# ELECTORAL CONTESTS WHEN VOTING IS COSTLY

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An advantaged candidate and a disadvantaged candidate compete in a large election. Candidates exert effort to improve their valences, and voters cast their votes costly. This paper characterizes the pure strategy equilibria in this kind of election games, and gives sufficient conditions for the existence of pure strategy equilibria. For most instances, there exists at least one and at most two pure strategy equilibria. On average a low voting cost causes high campaign efforts, but there also exists an interval of voting costs such that candidates' campaign efforts are strictly increasing on this interval. Moreover, when candidates become similar in terms of their productivity, their equilibrium valence choices also become similar.

KEYWORDS. Costly voting, Endogenous valence choice, Campaign effort, Election, Ethical-voters.
JEL CLASSIFICATION. C72, D72

## 1. INTRODUCTION

Should a candidate spend more money and effort to attract supporters when voters do not want to cast their ballots? Should a voter do more to support the candidate whom he prefers when this candidate is disadvantaged in the campaign? These questions will be answered in this paper. The answer to the first question is "*it depends on the voters' intention of avoiding voting*", and the answer to the second question is affirmative: yes, a voter goes out to vote for his disadvantaged candidate, even if his voting cost is high.

This paper investigates candidates' non-policy costly activities in large elections when candidates differ in productivity and voters cast their votes costly. The motivation of this study is the following stylized facts: (1) Candidates spend a fair amount of money and effort on campaigns, but the majority of the money and effort does not involve precise policy statements. (2) Candidates are not symmetric — they differ in their professional background, name knowing, productivity, etc. (3) Voting is costly, yet in most cases, voluntary, for voters.

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Scholars have found that under some circumstances candidates "becloud their policies in a fog of ambiguity" (Downs, 1957). For example, when there are party primary elections before general elections, candidates usually do not reveal their policy platforms precisely to voters. In Aragonès and Neeman (2000), a candidate uses ambiguous strategies because she then enjoys greater freedom in choosing her policy once she is elected; Meirowitz (2005) treats primaries as an opportunity of offering voters' preferences to candidates, and so candidates have incentive to keep their policy platforms vague. When there are no primaries, Callander and Wilson (2008) provide a context-dependent model to explain the ambiguity phenomenon.

A large body of empirical literature addresses that candidates take costly actions on impressionistic advertising or building reputations for charisma to build support. These non-policy characteristics of candidates are often referred to as *valence* in political economy (Stokes, 1963). Candidates persuade voters that they should win by emphasizing their professional backgrounds and previous political experience, by announcing that they have the quality to change and keep their campaign promise. This competition cannot be captured by the standard Downsian model (Downs, 1957), where candidates compete for voters by choosing policies from the real line $\mathbb{R}$.

Another feature in electoral competition is that candidates are asymmetric. They may differ in attracting campaigning contributions and using the contributions. They may also differ in ability to organize and communicate, and personality, etc. For example, an incumbent often has advantages and they tend to spend more and win more (Ashworth and Bueno de Mesquita, 2008). So it is interesting to know how the asymmetry between candidates affects their campaign behavior and voters' decisions when voting.

Finally, in most cases voting is costly for voters. If the election is relatively large, then full participation is a woefully poor approximation to any empirical reality. The question then arises, does the voting cost influence a candidates' campaign action?

In order to answer these questions, we need to consider an *integrated* model in which candidates and voters are all active: candidates compete with valences, and bear in mind that voting is costly for voters; a voter's decision is influenced by candidates' valence choices. Nevertheless, most existing models have treated candidates' behavior and voters' behavior in isolation. Some authors assume that voting is costless, and voters all participate and vote sincerely in the election. With this assumption, they consider how candidates invest to improve their competence in an election (e.g., Meirowitz 2008, Ashworth and Bueno de Mesquita 2009, Serra 2010, and Crutzen, Castanheira and Sahuguet 2010). In other models, candidates' policy platforms and valences are exogenously given, and in this setting the authors analyze how the voting cost influences voters' decisions (e.g., Börgers 2004, Feddersen and Sandroni

2006b, Degan and Merlo 2011). This isolated approach leaves the interaction between candidates and voters unclear.

In this paper I extend the *ethical-voters* model advocated by Feddersen and Sandroni (2006b) to an environment in which two candidates endogenously decide their valences through non-policy activities. These non-policy activities are costly for candidates and candidates are asymmetric in the sense that they differ in the marginal costs of implementing their valences. The realized valences together determine each candidate's fraction of supporters. Thus, candidates' valence competition makes each candidate's fraction of supporters, which is fixed in Feddersen and Sandroni (2006b), endogenous in this paper.

However, I suppose that candidates' valences do not affect voters' welfare directly. This modelling method attempts to capture the idea that voters have preferences over salient and widely agreed upon policy goals, e.g., justice, equality,[1] or human rights. Candidates' actions, including arguments, advertisements, promises, etc., make some voters believe that a particular candidate is the one who has the ability to deliver these goals and becomes a supporter for this candidate.[2] Keeping other things constant, if a candidate improves her valence, she improves the vote share for her, but it never happens that she attracts all voters unless her opponent's valence is zero. This is because, for example, different voters judge rival candidates' actions differently, or some voters are hard-core supporters for a particular candidate.

My treatment for valence is therefore different from the literature, where valence is treated as something including all the good qualities of a candidate, and enters a voter's utility function separable additively. However, this popular way is questionable. As Miller (2011) convincingly points out, valence is not necessarily synonymous with universal appeal. For instance, depending on the status quo, a voter who does not like the current policy is in favor of a challenger's quality of change, but another voter who is happy with the implementing policy does not. Because there is no generally accepted ways to decompose valence, I simply adopt the assumption that candidates' valences determine the fraction of each candidate's supporters, and voters' utility is from the election itself.

Throughout the paper, I focus exclusively on *pure strategy equilibria*. When valence accumulation is considered, there is usually no pure strategy equilibria and the mixed strategy equilibria is complicated. The only pure strategy equilibrium for endogenous valence accumulation known is in Ashworth and Bueno de Mesquita (2009, Lemma 2), where the authors show that when candidates' policy platforms are highly diverged, then each candidate implementing zero valence is a unique pure strategy equilibrium. However, this result quite contradicts the daily observation. Identifying how candidates use mixed

[1] de Tocqueville (1860) writes: "The passion of men for equality is ardent, insatiable, eternal, invincible."

[2] As Besley and Reynal-Querol (2011) find, a high quality candidate is easy to be selected.

strategies in valence choices is difficult. So it is important to know whether non trivial pure strategy equilibria of valence accumulation exists in election games? In this paper I show that pure strategy equilibria exists if we take voters' cost of voting into account.

The problem in previous literature is that there is no pure strategy equilibrium when valence choice is considered. This also appears in this model when voting is costless and voters all participate in the election. Nevertheless, if voting is costly, then I find that for most instances there exists at least one and at most two pure strategy equilibria.

As intuition suggests, the voting cost and asymmetry between candidates does influence candidates' valence accumulation. On average, a low voting cost induces high valences, but there exists an interval of voting costs such that candidates increase their valence accumulation as the voting cost increases over this interval. More importantly, when the voting cost is low enough or high enough, then candidates' valences are constant. On the other hand, when candidates become similar (in terms of their marginal costs), their equilibrium valences choices become similar, too.

This paper contributes the large literature of voting games in the following aspects. First of all, while a candidate's fraction of supporters, $k$, is exogenously given in Feddersen and Sandroni (2006b), it is endogenously decided by candidates here. Since $k$ is an important parameter in Feddersen and Sandroni (2006b), it is meaningful to see how this parameter is decided and what is the equilibrium value of $k$. Next, because the previous models either consider candidates' campaign behavior by fixing voters' voting behavior, or consider voters' voting behavior by fixing candidates' campaign behavior, these models actually provided *partial equilibrium* analysis for elections. But I consider candidates' and voters' behavior in a single model, so I analyze elections from the *general equilibrium* perspective.

I review the related work in Section 2, then present the basic model in Section 3. In Section 4, I show that there exists at least one pure strategy equilibrium for most elections and the comparative statics the model delivers. The paper is concluded in Section 5.

## 2. Literature Review

Since Stokes (1963) (also Stokes 1992), there have been numerous studies on the valence issue. Valence has be used to extend the Downisan model (e.g., Groseclose (2001), Aragones and Palfrey 2002, 2005, Hummel 2010, etc.), and to explain primary elections (e.g., Hummel 2009, Snyder and Ting 2011, Serra 2011, etc.) — in those models, a primary election is used to choose candidates with higher valence.

My paper is related to the literature of endogenous valence competition. Meirowitz (2008) considers a campaign in which candidates select effort levels to win. Meirowitz shows that there is no pure strategy in that game and candidates' efforts are chosen randomly. Ashworth and Bueno de Mesquita (2009) consider a game in which candidates first choose platforms and then choose valences. Recently, Crutzen, Castanheira and Sahuguet (2010) consider a model where candidates have to win a primary election before they compete in a general election. In their model, a candidate inputs effort to design the platform.

The papers that analyze political competition when candidates are asymmetric are Aragones and Palfrey (2002), Kartik and McAfee (2007), Meirowitz (2008), Hummel (2009), and Krasa and Polborn (2010), etc. Aragones and Palfrey (2002) consider a spatial model of two-candidate elections, in which one candidate is of high quality. They show that if the location of the median voter's ideal point is uncertain, then candidates choose their policy platforms useing pure strategy equilibria. Kartik and McAfee (2007) consider a model where a candidate with superior valence is exogenously committed to a policy. Meirowitz (2008) considers a campaign in which two candidates select effort levels to win an election, but their productivity is different. Hummel (2009) considers a model in which candidates with different given valences and they have to win a primary election before competing in the general election. In an interesting paper, Krasa and Polborn (2010) model a situation in which candidates with different productivities, in two policy areas, compete for voters by choosing how much money or effort they would allocate to each area if elected.

In literature about costly voting,[3] Börgers (2004) considers voting in a club (the number of voters is finite). In his model, the decision of a voter to appear in the election lowers the probability that any other voter is pivotal and thus reduces other voters' benefit from voting. It is this negative externality of voting that makes the author conclude that compulsory voting is undesirable. Coate and Conlin (2004), and Feddersen and Sandroni (2006b), follow Harsanyi (1977, 1980, 1992), consider votings in large elections. They assume that voters cast their ballot costly and people go out to vote because they are ethical — they think they should do their part. Most recently, Degan and Merlo (2011) consider an uncertainty voting model, where the cost of voting is from voters' uncertainty about candidates' platforms.

The paper will adopt Feddersen and Sandroni's *ethical-voters* model. This is because, firstly, it fits the empirical evidence (Coate and Conlin, 2004), and secondly, in this paper I study agents' behavior in large elections in which voters' cost of voting is exogenously given.

The two papers that are most related to mine are Herrera, Levine and Martinelli (2008) and Degan (2011). In Herrera et al. (2008), turnout is assumed to

[3] Merlo (2006) is an excellent review on this subject.

respond to campaign spending. However, in my model, spending determines supporter share and the turnout is determined by the supporter share and voting cost in turn. Degan (2011) considers a political advertising model. The main difference between her paper and mine is that in her paper candidates' qualities are unknown ex ante to voters and a candidate can spend money or exert effort in order to communicate her quality to voters; further, candidates are symmetric in her model. But in my paper, there is no uncertainty about candidates' qualities and they are asymmetric.

## 3. The Model

I consider a two-stage election game. At the first stage, two candidates choose their valences simultaneously; at the second stage, voters cast their votes, also simultaneously, after observing candidates' valence choices. The model is based on Feddersen and Sandroni (2006b).

This section is divided into two subsections. I present the basic model in the first subsection; in the next subsection I introduce each candidate's victory probability in an election.

Throughout this paper, I always use *she* to refer a typical candidate and *he* a typical voter.

### 3.1. *The Setting*

There are two candidates, labeled $A$ and $B$, who compete in an election. A typical candidate is denoted by $c$, and her rival is denoted by $-c$. There is a continuum of voters of measure one. Voters have to decide whether to vote or abstain. If they choose to vote, they vote for the candidate they support. The electoral outcome is decided by majority rule, where candidate $A$ wins if the fraction of voters in favor of $A$ exceeds the fraction of voters in favor of $B$. A tie is broken by tossing a fair coin.

