Now it is your turn to prove that Player 2's payoff in Matrix C-1 is the affine transformation from the game of Matching Pennies. First, write Player 2's payoff $\widetilde{u}_2(s_1, s_2)$ in Matrix C-1 as [you can use any parameters that you like]:

$$\widetilde{u}_2(s_1, s_2) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$$

where the first parameter should satisfy the following condition by the definition of an affine transformation:

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List all of the four equations and reduce the system of equations:

$$\begin{cases} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \end{cases} \iff \begin{cases} \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \\ \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_ \end{cases}$$

[Note that we need two equations since there are two unknown variables.] Solve the reduced system of equation, and you can get

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The final result is:

$$\widetilde{u}_2(s_1, s_2) = u_2(s_1, s_2) - 2, \quad \text{for any } s_1, s_2.$$

[See Matrix C-3].

Exercise 4. Find the set of Nash equilibria by yourself. Please refer Notes 1 in the event that you forget how to solve the game of BOS.

	L	R
T	$-3=-1-2$	$-1=1-2$
B	$-1=1-2$	$-3=-1-2$

Matrix C-3

Player 2

Player 1		L	R
	T	7, 1	1, 0
	B	1, 0	3, 3

Matrix D-1

Player 2

Player 1		L	R
	T	3, 1	0, 0
	B	0, 0	1, 3

BOS

Step 1 We keep using the notation as in Exercise 3. Write Player 1's payoffs in Matrix D-1 as

$$\widetilde{u}_1(s_1,s_2)=a\cdot u_1(s_1,s_2)+b,\quad a>0.$$

Thus,

$$\begin{cases}7=3\cdot a+b\\1=0\cdot a+b\\1=0\cdot a+b\\3=1\cdot a+b\end{cases}\iff\begin{cases}7=3\cdot a+b\\1=b\\3=1\cdot a+b\end{cases}\iff\begin{cases}a=2\\b=1,\end{cases}$$

that is,

$$\widetilde{u}_1(s_1,s_2)=2\cdot u_1(s_1,s_2)+1.$$

Step 2 Since Player 1's payoffs in Matrix D-1 are the affine transformations from the game of BOS, we know that his preference does not change. Obviously, Player 2's preference does not change.

Player 2

Player 1		L	M	R
	T	3, 4	5, 2	4, 3
	B	7, 0	1, 6	$\frac{7}{2}$, $\frac{7}{2}$

Matrix E-1

Exercise 5. We first introduce a simple method, where Player 1's payoffs are unchanged. I will introduce a more general method in Section 5.

Step 1 Consider an affine transformation for Player 2:

$$\widetilde{u}_2(s_1, s_2) = a \cdot u_2(s_1, s_2) + b, \quad a > 0.$$

Step 2 To transfer the game in Matrix E-1 into a zerosum game, we need the following outcome:

$$u_1(s_1, s_2) + \widetilde{u}_2(s_1, s_2) = 0, \quad \forall\, s_1, s_2.$$

Of course, there are six equations:

$$\begin{cases} 3 + (4 \cdot a + b) = 0, & \text{if } (s_1, s_2) = (T, L) \\ 5 + (2 \cdot a + b) = 0, & \text{if } (s_1, s_2) = (T, M) \\ \underline{\hspace{6cm}}, & \text{if } (s_1, s_2) = (T, R) \\ 7 + (0 \cdot a + b) = 0, & \text{if } (s_1, s_2) = \underline{\hspace{2cm}} \\ \underline{\hspace{6cm}}, & \text{if } (s_1, s_2) = \underline{\hspace{2cm}} \\ \underline{\hspace{6cm}}, & \text{if } (s_1, s_2) = \underline{\hspace{2cm}} \end{cases}$$

We pick any two district equations from the above system, that is, we suppose this system is compatible [it has a solution] and check ex post that it is true. For example, I choose the first two equations:

$$\begin{cases} 3 + (4 \cdot a + b) = 0 \\ 5 + (2 \cdot a + b) = 0, \end{cases}$$

which solves for

$$a = 1, \quad \text{and} \quad b = -7.$$

Step 3 Hence, Player 2's payoffs are transformed as

$$\widetilde{u}_2(s_1, s_2) = u_2(s_1, s_2) - 7, \quad \forall\, s_1, s_2,$$

and the game is now as in Matrix E-2. As you can see, it is really a zerosum game.

Player 2

		L	M	R
Player 1	T	$3, \; -3$	$5, \; -5$	$4, \; -4$
	B	$7, \; -7$	$1, \; -1$	$\frac{7}{2}, \; -\frac{7}{2}$

Matrix E-2

Remark (General Method). An alternative more general argument goes as follows. Let Player 1's affine transformation be

$$\widetilde{u}_1(s_1, s_2) = a_1 \cdot u_1(s_1, s_2) + b_1, \quad a_1 > 0;$$

let Player 2's affine transformation be

$$\widetilde{u}_2(s_1, s_2) = a_2 \cdot u_2(s_1, s_2) + b_2, \quad a_2 > 0.$$

Since we want to get a zerosum game, $\widetilde{u}_1(s_1, s_2)$ and $\widetilde{u}_2(s_1, s_2)$ should satisfy the following requirement:

$$\widetilde{u}_1(s_1, s_2) + \widetilde{u}_2(s_1, s_2) = 0, \quad \forall\, s_1, s_2,$$

that is,

$$\begin{aligned}&\left[a_1 \cdot u_1(s_1, s_2) + b_1\right] + \left[a_2 \cdot u_2(s_1, s_2) + b_2\right]\\ &\quad = \left[a_1 \cdot u_1(s_1, s_2) + a_2 \cdot u_2(s_1, s_2)\right] + [b_1 + b_2]\\ &\quad = 0.\end{aligned}$$

If we let $a_1 = a_2 = 1$, then the above equation becomes as

$$\left[u_1(s_1, s_2) + u_2(s_1, s_2)\right] + [b_1 + b_2] = 7 + (b_1 + b_2) = 0.$$

[Note that $u_1(s_1, s_2) + u_2(s_1, s_2) = 7$ throughout the game in Matrix E-1.] Hence, we can choose any b_1 and b_2 satisfying

$$b_1 + b_2 = -7.$$

For example, we can let $b_1 = -1$ and

$$\begin{aligned}b_2 &= -7 - b_1\\ &= -7 - (-1)\\ &= -6,\end{aligned}$$

and so

$$\begin{aligned}\widetilde{u}_1(s_1, s_2) &= u_1(s_1, s_2) - 1,\\ \widetilde{u}_2(s_1, s_2) &= u_2(s_1, s_2) - 6.\end{aligned}$$

Note that if we let $b_1 = 0$ and $b_2 = -7$, then we get the same result as in the former way.
Of course you can also let, for example, $a_1 = a_2 = \frac{1}{2}$, and obtain the results without difficulty [in this case, $b_1 + b_2 = -\frac{7}{2}$].
The more general case is let $a_1 = a_2 = a > 0$. Then

$$b_1 + b_2 = -7 \cdot a$$