Each candidate $c \in \{A, B\}$ chooses her valence $v_c \in \mathbb{R}_+$ before voters cast their votes. Voters observe candidates' valences, $v_A$ and $v_B$, and decide which candidate to support; that is, the pair of valences $(v_A, v_B)$ divides voters between supporters of $A$ (*type A* voters) and supporters of $B$ (*type B* voters). Formally, there is a function $\psi\colon \mathbb{R}_+ \to \mathbb{R}_+$, such that the fraction of type $c \in \{A, B\}$ voters is given by

$$k_c(v_c, v_{-c}) = \frac{\psi(v_c)}{\psi(v_c) + \psi(v_{-c})},$$

with $k_c(0,0) = 1/2$. I suppose that $\psi$ is concave, strictly increasing and satisfies $\psi(0) = 0$. Here, $k_c$ can be understood as a contest success function; see Corchón and Dahm (2010) for a discussion.

Each voter has a voting cost, given by $\bar{c} > 0$ multiplied by an independent uniformly distributed random variable supported on the interval $(0, 1)$.

Each voter's cost of voting is independent of any other random variable in this model. Each voter's cost of voting is his private information; that is, a voter knows his own realized voting cost, but not the realization of other voters' costs of voting.

Some voters are *ethical voters* and others are *abstainers*. Ethical voters are "group rule-utilitarians" (Harsanyi, 1980), where a group of voters are the same type, i.e., prefer the same candidate; abstainers always abstain. The fraction of ethical voters in each group type are $q_A$ and $q_B$, which are independent and uniformly distributed over $[0, 1]$.

According to Feddersen and Sandroni (2006b), voters have preferences about which candidate wins and the social cost of the election. In particular, type $c \in \{A, B\}$ voters have a utility function given by

$$w \cdot \mathbb{P}[c \text{ wins}] - \Phi, \tag{1}$$

where $w \in \mathbb{R}_{++}$ measures the importance of the election to a typical voter and $\Phi$ is the expected social cost of voting, which will be given later. For further reference, I define

$$e := \frac{\bar{c}}{w}.$$

Hence, $e$ is the ratio of voting cost to the importance of election. From now on, I will call an election $e$ if in which the upper bound of voting cost is $\bar{c}$, and the importance of the election for voters is $w$.

Let $\beta_c \in \mathbb{R}_{++}$ be the marginal cost of accumulating valence for candidate $c$. It is assumed that $\beta_A > \beta_B$ (except in Proposition 10, where I consider candidates' optimal valence choices when they have the same marginal cost). The benefit from winning office is normalized to be 1. Let the probability that candidate $c$ wins an election $e$ be $p_c(v_c, v_{-c}, e)$. Hence, the utility function for candidate $c$ is of the form

$$U_c(v_c, v_{-c}, e) = p_c(v_c, v_{-c}, e) - \beta_c v_c. \tag{2}$$

I now review the main results in Feddersen and Sandroni (2006b). Suppose that candidates implement valence $v_A$ and $v_B$, so the fraction of each voters type is $k_A$ and $k_B$. The authors introduce the concept of a *rule profile*, which is a pair of cutoff points $(\sigma_A, \sigma_B) \in [0, 1] \times [0, 1]$, such that an ethical type $c$ voter participates in the election and votes for candidate $c$, if and only if, his cost is below $\sigma_c \bar{c}$. In this context, candidate $A$ wins, if and only if, she receives the majority of votes, that is, $k_A q_A \sigma_A > k_B q_B \sigma_B$. Or equivalently,

$$\frac{q_B}{q_A} < \frac{k_A \sigma_A}{k_B \sigma_B}.$$

Let $F$ be the cumulative distribution function of $q_B/q_A$. It is evident to see that[4]

$$F(z) = \begin{cases} z/2 & \text{if } z \leqslant 1 \\ 1 - 1/(2z) & \text{if } z > 1. \end{cases}$$

Hence, if ethical voters use the rule profile $(\sigma_A, \sigma_B)$, then candidate $A$ is elected with probability

$$p_A(v_A, v_B, e) = p_A(\sigma_A, \sigma_B) = F\left(\frac{k_A\sigma_A}{k_B\sigma_B}\right).$$

Now, the expected social cost of voting from a rule profile $(\sigma_A, \sigma_B)$ can be obtained as follows:

$$\begin{aligned} \Phi(\sigma_A, \sigma_B) &= \bar{c}\left[k_A\mathbb{E}[q_A]\int_0^{\sigma_A} x\,\mathrm{d}x + k_B\mathbb{E}[q_B]\int_0^{\sigma_B} x\,\mathrm{d}x\right] \\ &= \frac{\bar{c}}{4}\left[k_A\sigma_A^2 + k_B\sigma_B^2\right]. \end{aligned}$$

Given voters preferences in (1), it follows that if ethical voters act according to the rule profile $(\sigma_A, \sigma_B)$, then the induced expected payoff for voters of type $c \in \{A, B\}$ is

$$V_c(\sigma_A, \sigma_B) = wp_c(\sigma_A, \sigma_B) - \Phi(\sigma_A, \sigma_B).$$

Feddersen and Sandroni (2006b) introduce the following concept of *consistent rules*:[5]

Definition 1 (Feddersen and Sandroni 2006b, Definition 1). The pair $(\sigma_A^*, \sigma_B^*) \in (0, 1] \times (0, 1]$ is a consistent rule profile if $V_c(\sigma_c^*, \sigma_{-c}^*) \geqslant V_c(\sigma_c, \sigma_{-c}^*)$ for all $\sigma_c \in [0, 1]$ and every $c \in \{A, B\}$.

Therefore, if everybody follows the rule profile $(\sigma_A^*, \sigma_B^*)$, then each ethical-voter of the same type achieves the best outcome. The consistent rule profile $(\sigma_A^*, \sigma_B^*)$ obtained in Feddersen and Sandroni (2006b, Table 1) are summarized in Figure 1, where, for example, the pair

$$\boldsymbol{\sigma}^* = \left(\left(\frac{1}{e^2 k_A k_B}\right)^{1/4}, \left(\frac{k_A}{e^2 k_B^3}\right)^{1/4}\right)$$

is the consistent rules for a triple $(e, k_A, k_B)$ satisfying $e > [1/(k_A k_B)]^{1/2}$. Those rules determine candidate $A$'s (and consequently, candidate $B$'s) probabilities of victory according to the distribution function $F[k_A\sigma_A^*/(k_B\sigma_B^*)]$, which are also depicted in Figure 1.

[4] If $z \leqslant 1$, the $F(z) = \int_0^1 zq_A\,\mathrm{d}q_A = z/2$; if $z > 1$, then $F(z) = \int_0^{1/z} zq_A\,\mathrm{d}q_A + \int_{1/z}^1 \mathrm{d}q_A = 1 - 1/(2z)$.

[5] Feddersen and Sandroni (2006a) provide a justification for the concept of consistent rules.

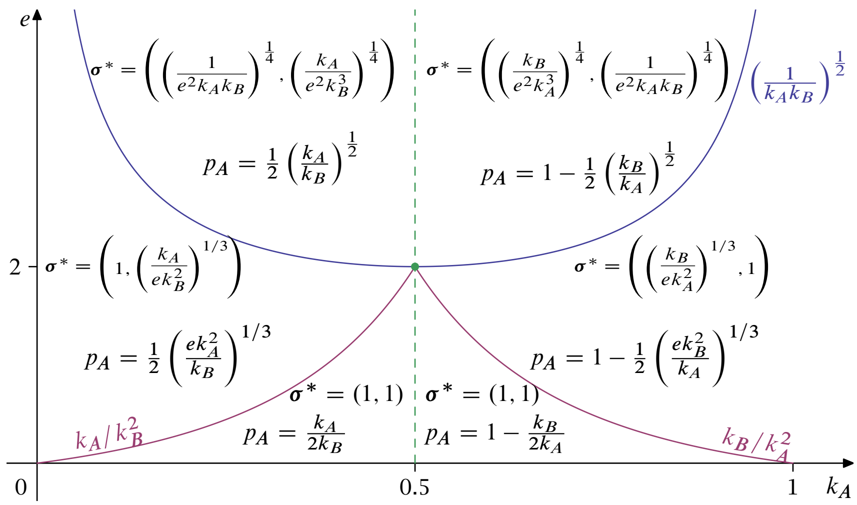



FIGURE 1. The consistent rule profile $\boldsymbol{\sigma}^* = (\sigma_A^*, \sigma_B^*)$, and candidate $A$'s victory probabilities, $p_A$.

### 3.2. *Victory Probabilities*

To analyze candidates' equilibrium valence choices, we first need to do some groundwork, that is, to derive candidate $c$'s victory probability $p_c(v_c, v_{-c}, e)$ in an election $e$ when the pair of valences is $(v_c, v_{-c})$.

Notice that Figure 1 is based on $k_A$, the fraction of supporters for candidate $A$. Since $k_A$ is endogenously determined by a pair of valences $(v_c, v_{-c})$, Figure 1 can be transferred to a new figure based on $(v_c, v_{-c})$. Formally, inserting $k_c(v_c, v_{-c})$, where $v_{-c} > 0$, into the function $[1/(k_A k_B)]^{1/2}$ in Figure 1 yields the following function $\Gamma \colon \mathbb{R}_+^2 \to \mathbb{R}$ defined by

$$\Gamma(v_c, v_{-c}) = \frac{\psi(v_c) + \psi(v_{-c})}{\sqrt{\psi(v_c)\psi(v_{-c})}}.$$

Similarly, inserting $k_c(v_c, v_{-c})$ into $k_c/k_{-c}^2$ we get the function $\Theta_c \colon \mathbb{R}_+^2 \to \mathbb{R}$, for each $c \in \{A, B\}$, defined by

$$\Theta_c(v_c, v_{-c}) = \frac{\psi(v_c)\left[\psi(v_c) + \psi(v_{-c})\right]}{\psi^2(v_{-c})}.$$

These three functions are depicted in Figure 2.

Fix an election $\hat{e} \in (2, \infty)$. The constant function $e = \hat{e}$ intersects $\Gamma(v_c, v_{-c})$ at two points, $a_1$ and $a_2$, respectively. The projections of these two points on the $v_c$-axis play a crucial role for deriving the victory probability of candidate $c$. Denote the projection of $a_1$ on the $v_c$-axis by $L(v_{-c}, \hat{e})$, which is candidate $c$'s *minimal* valence such that the value of the function $\Gamma(\cdot, v_{-c})$ is $\hat{e}$. Also, denote the projection of $a_2$ on the $v_c$-axis by $R(v_{-c}, \hat{e})$, which is candidate $c$'s *maximal* valence such that $\Gamma(v_c, v_{-c}) = \hat{e}$.

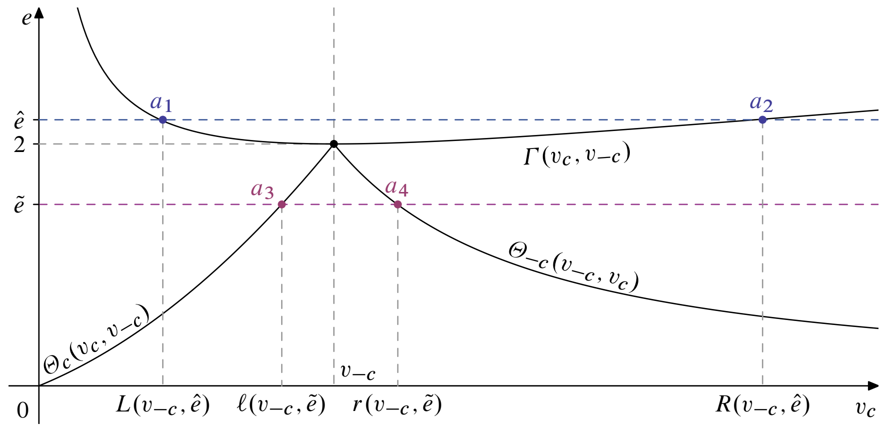



FIGURE 2. Some points, where $v_{-c}$ is given.

Analogously, for some election $\tilde{e} \in (0, 2)$, the function $e = \tilde{e}$ intersects $\Theta_c$ at the point $a_3$ and intersects $\Theta_{-c}$ at the point $a_4$. Call the projection of $a_3$ ($a_4$, resp.) on the $v_c$-axis $\ell(v_{-c}, \tilde{e})$ ($r(v_{-c}, \tilde{e})$, resp.).

The general form of $L(\cdot, \cdot)$, $R(\cdot, \cdot)$, $\ell(\cdot, \cdot)$ and $r(\cdot, \cdot)$ are given in Appendix A. There are pleasant connections between these functions (the proof is also in Appendix A): for $e \in (2, \infty)$,

$$v_c < L(v_{-c}, e) \iff R(v_c, e) < v_{-c}. \tag{3}$$

and for $e \in (0, 2)$,

$$v_c < \ell(v_{-c}, e) \iff r(v_c, e) < v_{-c}. \tag{4}$$

The above (3) means that for every election $e \in (2, \infty)$ and and every pair of valences $(v_c, v_{-c})$, if $v_c$ locates at the left-side of the point $L(v_{-c}, e)$, then $v_{-c}$ must locate at the right-side of the point $R(v_c, e)$, and vise versa. One can interpret (4) similarly.

We are now in a position to obtain the victory probability for candidate $c$ from a pair of valences $(v_c, v_{-c})$. For every $e \in (2, \infty)$ and $v_{-c} > 0$, the points $L(v_{-c}, e)$, $v_{-c}$ and $R(v_{-c}, e)$ divides $p_c(v_c, v_{-c}, e)$ into four parts. As an example, if $v_c \in [0, L(v_{-c}, e)]$, then

$$\begin{aligned} p_c(v_c, v_{-c}, e) = \frac{1}{2}\left(\frac{ek_c^2}{k_{-c}}\right)^{1/3} &= \frac{1}{2}\left[\frac{e\psi^2(v_c)/[\psi(v_c)+\psi(v_{-c})]^2}{\psi(v_{-c})/[\psi(v_c)+\psi(v_{-c})]}\right]^{1/3} \\ &= \frac{1}{2}\left[\frac{e\psi^2(v_c)}{\psi(v_{-c})\left[\psi(v_c)+\psi(v_{-c})\right]}\right]^{1/3}. \end{aligned}$$

Now insert $k_c(v_c, v_{-c}) = \psi(v_c)/[\psi(v_c) + \psi(v_{-c})]$ into the probabilities in Figure 1, and then define (just for the sake of expression):

- For $v_c \leqslant v_{-c}$:

$$\mu_c(v_c, v_{-c}, e) := \frac{1}{2}\left[\frac{e\psi^2(v_c)}{\psi(v_{-c})\left[\psi(v_c) + \psi(v_{-c})\right]}\right]^{1/3},$$

$$\pi_c(v_c, v_{-c}) := \frac{1}{2}\left[\frac{\psi(v_c)}{\psi(v_{-c})}\right]^{1/2},$$

$$\rho_c(v_c, v_{-c}) := \frac{1}{2}\left[\frac{\psi(v_c)}{\psi(v_{-c})}\right].$$

- For $v_c > v_{-c}$:

$$\widetilde{\mu}_c(v_c, v_{-c}, e) := 1 - \mu_{-c}(v_{-c}, v_c, e),$$

$$\widetilde{\pi}_c(v_c, v_{-c}) := 1 - \pi_{-c}(v_{-c}, v_c),$$

$$\widetilde{\rho}_c(v_c, v_{-c}) := 1 - \rho_{-c}(v_{-c}, v_c).$$

LEMMA 2. *Consider an election $e$ and a pair of valences $(v_c, v_{-c})$ with $v_{-c} > 0$.*

- *If $e \in [2, \infty)$, then candidate $c \in \{A, B\}$ is elected in the election with probability*

$$p_c(v_c, v_{-c}, e) = \begin{cases} \mu_c(v_c, v_{-c}, e) & \text{if } v_c \in [0, L(v_{-c}, e)] \\ \pi_c(v_c, v_{-c}) & \text{if } v_c \in (L(v_{-c}, e), v_{-c}] \\ \widetilde{\pi}_c(v_c, v_{-c}) & \text{if } v_c \in (v_{-c}, R(v_{-c}, e)] \\ \widetilde{\mu}_c(v_c, v_{-c}, e) & \text{if } v_c \in (R(v_{-c}, e), \infty). \end{cases}$$

- *If $e \in (0, 2)$, then candidate $c \in \{A, B\}$ is elected in the election with probability*

$$p_c(v_c, v_{-c}, e) = \begin{cases} \mu_c(v_c, v_{-c}, e) & \text{if } v_c \in [0, \ell(v_{-c}, e)] \\ \rho_c(v_c, v_{-c}) & \text{if } v_c \in (\ell(v_{-c}, e), v_{-c}] \\ \widetilde{\rho}_c(v_c, v_{-c}) & \text{if } v_c \in (v_{-c}, r(v_{-c}, e)] \\ \widetilde{\mu}_c(v_c, v_{-c}, e) & \text{if } v_c \in (r(v_{-c}, e), \infty). \end{cases}$$

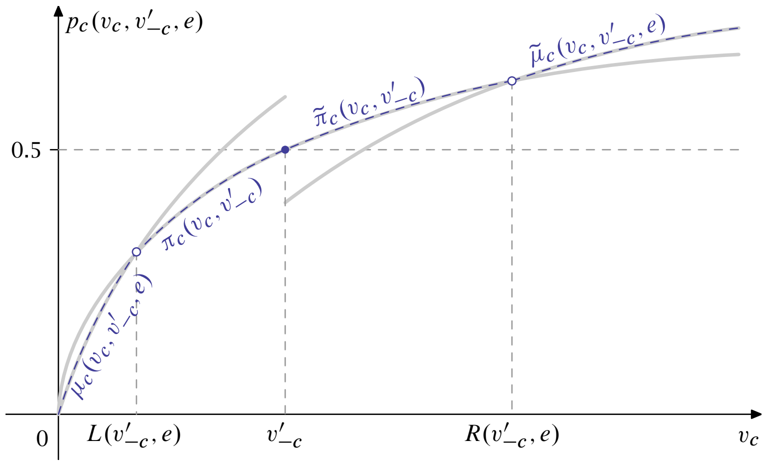
The victory probabilities for candidate $c$ derived in Lemma 2 are illustrated in Figure 2, where candidate $-c$'s campaign effort is fixed at $v'_{-c} > 0$.

The probability $p_c(v_c, v_{-c}, e)$ can be reformulated as follows:

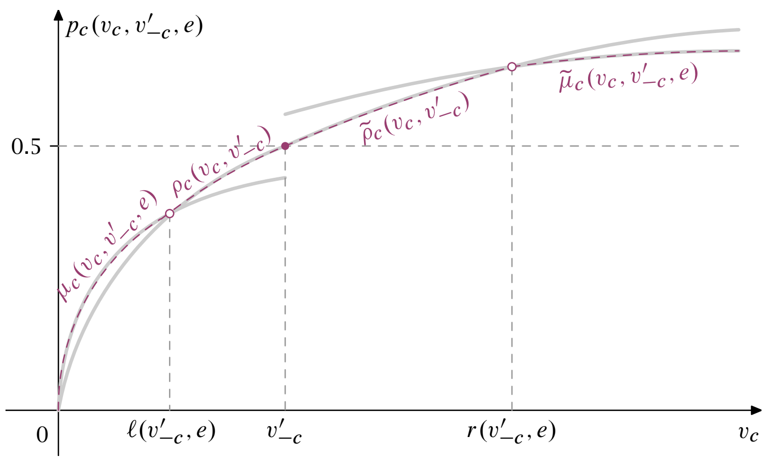
COROLLARY 3. *Candidate $c$ wins in an election $e$ with probability*

$$p_c(v_c, v_{-c}, e) = \begin{cases} \min\{\mu_c(v_c, v_{-c}, e), \pi_c(v_c, v_{-c})\} & \text{if } e \in [2, \infty) \text{ and } v_c \leqslant v_{-c} \\ \max\left\{\widetilde{\pi}_c(v_c, v_{-c}), \widetilde{\mu}_c(v_c, v_{-c}, e)\right\} & \text{if } e \in [2, \infty) \text{ and } v_c > v_{-c} \\ \max\{\mu_c(v_c, v_{-c}, e), \rho_c(v_c, v_{-c})\} & \text{if } e \in (0, 2) \text{ and } v_c \leqslant v_{-c} \\ \min\left\{\widetilde{\rho}_c(v_c, v_{-c}), \widetilde{\mu}_c(v_c, v_{-c}, e)\right\} & \text{if } e \in (0, 2) \text{ and } v_c > v_{-c}. \end{cases}$$

PROOF. All proofs are relegated to Appendix B. □




(a) Candidate $c$'s victory probability when $e \in (2, \infty)$.




(b) Candidate $c$'s victory probability when $e \in (0, 2)$.

Figure 3. Candidate $c$'s victory probabilities in an election $e$ when her opponent's campaign effort is $v'_{-c}$.

According to Corollary 3, Figure 3 illustrates candidate $c$'s victory probabilities, where candidate $-c$'s actions is fixed as $v'_{-c} > 0$.

Notation. Let $f(x_1, x_2, \cdot)$ be a real-valued function, where $(x_1, x_2) \in \mathbb{R}^2$; let $\mathbf{a}$ be a point in the domain of $f$; let $(i, j) \in \{1, 2\} \times \{1, 2\}$. I denote $\mathbb{D}_i f(\mathbf{a})$ the $i^{\text{th}}$ partial derivative of $f$ at $\mathbf{a}$ if it exists, and $\mathbb{D}_{ij} f(\mathbf{a}) := \mathbb{D}_j[\mathbb{D}_i f(\mathbf{a})]$ the mixed partial derivative of $f$ at $\mathbf{a}$ if it exists. Therefore, in terms of the Leibnitz notation, $\mathbb{D}_i f(\mathbf{a}) = \partial f(\mathbf{a})/\partial x_i$, and $\mathbb{D}_{ij} f(\mathbf{a}) = \partial^2 f(\mathbf{a})/\partial x_i \partial x_j$.

It will be useful to characterize the properties of $\mathbb{D}_1 p_c(v_c, v_{-c}, e)$, the marginal probability that candidate $c \in \{A, B\}$ wins in an election $e \in (0, \infty)$. These properties are summarized in the following Lemma 4.

LEMMA 4. *Consider an election $e$. For every $v_{-c} > 0$, the partial derivative $\mathbb{D}_1 p_c(v_c, v_{-c}, e)$ exists except for the points $L(v_{-c}, e)$ and $R(v_{-c}, e)$ if $e \in (2, \infty)$, or $\ell(v_{-c}, e)$ and $r(v_{-c}, e)$ if $e \in (0, 2)$. Whenever $p_c(v_c, v_{-c}, e)$ is differentiable, the following properties hold:*

- $\mathbb{D}_1 p_c(v_c, v_{-c}, e) > 0$ *and* $\mathbb{D}_{11} p_c(v_c, v_{-c}, e) < 0$.
- $\mathbb{D}_{12} p_c(v_c, v_{-c}, e) < 0$ *if* $v_c \leqslant v_{-c}$.
- $\mathbb{D}_{12} p_c(v_c, v_{-c}, e) > 0$ *if* $v_c > v_{-c}$.

## 4. RESULTS

The appropriate equilibrium concept for the election game is the modified *subgame perfect Nash equilibrium*, which consists of a pair of valences $(v_c^*, v_{-c}^*)$ chosen by the two candidates, and a pair of consistent rules $(\sigma_A^*, \sigma_B^*)$ chosen by voters. Since the consistent rules have been derived by Feddersen and Sandroni (2006b), in the remainder of this paper I use the term *equilibrium* to mean a pair of valences $(v_c^*, v_{-c}^*)$ such that for each candidate $c \in \{A, B\}$, her choice $v_c^*$ maximizes the expected utility $U_c(v_c, v_{-c}^*, e)$ given her opponent's choice $v_{-c}^*$ and voters' consistent rules $(\sigma_A^*, \sigma_B^*)$.

In this section, I first characterize the pure strategy equilibria in the election game. Then I identify sufficient conditions so that there exists at least one pure strategy equilibrium. The major finding of this section is that in most environments there exists at least one pure strategy equilibrium.

### 4.1. *Existence of Equilibrium*

With the interpretation of equilibrium at the beginning of this section, an election $e$ can be represented by a normal form game $\mathcal{G}_e = (S_c', U_c)_{c \in \{A,B\}}$, where $S_c'$ is candidate $c$'s action space. For each $v_{-c} > 0$, candidate $c$'s expected payoff is $U_c(v_c, v_c, e) = p_c(v_c, v_{-c}, e) - \beta_c v_c$. Since $p_c \in [0, 1]$, we have $U_c(v_c, v_{-c}, e) \leqslant 1 - \beta_c v_c$; therefore, candidate $c$ never chooses $v_c > 1/\beta_c$. So we can restrict $S_c' = [0, 1/\beta_c]$, a convex and compact subset in $\mathbb{R}$.

Two significant features of the game $\mathcal{G}_e$ is that (i) $U_c$ is not upper semicontinuous at the point $(0, 0)$ and (ii) $U_c$ is not quasiconcave.

CLAIM 1. *The function $U_c(\cdot, \cdot, e)\colon S_c' \times S_{-c}' \to \mathbb{R}$ is not upper semicontinuous at $(0, 0)$ for each candidate $c \in \{A, B\}$.*

To see this, notice that $p_c(0, 0, e) = 1/2$ by assumption.[6] Then $U_c(0, 0, e) = p_c(0, 0, e) = 1/2$. Take a sequence $\{1/n\}_{n=1}^{\infty}$. Then $\{(1/n, 0)\} \to (0, 0)$. For each

[6]The assumption that each candidate wins with equal probability if $v_c = v_{-c} = 0$ is non-essential.

$n$, candidate $c$ wins with probability 1; hence,

$$U_c(1/n, 0, e) = 1 - \frac{\beta_c}{n} \to 1.$$

Claim 2. *$U_c(\cdot, v_{-c}, e)$: $S'_c \to \mathbb{R}$ is not quasiconcave for each candidate $c \in \{A, B\}$ when $e \neq 2$ and $v_{-c} > 0$.*

Perhaps the easiest way to see that $U_c$ fails to be quasiconcave is to notice that when $e \in (2, \infty)$ and $v_c > v_{-c} > 0$, candidate $c$'s expected utility function $U_c(\cdot, v_{-c}, e)$ is a concatenation of two strictly concave functions.

These two features mean the classical existence theorems cannot be applied here. For example, Dasgupta and Maskin (1986) require $U_c$ to be upper semicontinuous, and (Reny, 1999) and its strengthening form in McLennan, Monteiro and Tourky (2011) require $U_c$ to be quasiconcave.

To make things as simple as possible, I adopt a mild assumption:

Assumption 1. A candidate $c \in \{A, B\}$ has to implement a valence at least $\xi_c > 0$, where $\mathbb{D}_1 \mu_c(\xi_c, v_{-c}, e) > \beta_c$.

The above Assumption 1 is actually a moderate and natural assumption. It just says that a candidate cannot quit the electoral competition once she has participated in: she has to take part in at least one campaign debate, make an advertisement, etc.

Now, candidate $c$'s action space is $S_c = [\xi_c, 1/\beta_c]$, which is compact (by Heine-Borel Theorem). Furthermore, for each $e \in (0, \infty)$, candidate $c$'s payoff function $U_c(\cdot, \cdot, e)$ is continuous on $[\xi_c, 1/\beta_c] \times [\xi_{-c}, 1/\beta_{-c}]$. It follows from the Glicksberg-Fan Fixed Point Theorem (Glicksberg, 1952; Fan, 1952) that:

Lemma 5. *There exists at least one equilibrium (possibly mixed) in the election game for every $e \in (0, \infty)$.*

An equilibrium I consider hereafter is a pure strategy equilibrium otherwise stated explicitly.

### 4.2. *Preliminaries*

In this subsection, I derive some primary results and consider the equilibria under special environments. In particular, I show that there is no pure strategy equilibrium when voting is costless and there is a unique pure strategy equilibrium when $e = 2$.

I first analyze candidates' behavior in the setting where voters cast their votes without cost and all participate in the election. Under this setting, candidate $c \in \{A, B\}$ wins with probability 1 if and only if she receives the majority of votes, that is, if and only if

$$k_c(v_c, v_{-c}) > k_{-c}(v_{-c}, v_c) \iff \frac{\psi(v_c)}{\psi(v_c)+\psi(v_{-c})} > \frac{\psi(v_{-c})}{\psi(v_c)+\psi(v_{-c})}$$
$$\iff v_c > v_{-c}.$$

PROPOSITION 6. *There exists no pure strategy equilibrium if voting is costless.*

REMARK. When valences are decided by candidates endogenously, there is usually no pure strategy equilibrium; see Groseclose (2001), Meirowitz (2008), and Ashworth and Bueno de Mesquita (2008), etc.

From now on, I suppose that $\bar{c} > 0$. In what follows, I show that when the voting cost is taken into account, there exists pure strategy equilibria for most instances.

For a candidate $c \in \{A, B\}$, let $\mathrm{BR}_c : S_{-c} \times \mathbb{R}_{++} \twoheadrightarrow S_c$ be her *best response correspondence*, where $\mathrm{BR}_c(v_{-c}, e)$ is the set of candidate $c$'s optimal valence choices in an election $e \in (0, \infty)$ when her opponent's choice is $v_{-c}$. The following Lemma 7 establishes some useful properties of $\mathrm{BR}_c$.

LEMMA 7. *For every candidate $c \in \{A, B\}$ and every $v_{-c} > 0$, (i) if $e \in (2, \infty)$, then $R(v_{-c}, e) \notin \mathrm{BR}_c(v_{-c}, e)$, and (ii) if $e \in (0, 2)$, then $L(v_{-c}, e) \notin \mathrm{BR}_c(v_{-c}, e)$. Hence, if a pair of valences $(v_A^*, v_B^*)$ consists of an equilibrium for an election $e \in (0, \infty)$, then $\mathbb{D}_1 p_c(v_c^*, v_{-c}^*, e)$ exists for each candidate $c \in \{A, B\}$.*

In an election $e \in (0, \infty)$, candidate $c \in \{A, B\}$ chooses $v_c$, given her opponent's valence choice $v_{-c}$, to maximize her expected payoff; that is,

$$\max_{v_c \in \mathbb{R}_+} \{p_c(v_c, v_{-c}, e) - \beta_c v_c\}.$$

If $p_c(v_c, v_{-c}, e)$ is differentiable at $(v_c, v_{-c})$, then the first-order condition for candidate $c$'s objective function yields

$$\mathbb{D}_1 p_c(v_c, v_{-c}, e) = \beta_c. \tag{5}$$

The following Lemma 8 gives the necessary condition for the existence of equilibrium. Let $\Lambda := \psi'/\psi$. Notice that $\Lambda(x)$ is well-defined, strictly decreasing with $x$, $\lim_{x\to\infty} \Lambda(x) = 0$, and $\lim_{x\to 0^+} \Lambda(x) = \infty$.[7]

LEMMA 8. *If a pair of valences $(v_A, v_B)$ consists of an equilibrium, then $(v_A, v_B) \neq (0, 0)$, $v_A < v_B$, and*

$$\frac{\Lambda(v_A)}{\Lambda(v_B)} = \frac{\beta_A}{\beta_B}. \tag{6}$$

The following inequality will be used frequently, so it is relevant to present it here now.

[7]The notation $\lim_{x\to 0^+}$ denotes $x$ approaches 0 across through the positive values.

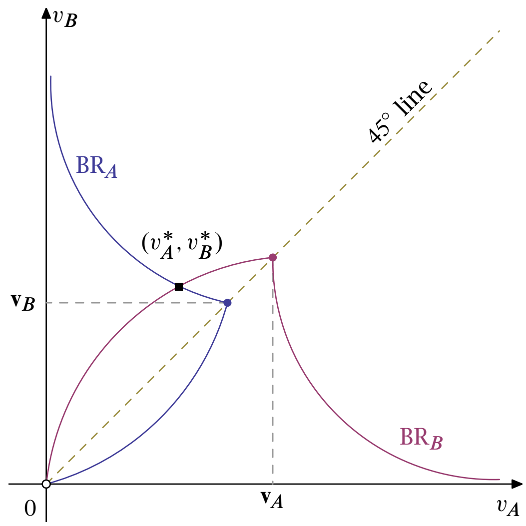



FIGURE 4. Best response functions and equilibrium for $e = 2$.

LEMMA 9. *For every election* $e \in (2, \infty)$ *and every valence* $v_{-c} > 0$*, the following inequality holds:*

$$\mathbb{D}_1 \widetilde{\mu}_c \left( R(v_{-c}, e), v_{-c}, e \right) < \mathbb{D}_1 \widetilde{\pi}_c \left( v_{-c}, v_{-c} \right). \tag{7}$$

The above inequality (7) simply says that the partial derivative of $\widetilde{\mu}_c$ at the indifferentiable point $R(v_{-c}, e)$[8] is strictly less than the partial derivative of $\widetilde{\pi}_c$ at the point where candidate $c$ chooses the same valence as her opponent $-c$. It is this inequality that facilitates the characterization of incentive compatiblility conditions later.

Before leaving this subsection, I consider the special case that $e = 2$. In this case, candidate $c \in \{A, B\}$ is elected with probability

$$p_c(v_c, v_{-c}, 2) = \begin{cases} \mu_c(v_c, v_{-c}, 2) & \text{if } v_c \leqslant v_{-c} \\ \widetilde{\mu}_c(v_c, v_{-c}, 2) & \text{if } v_c > v_{-c}. \end{cases}$$

Therefore, $p_c(v_c, v_{-c}, 2)$ is differentiable everywhere for each $c \in \{A, B\}$, and so it is straightforward to characterize the best response functions $\mathrm{BR}_c$. In Figure 4 the best response functions and the unique equilibrium are depicted, where the shapes of $\mathrm{BR}_A$ and $\mathrm{BR}_B$ follow from **??** immediately, and their shapes induce the conclusion that they must intersect at one and only one point, $(v_A^*, v_B^*)$ — the equilibrium for $e = 2$.

[8]Some comments about the term *indifferentiability* is in order. Clearly, $\mathbb{D}_1 \widetilde{\mu}_c(v_c, v_{-c}, e)$ always exists. What I mean of *indifferentiablity* is that at some point $\mathbb{D}_1 p_c(v_c, v_{-c}, e)$ fails to exist. For instance, at $R(v_{-c}, e)$ we have $\mathbb{D}_1 \widetilde{\pi}_c(R(v_{-c}, e), v_{-c}) \neq \mathbb{D}_1 \widetilde{\mu}_c(R(v_{-c}, e), v_{-c})$.

PROPOSITION 10. *In the election* $e = 2$*, there exists a unique equilibrium* $(v_A^*, v_B^*)$*, where* $v_A^* < v_B^*$*, and* $(\mathbb{D}_1\mu_A(v_A^*, v_B^*, 2), \mathbb{D}_1\widetilde{\mu}_B(v_B^*, v_A^*, 2)) = (\beta_A, \beta_B)$.

### 4.3. *Equilibria for* $e \in (2, \infty)$

In this subsection I discuss the sufficient conditions so that pure strategy equilibria exist for elections $e \in (2, \infty)$. Let $v_{c,e}$ be candidate $c$'s valence choice in an election $e$.

NOTATION. Since the notation $v_{c,e}$ already indicates clearly which election we are talking about, I will simply denote $f(v_{c,e}, e)$ as $f(v_{c,e})$ for a function $f$ which also depends on $e$. But where there is a possibility of confusion, then I shall revert back to the more descriptive notation $f(v_{c,e}, e)$.

Obviously, for an election $e \in (2, \infty)$ and a pair of valences $(v_c, v_{-c})$, if candidate $c$ wins with probability $\mu_c(v_c, v_{-c}, e)$, then candidate $-c$ wins with probability $1 - \mu_c(v_c, v_{-c}, e) = \widetilde{\mu}_{-c}(v_{-c}, v_c, e)$. Similarly, if candidate $c$ wins with probability $\pi_c(v_c, v_{-c})$, then candidate $-c$ wins with probability $1-\pi_c(v_c, v_{-c}) = \widetilde{\pi}_{-c}(v_{-c}, v_c)$. Therefore, for an election $e \in (2, \infty)$, there are two possible pure strategy equilibria. In the first equilibrium $(v_{A,e}^*, v_{B,e}^*)$, candidate $A$ and $B$ wins with probability $\mu_A(v_{A,e}^*, v_{B,e}^*)$ and $\widetilde{\mu}_B(v_{B,e}^*, v_{A,e}^*)$, respectively. In the second equilibrium $(v_A^*, v_B^*)$, candidate $A$ and $B$ wins with probability $\pi_A(v_A^*, v_B^*)$ and $\widetilde{\pi}_B(v_B^*, v_A^*)$, respectively. I denote the first equilibrium $\mathbf{E}_{\widetilde{\mu}e}$, and the second equilibrium $\mathbf{E}_\pi$ (it will be shown shortly that this equilibrium is independent of the particular election $e$). Let

$$U_B^{\widetilde{\pi}}(v_B, v_A) := \widetilde{\pi}_B(v_B, v_A) - \beta_B v_B,$$
$$U_B^{\widetilde{\mu}}(v_B, v_A, e) := \widetilde{\mu}_B(v_B, v_A, e) - \beta_B v_B.$$

The next proposition characterizes these two equilibria:

PROPOSITION 11. *Consider an election* $e \in (2, \infty)$.

1. *A pair of valences* $(v_{A,e}^*, v_{B,e}^*)$ *is an* $\mathbf{E}_{\widetilde{\mu}e}$*-equilibrium if and only if the following three conditions are satisfied:*

$$(\mathbf{E}_{\widetilde{\mu}e}\text{-a}) \qquad \left(\mathbb{D}_1\mu_A\left(v_{A,e}^*, v_{B,e}^*\right), \mathbb{D}_1\widetilde{\mu}_B\left(v_{B,e}^*, v_{A,e}^*\right)\right) = (\beta_A, \beta_B).$$

$$(\mathbf{E}_{\widetilde{\mu}e}\text{-b}) \qquad v_{A,e}^* < L\left(v_{B,e}^*\right).$$

$$(\mathbf{E}_{\widetilde{\mu}e}\text{-c}) \qquad U_B^{\widetilde{\mu}}\left(v_{B,e}^*, v_{A,e}^*\right) \geqslant \max_{v_B \in \left(v_{A,e}^*, R(v_{A,e}^*)\right)} \left\{U_B^{\widetilde{\pi}}\left(v_B, v_{A,e}^*\right)\right\}.$$

*In this equilibrium, both* $v_{A,e}^*$ *and* $v_{B,e}^*$ *are continuous and strictly increasing with* $e$.

2. *A pair of valences* $(v_A^*, v_B^*)$ *is an* $\mathbf{E}_\pi$*-equilibrium if and only if the following three conditions are satisfied:*

$$(\mathbf{E}_\pi\text{-a}) \qquad \left(\mathbb{D}_1\pi_A\left(v_A^*, v_B^*\right), \mathbb{D}_1\widetilde{\pi}_B\left(v_B^*, v_A^*\right)\right) = (\beta_A, \beta_B).$$

$$(\mathbf{E}_\pi\text{-b}) \qquad v_A^* > L\left(v_B^*, e\right).$$

$$(\mathbf{E}_\pi\text{-c}) \qquad U_B^{\widetilde{\pi}}\left(v_B^*, v_A^*\right) \geqslant \max_{v_B > R(v_A^*, e)} \left\{U_B^{\widetilde{\mu}}\left(v_B, v_A^*, e\right)\right\}.$$

*In this equilibrium, both* $v_A^*$ *and* $v_B^*$ *are independent of* $e$*.*

In Proposition 11, the condition ($\mathbf{E}_{\widetilde{\mu}e}$-a) is just the first-order conditions and ($\mathbf{E}_{\widetilde{\mu}e}$-b) says that given $e$ and $v_{B,e}^*$, candidate $A$'s optimal valence choice $v_{A,e}^*$ locates at the left-side of the indifferentiable point $L(v_{B,e}^*)$. Finally, ($\mathbf{E}_{\widetilde{\mu}e}$-c) is candidate $B$'s incentive compatibility condition: given $e$ and $v_{A,e}^*$, candidate $B$ does not want to decrease her valence to any point in the interval $(v_{A,e}^*, R(v_{A,e}^*))$. When conditions ($\mathbf{E}_{\widetilde{\mu}e}$-a) — ($\mathbf{E}_{\widetilde{\mu}e}$-c) hold, no candidate has incentive to deviate unilaterally. The interpretation of conditions ($\mathbf{E}_\pi$-a) — ($\mathbf{E}_\pi$-c) is similar.

Remark. Some comments on the unique profitable deviation for candidate $B$ in $\mathbf{E}_{\widetilde{\mu}e}$ are in order. Notice that by (7) and Corollary 3 the following inequalities hold:

$$\mathbb{D}_1\widetilde{\pi}_B\left(R(v_{A,e}^*), v_{A,e}^*\right) < \mathbb{D}_1\widetilde{\mu}_B\left(R(v_{A,e}^*), v_{A,e}^*\right) < \mathbb{D}_1\widetilde{\pi}_B\left(v_{A,e}^*, v_{A,e}^*\right);$$

see Figure 5. If the pair $(v_{A,e}^*, v_{B,e}^*)$ satisfies

$$\mathbb{D}_1\widetilde{\mu}_B\left(v_{B,e}^*, v_{A,e}^*\right) \in \left(\mathbb{D}_1\widetilde{\pi}_B\left(R(v_{A,e}^*), v_{A,e}^*\right), \mathbb{D}_1\widetilde{\mu}_B\left(R(v_{A,e}^*), v_{A,e}^*\right)\right),$$

then there exists a point $v_B' \in (v_{A,e}^*, R(v_{A,e}^*))$ such that

$$\mathbb{D}_1\widetilde{\pi}_B\left(v_B', v_{A,e}^*\right) = \mathbb{D}_1\widetilde{\mu}_B\left(v_{B,e}^*, v_{A,e}^*\right) = \beta_B.$$

By deviating to $v_B' \in (v_{A,e}^*, R(v_{A,e}^*))$, candidate $B$ decreases her victory probability, but also saves $\beta_B \cdot (v_{B,e}^* - v_B')$ of the campaign cost. So it is possible that this deviation is desirable for candidate $B$. Therefore, ($\mathbf{E}_{\widetilde{\mu}e}$-c) is necessary for preventing such a deviation.

Unfortunately, there exists election $e$ and functional form $\psi$ such that neither $\mathbf{E}_{\widetilde{\mu}e}$ nor $\mathbf{E}_\pi$ is an equilibrium. (See Example 12 following.) So I seek conditions that guarantee the existence of pure strategy equilibria. This is done by the following Proposition 13. Roughly speaking, the proposition says that if $e$ is large enough or is sufficiently closed to 2, then there always exists at least one pure strategy equilibrium. But before moving to the general analysis, let us consider an example first.

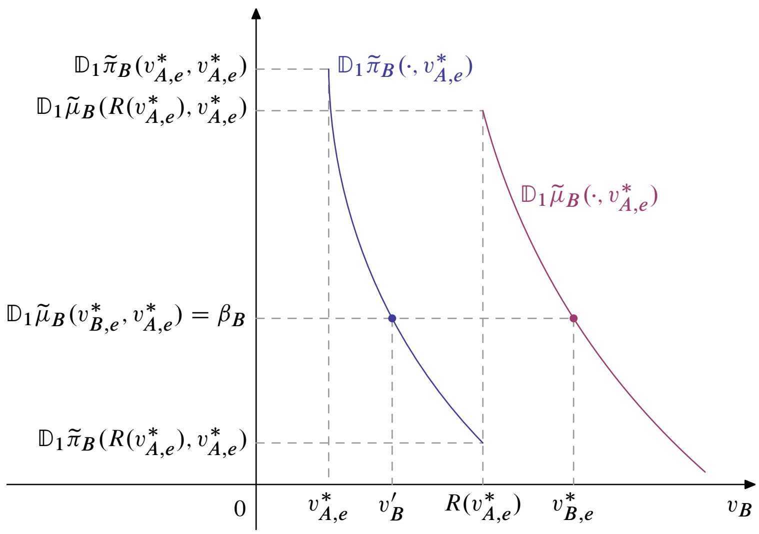



FIGURE 5. Illustration of ($\mathbf{E}_{\widetilde{\mu}e}$-c).

EXAMPLE 12. Let $\psi(x) = x^{1/2}$, $\beta_A = 4$ and $\beta_B = 1$. Then $\Lambda(x) = \psi'(x)/\psi(x) = 1/(2x)$ and the necessary condition (6) implies that $v_{B,e} = 4v_{A,e}$ for every equilibrium.

**The $\mathbf{E}_{\widetilde{\mu}e}$-equilibrium.** When $v_{B,e} = 4v_{A,e}$ we have

$$\mu_A(v_{A,e}, v_{B,e}) = \frac{1}{2}\left[\frac{e v_{A,e}}{(v_{A,e}v_{B,e})^{1/2} + v_{B,e}}\right]^{1/3} = \frac{(e/6)^{1/3}}{2},$$

and

$$\mathbb{D}_1\mu_A(v_{A,e}, v_{B,e}) = \frac{5(e/6)^{1/3}}{36v_{A,e}}.$$

Hence, ($\mathbf{E}_{\widetilde{\mu}e}$-a) implies that

$$v^*_{A,e} = \frac{5(e/6)^{1/3}}{144} \quad \text{and} \quad v^*_{B,e} = \frac{5(e/6)^{1/3}}{36}.$$

I then consider ($\mathbf{E}_{\widetilde{\mu}e}$-b). Evidently, there exists a unique point $e_{\widetilde{\mu}1}$ such that $L(v_{B,e_{\widetilde{\mu}1}}) = v_{A,e_{\widetilde{\mu}1}}$, i.e.,

$$\begin{aligned} \psi\left[L\left(v^*_{B,e_{\widetilde{\mu}1}}\right)\right] = X_L\left(e_{\widetilde{\mu}1}\right)\cdot\psi\left(v^*_{B,e_{\widetilde{\mu}1}}\right) = \psi\left(v^*_{A,e_{\widetilde{\mu}1}}\right) \iff& X_L\left(e_{\widetilde{\mu}1}\right) = 1/2 \\ \iff& e_{\widetilde{\mu}1} = \frac{3\sqrt{2}}{2}. \end{aligned}$$

Now if $e \in (2, e_{\widetilde{\mu}1})$, then

$$\begin{aligned}\psi\left[L\left(v^*_{B,e}\right)\right] - \psi\left(v^*_{A,e}\right) &= X_L(e)\cdot\psi\left(v^*_{B,e}\right) - \psi\left(v^*_{A,e}\right)\\ &> X_L\left(e_{\widetilde{\mu}1}\right)\cdot\psi\left(v^*_{B,e}\right) - \psi\left(v^*_{A,e}\right)\\ &= 0;\end{aligned}$$

that is, ($\mathbf{E}_{\widetilde{\mu}e}$-b) holds for all $e \in (2, e_{\widetilde{\mu}1})$.

Finally, consider condition ($\mathbf{E}_{\widetilde{\mu}e}$-c). Given $v^*_{A,e}$, let us consider the relaxed optimization problem for candidate $B$:[9]

$$\max_{\hat{v}_{B,e} > v^*_{A,e}} \left\{\widetilde{\pi}_B\left(\hat{v}_{B,e}, v^*_{A,e}\right) - \beta_B \hat{v}_{B,e}\right\}$$

Then $\hat{v}_{B,e}$ should satisfy the following first-order condition:

$$\mathbb{D}_1\widetilde{\pi}_B\left(\hat{v}_{B,e}, v^*_{A,e}\right) = \beta_B \iff \frac{\pi_A\left(v^*_{A,e}, \hat{v}_{B,e}\right)}{2}\Lambda\left(\hat{v}_{B,e}\right) = \frac{1}{4}\left(\frac{v^*_{A,e}}{\hat{v}_{B,e}}\right)^{1/4}\frac{1}{2\hat{v}_{B,e}} = 1, \tag{8}$$

i.e.,

$$\hat{v}_{B,e} = \frac{1}{4\times 2^{2/5}}\left(v^*_{A,e}\right)^{1/5}.$$

Hence,

$$U^{\widetilde{\pi}}_B\left(\hat{v}_{B,e}, v^*_{A,e}\right) = \widetilde{\pi}_B\left(\hat{v}_{B,e}, v^*_{A,e}\right) - \hat{v}_{B,e} = 1 - 5\hat{v}_{B,e} = 1 - \frac{5}{4\times 2^{2/5}}\left(v^*_{A,e}\right)^{1/5},$$

where the first equality holds because, by (8),

$$\pi_A\left(v^*_{A,e}, \hat{v}_{B,e}\right) = 4\hat{v}_{B,e}.$$

Therefore,

$$U^{\widetilde{\pi}}_B\left(\hat{v}_{B,e}, v^*_{A,e}\right) - U^{\widetilde{\mu}}_B\left(v^*_{B,e}, v^*_{A,e}\right) = \frac{5}{4\times 2^{2/5}}\left(v^*_{A,e}\right)^{1/5} - \frac{23(e/6)^{1/3}}{36}.$$

The above difference is strictly decreasing with $e$, and is equivalent if and only if $e = 2.11412$. Let

$$e_{\widetilde{\mu}2} = 2.11412.$$

Then ($\mathbf{E}_{\widetilde{\mu}e}$-c) holds whenever $e \in (2, e_{\widetilde{\mu}2})$.

Let $e_{\widetilde{\mu}} = \min\{e_{\widetilde{\mu}1}, e_{\mu 2}\} = 2.11412$. Then for every election $e \in (2, e_{\widetilde{\mu}})$, there exists an equilibrium $(v^*_{A,e}, v^*_{B,e})$.

These outcomes can be explained using Figure 6(a) and (b). If $e = e_{\widetilde{\mu}1}$, then $v^*_{B,e}$ and $R(v^*_{A,e})$ coincide and $U^{\widetilde{\mu}}_B(v_B, v^*_{A,e})$ obtains its maximum at the point $R(v^*_{A,e})$. In this case, candidate $B$ will deviate and choose some point $\hat{v}_{B,e_{\widetilde{\mu}1}} < v^*_{B,e_{\widetilde{\mu}1}}$. However, as $e$ decreases, this incentive eventually disappears (Figure 6(b)).

[9] The choice of candidate $B$ is actually restricted on the interval $(v^*_{A,e}, R(v^*_{A,e})]$. By considering the relaxed optimization problem, I avoid discussing the possibility that $\hat{v}_{B,e} = R(v^*_{A,e})$.

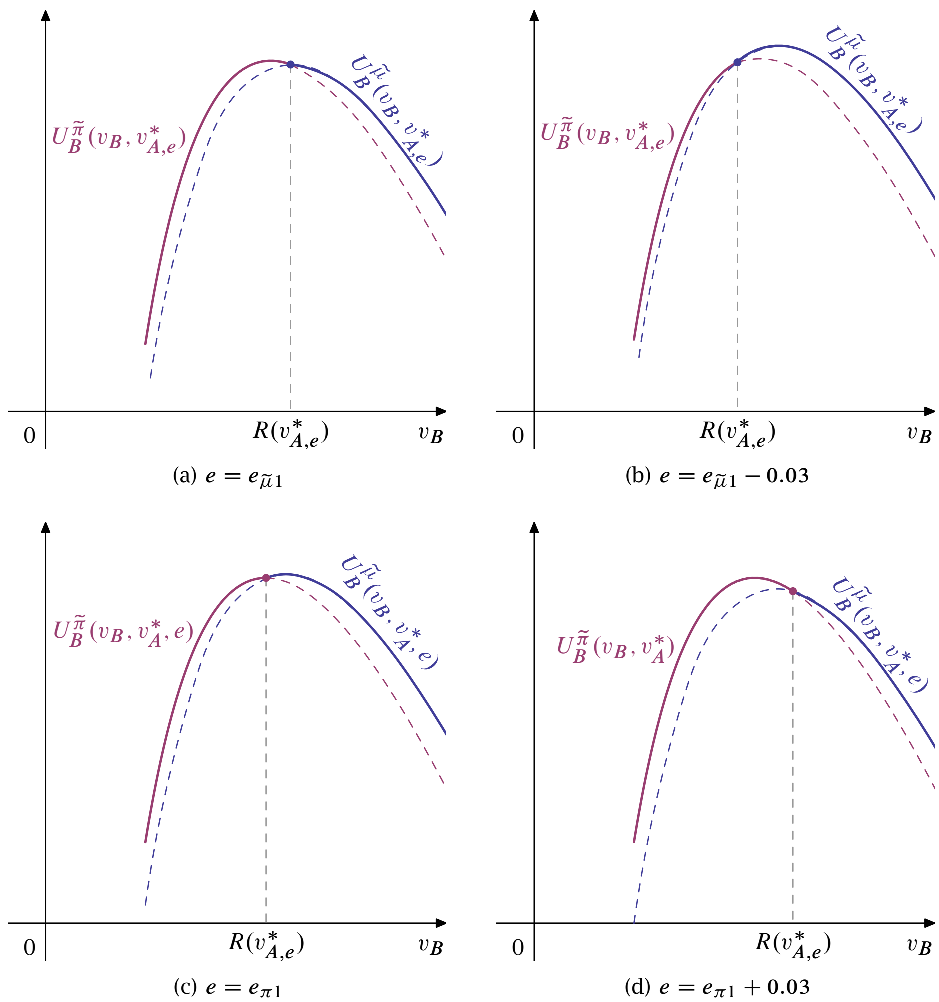



FIGURE 6. The equilibria in Example 12.

**The $\mathbf{E}_\pi$-equilibrium.** The first-order conditions yield $v_A^* = 1/(32\sqrt{2})$ and $v_B^* = 1/(8\sqrt{2})$. Consequently,

$$U_B^{\widetilde{\pi}}\left(v_A^*, v_B^*, e\right) = 1 - \frac{1}{2\sqrt{2}} - \frac{1}{8\sqrt{2}} = 1 - \frac{5\sqrt{2}}{16}.$$

The condition ($\mathbf{E}_\pi$-b) yields a cutoff point $e_{\pi 1} = 3\sqrt{2}/2$. Next, notice that when $e = e_{\pi 1} - 0.01$, then

$$\max_{v_B > R(v_A^*, e_{\pi 1} - 0.01)} U_B^{\widetilde{\mu}}\left(v_B, v_A^*, e\right) \approx 0.5489 < U_B^{\widetilde{\pi}}\left(v_B^*, v_A^*, e\right).$$

Thus, there exists an $e_{\pi 2} \in (e_{\pi 1}, e_{\pi 1} - 0.01)$ such that ($\mathbf{E}_\pi$-c) holds. Since $U_B^{\widetilde{\mu}}(v_B, v_A^*, e)$ is strictly decreasing with $e$ (notice that this conclusion holds because $v_A^*$ is independent of $e$), we know that ($\mathbf{E}_\pi$-c) holds for all $e \in (e_{\pi 2}, \infty)$.

Let $e_\pi = \max\{e_{\pi 1}, e_{\pi 2}\}$. Then $(v_A^*, v_B^*)$ is an equilibrium in all elections $e \in (e_\pi, \infty)$.

According to Figure 6(c) and (d), when $e = e_{\pi 1}$, the points $v_B^*$ and $R(v_A^*, e_{\pi 1})$ coincide and candidate $B$ wants to increase her valence from $v_B^*$. However, as $e$ increases, this deviation incentive disappears.

It is easy to see that in this example, the (pure strategy) equilibrium is unique. ◇

The results obtained in Example 12 can be generalized as follows:

Proposition 13. *In the election game, the following properties hold:*

1. *There exists a cutoff point $e_{\widetilde{\mu}} > 2$ such that $\mathbf{E}_{\widetilde{\mu}e}$ is an equilibrium for every election $e \in [2, e_{\widetilde{\mu}})$.*
2. *There exists a cutoff point $e_\pi > 2$ such that $\mathbf{E}_\pi$ is an equilibrium for every election $e \in (e_\pi, \infty)$; there exists a cutoff point $e'_\pi \geqslant e_\pi$ such that $\mathbf{E}_\pi$ is the unique equilibrium for every election $e \in (e'_\pi, \infty)$, and there also exists a cutoff point $e''_\pi$ such that $\mathbf{E}_\pi$ is not an equilibrium for all $e \in (2, e''_\pi)$.*

It follows from Proposition 13 that when $e$ is large enough (larger than $e'_\pi$), then there always exists a unique pure strategy equilibrium. Also, the magnitude of $e$ does not affect candidates' equilibrium valence choices any more (of course the presence of $e$ matters): their valences are constant across all $e \in (e'_\pi, \infty)$.

Remark. In most election games with endogenous campaign effort decisions, there is no pure strategy equilibrium and the mixed strategies are usually difficult to describe (e.g., Aragones and Palfrey 2002, Meirowitz 2008, Ashworth and Bueno de Mesquita 2009, and Hummel 2010, etc.) Hence, Proposition 13 purifies the mixed strategies in the previous models by considering voting cost (Harsanyi, 1973).

Particularly, when $\psi(x) = x^\gamma$ with $\gamma \in (0, 1]$, a commonly used contest function, then the proof of Proposition 13(1) becomes much easier.

Corollary 14. *Suppose that $\psi(x) = x^\gamma$, where $\gamma \in (0, 1]$. Then there exists a cutoff point $e_{\widetilde{\mu}} > 2$ such that $\mathbf{E}_{\widetilde{\mu}e}$ is an equilibrium for every election $e \in (2, e_{\widetilde{\mu}}]$.*

### 4.4. *Equilibria for $e \in (0, 2)$*

The analysis for $e \in (0, 2)$ is analogous to the one in the previous subsection, but the role of candidate $A$ and $B$ need to be exchanged; see Figure 3.

Similar to the instances for $e \in (2, \infty)$, there exists two possible equilibria for $e \in (0, 2)$, too. The first equilibrium is a pair $(v_{A,e}^*, v_{B,e}^*)$, so that candidate $A$ and $B$ wins with probability $\mu_A(v_A^*, v_B^*, e)$ and $\widetilde{\mu}_B(v_{A,e}^*, v_{B,e}^*)$, respectively; the second equilibrium is a pair $(v_A^*, v_B^*)$, such that candidate $A$ and $B$ wins with probability $\rho_A(v_A^*, v_B^*)$ and $\widetilde{\rho}_B(v_A^*, v_B^*)$, respectively. I call the first equilibrium an $\mathbf{E}_{\mu e}$-equilibrium and the second an $\mathbf{E}_\rho$-equilibrium.

The proofs for the following two propositions are similar to Propositions 11 and 13 and thus are omitted. Let

$$\begin{aligned} U_A^\mu(v_A, v_B, e) &:= \mu_A(v_A, v_B, e) - \beta_A v_A, \\ U_A^\rho(v_A, v_B) &:= \rho_A(v_A, v_B) - \beta_A v_A. \end{aligned}$$

Proposition 15. *Consider an election $e \in (0, 2)$.*

1. *A pair of valences $(v_{A,e}^*, v_{B,e}^*)$ is an $\mathbf{E}_{\mu e}$-equilibrium if and only if the following three conditions are satisfied:*

$$(\mathbf{E}_{\mu e}\text{-}1) \qquad \left(\mathbb{D}_1 \mu_A\left(v_{A,e}^*, v_{B,e}^*\right), \mathbb{D}_1 \widetilde{\mu}_B\left(v_{B,e}^*, v_{A,e}^*\right)\right) = \left(\beta_A, \beta_B\right).$$

$$(\mathbf{E}_{\mu e}\text{-}2) \qquad v_{A,e}^* < \ell\left(v_{B,e}^*\right).$$

$$(\mathbf{E}_{\mu e}\text{-}3) \qquad U_A^\mu\left(v_{A,e}^*, v_{B,e}^*\right) \geqslant \max_{v_A \in \left(\ell(v_{B,e}^*), v_{B,e}^*\right)} \left\{U_A^\rho\left(v_A, v_{B,e}^*\right)\right\}.$$

*In this equilibrium, both $v_{A,e}^*$ and $v_{B,e}^*$ are continuous and strictly increasing with $e$.*

2. *A pair of valences $(v_A^*, v_B^*)$ is an $\mathbf{E}_\rho$-equilibrium if and only if the following three conditions are satisfied:*

$$(\mathbf{E}_\rho\text{-}1) \qquad \left(\mathbb{D}_1 \rho_A\left(v_A^*, v_B^*\right), \mathbb{D}_1 \widetilde{\rho}_B\left(v_B^*, v_A^*\right)\right) = \left(\beta_A, \beta_B\right).$$

$$(\mathbf{E}_\rho\text{-}2) \qquad v_A^* > \ell\left(v_B^*, e\right).$$

$$(\mathbf{E}_\rho\text{-}3) \qquad U_A^\rho\left(v_A^*, v_B^*\right) \geqslant \max_{v_A \in \left(0, \ell(v_B^*, e)\right)} \left\{U_A^\mu\left(v_A, v_B^*, e\right)\right\}.$$

*In this equilibrium, both $v_A^*$ and $v_B^*$ are independent of $e$.*

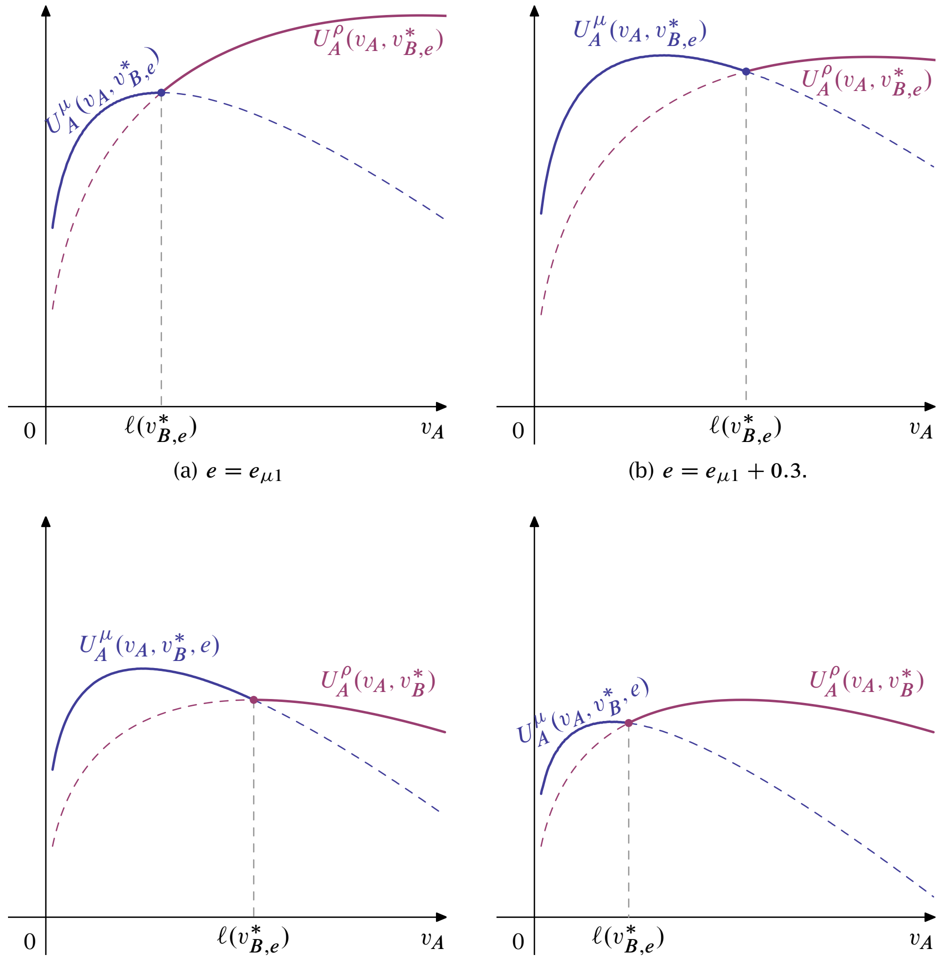
Proposition 16. *In the election game, the following properties hold:*

1. *There exists a cutoff point $e_\rho > 0$ such that $\mathbf{E}_\rho$ is an equilibrium for every election $e \in (0, e_\rho)$.*

2. *There exists a cutoff point $e_\mu > 0$ such that $\mathbf{E}_\mu$ is an equilibrium for every election $e \in (e_\mu, 2)$.*

Now let us compare the two equilibria $\mathbf{E}_\rho$ and $\mathbf{E}_\pi$. In particular, let $(v_A, v_B)$ be an $\mathbf{E}_\rho$-equilibrium for some $e \in (0, e_\rho)$ and let $(\tilde{v}_A, \tilde{v}_B)$ be an $\mathbf{E}_\pi$-equilibrium for some $e \in (e_\pi, \infty)$. Then,

$$\mathbb{D}_1 \rho_A(v_A, v_B) = \frac{1}{2}\Lambda(v_A) = \beta_A = \mathbb{D}_1 \pi_A(\tilde{v}_A, \tilde{v}_B) = \frac{\pi_A(\tilde{v}_A, \tilde{v}_B)}{2}\Lambda(\tilde{v}_A).$$




(a) $e = e_{\mu 1}$

(b) $e = e_{\mu 1} + 0.3$.

(c) $e = e_{\pi 1}$.

(d) $e = e_{\pi 1} - 0.3$

FIGURE 7. The equilibria in Example 18.

Since $\pi_A(\tilde{v}_A, \tilde{v}_B) < 1$, we know that

$$\Lambda(v_A) < \Lambda(\tilde{v}_A) \iff v_A > \tilde{v}_A,$$

since $\Lambda$ is strictly decreasing. Similarly, $v_B > \tilde{v}_B$. Therefore, the following Proposition 17 holds.

PROPOSITION 17. *Both candidates' valence choices when $e \in (0, e_\rho)$ are larger than their choices when $e \in (e_\pi, \infty)$.*

I close this subsection by continuing Example 12.

EXAMPLE 18. Let $\psi(x) = x^{1/2}$, $\beta_A = 4$, and $\beta_B = 1$. So we still have $v_{B,e} = 4v_{A,e}$ for every equilibrium.

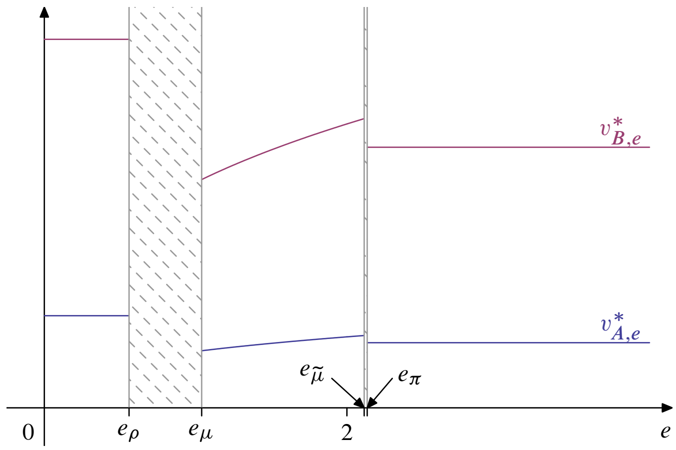



FIGURE 8. Equilibria in Examples 12 and 18.

I first consider the $\mathbf{E}_\rho$-equilibrium. In this case, $(v_A^*, v_B^*) = (1/32, 1/8)$. Then $1/32 = \ell(1/8, e_{\rho 1})$ gives:

$$e_{\rho 1} = 3/4.$$

It can be seen that $1/32 > \ell(1/8, e)$ for all $e \in (0, e_{\rho 1})$.

Now consider candidate $A$. Her expected payoff from $(1/32, 1/8)$ is

$$\rho_A(1/32, 1/8) - 4 \times \frac{1}{32} = \frac{1}{8}.$$

Then, when $e = 0.56$,

$$\underset{v_A \in (0, \ell(1/8, 0.56))}{\operatorname{argmax}} \left\{\mu_A\left(v_A, 1/32, 0.56\right) - 4v_A\right\} \approx 0.01289 < \ell(1/8, 0.56) = 0.05,$$

and

$$\max_{v_A \in (0, \ell(1/8, 0.56))} \left\{\mu_A\left(v_A, 1/32, 0.56\right) - 4v_A\right\} \approx 0.1246 < 1/8$$

Let $e_\rho = \min\{0.75, 0.56\} = 0.56$. Then $\mathbf{E}_\rho$ is an equilibrium for all $e \in (0, 0.56)$.

Now consider $\mathbf{E}_{\mu e}$. In this case,

$$v^*_{A,e} = \frac{5(e/6)^{1/3}}{144}, \quad \text{and} \quad v^*_{B,e} = \frac{5(e/6)^{1/3}}{36}.$$

Notice that $e_{\mu 1} = 3/4$. Pick $e_{\mu 2}$ such that

$$\mathbb{D}_1 \rho_A \left(\ell\left(v_{B, e_{\mu 2}}, e_{\mu 2}\right), v_{B, e_{\mu 2}}\right) = \beta_A = 4.$$

This gives

$$e_{\mu 2} \approx 1.04.$$

Take $e_\mu = \max\{e_{\mu 1}, e_{\mu 2}\}$. Then $\mathbf{E}_{\mu e}$ is an equilibrium for all $e \in (1.04, 2)$. The above arguments are illustrated in Figure 7.

Figure 8 combines the findings in Examples 12 and 18. It can be seen from this figure that there exists pure strategy equilibrium when $e \in (0, e_\rho) \cup (e_\mu, e_{\tilde{\mu}}) \cup (e_\pi, \infty)$. More importantly, both candidates's valences on the interval

$[0, e_\rho)$ and $(e_\pi, \infty)$ are constant, and strictly increasing on the interval $(e_\mu, e_{\tilde{\mu}})$. Finally, they implement the highest valences on the interval $(0, e_\rho)$. ◇

### 4.5. *Asymmetry between Candidates*

The final order of business is to investigate how the asymmetry between candidates influences their decisions. Let us fix candidate $A$'s marginal cost $\beta_A$ throughout this subsection. Let $\Delta v_e^* := v_{B,e}^* - v_{A,e}^*$, the difference between candidates' valence choices in equilibrium.

PROPOSITION 19. *Fix $\beta_A$. Then the equilibrium valences $(v_{A,e}^*, v_{B,e}^*)$ satisfy the following properties:*

- *Candidate $A$'s equilibrium valence choice $v_{A,e}^*$ increases with $\beta_B$.*
- *The difference $\Delta v_e^*$ decreases with $\beta_B$.*

According to Proposition 19, when candidates become similar (in terms of their marginal costs $\beta_A$ and $\beta_B$), then their equilibrium choices become similar, too: as candidate $B$'s marginal cost $\beta_B$ increases (but still less than $\beta_A$), candidate $A$ increases her valence in a rate faster than candidate $B$.

## 5. Conclusion

I considered a model in which two asymmetric candidates compete for voters through non-policy activities. When voting is costly for voters, there exist at least one pure strategy equilibrium for most cases. This model predicts that candidates' valence choices are influenced by voters' voting cost and the productivity difference between the two candidates.

The current paper can be extended in the following directions: (1) I only considered pure strategy equilibrium, and shown that there exist elections where no such equilibria exist. While I have shown that there exists at least on equilibrium (possibly mixed) under Assumption 1, it is important to work out the explicit form of the mixed equilibria. (2) I supposed that candidates' costly actions do not influence the importance of the elections for voters. This assumption can be relaxed.

## Appendix A. The Indifferentiable Points

- The function $L\colon \mathbb{R}_{++} \times [2, \infty) \to \mathbb{R}_+$ is defined by letting $L(v_{-c}, e) = \min\{v_c \in \mathbb{R}_+ : \Gamma(v_c, v_{-c}) = e\}$; that is,

$$L(v_{-c}, e) = \psi^{-1}\left[\frac{e^2 - 2 - e\sqrt{e^2 - 4}}{2} \cdot \psi(v_{-c})\right] =: \psi^{-1}\left[X_L(e) \cdot \psi(v_{-c})\right].$$

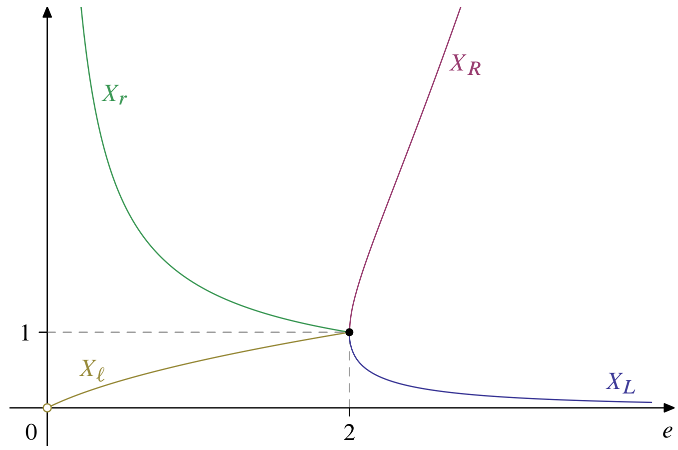



FIGURE A.1. $X_L(e)$, $X_R(e)$, $X_\ell(e)$, and $X_r(e)$.

- The function $R\colon \mathbb{R}_{++} \times [2, \infty) \to \mathbb{R}_+$ is defined by letting $R(v_{-c}, e) = \max\{v_c \in \mathbb{R}_+ : \Gamma(v_c, v_{-c}) = e\}$; that is,

$$R(v_{-c}, e) = \psi^{-1}\left[\frac{e^2 - 2 + e\sqrt{e^2 - 4}}{2} \cdot \psi(v_{-c})\right] =: \psi^{-1}\left[X_R(e) \cdot \psi(v_{-c})\right].$$

- The function $\ell\colon \mathbb{R}_{++} \times (0, 2] \to \mathbb{R}_+$ is defined by letting $\Theta_c(\ell(v_{-c}, e), v_{-c}) = e$; that is,

$$\ell(v_{-c}, e) = \psi^{-1}\left[\frac{\sqrt{4e + 1} - 1}{2} \cdot \psi(v_{-c})\right] =: \psi^{-1}\left[X_\ell(e) \cdot \psi(v_{-c})\right].$$

- The function $r\colon \mathbb{R}_{++} \times (0, 2] \to \mathbb{R}_+$ is defined by setting $\Theta_{-c}(v_{-c}, \ell(v_{-c}, e)) = e$; that is,

$$r(v_{-c}, e) = \psi^{-1}\left[\frac{\sqrt{4e + 1} + 1}{2e} \cdot \psi(v_{-c})\right] =: \psi^{-1}\left[X_r(e) \cdot \psi(v_{-c})\right].$$

All of these functions are well-defined and increasing with $v_{-c}$. Notice that if $e \neq 2$, then $L(v_{-c}, e), \ell(v_{-c}, e) < v_{-c}$, and $R(v_{-c}, e), r(v_{-c}, e) > v_{-c}$; if $e = 2$, then $L(v_{-c}, 2) = R(v_{-c}, 2) = \ell(v_{-c}, 2) = r(v_{-c}, 2) = v_{-c}$. Also, $R$ and $\ell$ are increasing with $e$, and $L$ and $r$ are decreasing with $e$. These properties can be seen from Figure A.1, where $X_i(e)$, $i \in \{L, R, \ell, r\}$, are depicted.

I now show (3). This property holds because $X_L(e) \cdot X_R(e) = 1$, and so

$$\begin{aligned} v_c < L(v_{-c}, e) &\iff \psi(v_c) < \psi(L(v_{-c}, e)) = X_L(e) \cdot \psi(v_{-c}) \\ &\iff X_R(e) \cdot \psi(v_c) < \left[X_R(e) \cdot X_L(e)\right] \cdot \psi(v_{-c}) \\ &\iff X_R(e) \cdot \psi(v_c) < \psi(v_{-c}) \\ &\iff R(v_c, e) < v_{-c}. \end{aligned}$$

Notice that (4) can be proved using the exactly same way.

## Appendix B. Proofs

Proof of Corollary 3. I first show that:

- For an election $e \in [2, \infty)$:

$$\text{(B.1)} \qquad \begin{cases} \mu_c(v_c, v_{-c}, e) < \pi_c(v_c, v_{-c}) & \text{if } \; v_c \in (0, L(v_{-c}, e)) \\ \mu_c(v_c, v_{-c}, e) = \pi_c(v_c, v_{-c}) & \text{if } \; v_c \in \{0, L(v_{-c}, e)\} \\ \mu_c(v_c, v_{-c}, e) > \pi_c(v_c, v_{-c}) & \text{if } \; v_c \in (L(v_{-c}, e), v_{-c}], \end{cases}$$

and

$$\text{(B.2)} \qquad \begin{cases} \widetilde{\mu}_c(v_c, v_{-c}, e) < \widetilde{\pi}_c(v_c, v_{-c}) & \text{if } \; v_c \in (v_{-c}, R(v_{-c}, e)) \\ \widetilde{\mu}_c(v_c, v_{-c}, e) = \widetilde{\pi}_c(v_c, v_{-c}) & \text{if } \; v_c = R(v_{-c}, e) \\ \widetilde{\mu}_c(v_c, v_{-c}, e) > \widetilde{\pi}_c(v_c, v_{-c}) & \text{if } \; v_c \in (R(v_{-c}, e), \infty). \end{cases}$$

- For an election $e \in (0, 2)$:

$$\text{(B.3)} \qquad \begin{cases} \mu_c(v_c, v_{-c}, e) > \rho_c(v_c, v_{-c}) & \text{if } \; v_c \in (0, \ell(v_{-c}, e)) \\ \mu_c(v_c, v_{-c}, e) = \rho_c(v_c, v_{-c}) & \text{if } \; v_c \in \{0, \ell(v_{-c}, e)\} \\ \mu_c(v_c, v_{-c}, e) < \rho_c(v_c, v_{-c}) & \text{if } \; v_c \in (\ell(v_{-c}, e), v_{-c}], \end{cases}$$

and

$$\text{(B.4)} \qquad \begin{cases} \widetilde{\mu}_c(v_c, v_{-c}, e) > \widetilde{\rho}_c(v_c, v_{-c}) & \text{if } \; v_c \in (v_{-c}, r(v_{-c}, e)) \\ \widetilde{\mu}_c(v_c, v_{-c}, e) = \widetilde{\rho}_c(v_c, v_{-c}) & \text{if } \; v_c = r(v_{-c}, e) \\ \widetilde{\mu}_c(v_c, v_{-c}, e) < \widetilde{\rho}_c(v_c, v_{-c}) & \text{if } \; v_c \in (r(v_{-c}, e), \infty). \end{cases}$$

To verify (B.1), notice that when $v_c \leqslant v_{-c}$,

$$\frac{\mu_c(v_c, v_{-c}, e)}{\pi_c(v_c, v_{-c})} = \left( \frac{e^2 \psi(v_c) \psi(v_{-c})}{\left[\psi(v_c) + \psi(v_{-c})\right]^2} \right)^{1/6}.$$

Hence,

$$\frac{\mu_c\left(L(v_{-c}, e), v_{-c}, e\right)}{\pi_c\left(L(v_{-c}, e), v_{-c}\right)} = \left( \frac{e^2 \cdot X_L(v_{-c}, e) \cdot \psi^2(v_{-c})}{\left[X_L(v_{-c}, e) + 1\right]^2 \psi^2(v_{-c})} \right)^{1/6} = 1.$$

Since

$$\mathbb{D}_1 \left[ \frac{\mu_c(v_c, v_{-c}, e)}{\pi_c(v_c, v_{-c})} \right] = \frac{e \pi_c(v_c, v_{-c}) \left[\psi(v_{-c}) - \psi(v_c)\right] \psi'(v_c)}{12 f^2(v_c, v_{-c}, e) \left[\psi(v_c) + \psi(v_{-c})\right]^2} > 0,$$

it follows from the above inequality that $\mu_c(v_c, v_{-c}, e)/\pi_c(v_c, v_{-c}) < 1$ when $v_c \in (0, L(v_{-c}, e))$, and $\mu_c(v_c, v_{-c}, e)/\pi_c(v_c, v_{-c}) > 1$ when $v_c \in (L(v_{-c}, e), v_{-c})$.

To verify (B.3), notice that when $v_c \leqslant v_{-c}$ and $e \in (0, 2)$,

$$\frac{\mu_c(v_c, v_{-c}, e)}{\rho_c(v_c, v_{-c})} = \left[ \frac{e \psi^2(v_{-c})}{\psi^2(v_c) + \psi(v_c)\psi(v_{-c})} \right]^{1/3}.$$

Hence,

$$\frac{\mu_c\big(L(v_{-c},e),v_{-c},e\big)}{\rho_c\big(L(v_{-c},e),v_{-c}\big)}=\left(\frac{e\psi^2(v_{-c})}{\Big[X_\ell^2(v_{-c})+X_\ell(v_{-c})\Big]\cdot\psi^2(v_{-c})}\right)^{1/3}=1.$$

Since

$$\mathbb{D}_1\left[\frac{\mu_c(v_c,v_{-c},e)}{\rho_c(v_c,v_{-c})}\right]=-\left[\frac{e\psi^2(v_{-c})}{\psi^2(v_c)+\psi(v_c)\psi(v_{-c})}\right]^{4/3}\frac{2\psi(v_c)+\psi(v_{-c})}{3e\psi^2(v_{-c})}\psi'(v_c)<0,$$

we get the desirable results.

To verify (B.2), rewrite the first inequality in (B.1) as follows:

(B.1.1) $\qquad \mu_{-c}(v_{-c},v_c,e)<\pi_{-c}(v_{-c},v_c)\quad\text{if}\quad v_{-c}\in(0,L(v_c,e)).$

Observe that $v_{-c}\in(0,L(v_c,e))$ if and only if $v_c\in(R(v_{-c},e),\infty)$ by (3). Then it follows from (B.1.1) that

$$\begin{aligned}\widetilde{\mu}_c(v_c,v_{-c},e)=1-\mu_{-c}(v_{-c},v_c,e)>1-\pi_{-c}(v_{-c},v_c)&\\ =\widetilde{\pi}_c(v_c,v_{-c})\quad\text{if}\quad v_c\in(R(v_{-c},e),\infty).&\end{aligned}$$

This proves the third inequality in (B.2). All the remaining claims can be proved similarly. □

Proof of Lemma 4. It follows readily from calculus that:

- For $\mu_c(v_c,v_{-c},e)$:

$$\mathbb{D}_1\mu_c(v_c,v_{-c},e)=\frac{8\psi(v_{-c})\big[\psi(v_c)+2\psi(v_{-c})\big]\mu_c^4(v_c,v_{-c},e)\psi'(v_c)}{3e\psi^3(v_c)}>0,$$

$$\begin{aligned}\mathbb{D}_{11}\mu_c(v_c,v_{-c},e)=&\frac{\mu_c(v_c,v_{-c},e)}{9\psi^2(v_c)\big[\psi(v_c)+\psi(v_{-c})\big]^2}\\ &\times\Big\{\Big[3\psi^3(v_c)+6\psi(v_c)\psi^2(v_{-c})+9\psi^2(v_c)\psi(v_{-c})\Big]\psi''(v_c)\\ &\quad-\Big[2\psi^2(v_c)+8\psi(v_c)\psi(v_{-c})+2\psi^2(v_{-c})\Big]\psi'^2(v_c)\Big\}<0,\end{aligned}$$

$$\begin{aligned}\mathbb{D}_{12}\mu_c(v_c,v_{-c},e)=&\frac{-e\psi(v_c)\big[\psi^2(v_c)+\psi(v_c)\psi(v_{-c})+4\psi^2(v_{-c})\big]\psi'(v_c)\psi'(v_{-c})}{72\psi^2(v_{-c})\big[\psi(v_c)+\psi(v_{-c})\big]^3\mu_c^2(v_c,v_{-c},e)}\\ &<0.\end{aligned}$$

- For $\pi_c(v_c,v_{-c})$:

$$\mathbb{D}_1\pi_c(v_c, v_{-c}) = \frac{\pi_c(v_c, v_{-c})\psi'(v_c)}{2\psi(v_c)} > 0,$$

$$\mathbb{D}_{11}\pi_c(v_c, v_{-c}) = \frac{\left[2\psi(v_c)\psi''(v_c) - \psi'^2(v_c)\right]\pi_c(v_c, v_{-c})}{4\psi^2(v_c)} < 0,$$

$$\mathbb{D}_{12}\pi_c(v_c, v_{-c}) = \frac{-\psi'(v_c)\psi'(v_{-c})}{16\psi^2(v_{-c})\pi_c(v_c, v_{-c})} < 0.$$

- For $\rho_c(v_c, v_{-c})$:

$$\mathbb{D}_1\rho_c(v_c, v_{-c}) = \frac{\psi'(v_c)}{2\psi(v_{-c})} > 0,$$

$$\mathbb{D}_{11}\rho_c(v_c, v_{-c}) = \frac{\psi''(v_c)}{2\psi(v_{-c})} < 0,$$

$$\mathbb{D}_{12}\rho_c(v_c, v_{-c}) = \frac{-\psi'(v_c)\psi'(v_{-c})}{2\psi^2(v_{-c})} < 0.$$

- For $\widetilde{\mu}_c(v_c, v_{-c}, e) = 1 - \mu_{-c}(v_{-c}, v_c, e)$:

$$\mathbb{D}_1\widetilde{\mu}_c(v_c, v_{-c}, e) = \frac{8\left[2\psi(v_c) + \psi(v_{-c})\right]\mu_{-c}^4(v_{-c}, v_c, e)\psi'(v_c)}{3e\psi^2(v_{-c})} > 0,$$

$$\begin{aligned}\mathbb{D}_{11}\widetilde{\mu}_c(v_c, v_{-c}, e) = {} & \frac{\mu_{-c}(v_{-c}, v_c, e)}{9\psi^2(v_c)\left[\psi(v_c) + \psi(v_{-c})\right]^2} \\ & \times \Big\{\Big[6\psi^3(v_c) + 3\psi(v_c)\psi^2(v_{-c}) + 9\psi^2(v_c)\psi(v_{-c})\Big]\psi''(v_c) \\ & - \Big[4\psi^2(v_{-c}) + 10\psi(v_c)\psi(v_{-c}) + 10\psi^2(v_c)\Big]\psi'^2(v_c)\Big\} < 0,\end{aligned}$$

$$\begin{aligned}\mathbb{D}_{12}\widetilde{\mu}_c(v_c, v_{-c}, e) = {} & \frac{e\psi(v_{-c})\left[4\psi^2(v_c) + \psi(v_c)\psi(v_{-c}) + \psi^2(v_{-c})\right]\psi'(v_c)\psi'(v_{-c})}{72\psi^2(v_c)\left[\psi(v_c) + \psi(v_{-c})\right]^3\mu_{-c}^2(v_{-c}, v_c, e)} \\ & > 0.\end{aligned}$$

- For $\widetilde{\pi}_c(v_c, v_{-c}) = 1 - \pi_{-c}(v_{-c}, v_c)$:

$$\mathbb{D}_1\widetilde{\pi}_c(v_c, v_{-c}) = \frac{\pi_{-c}(v_{-c}, v_c)\psi'(v_c)}{2\psi(v_c)} > 0,$$

$$\mathbb{D}_{11}\widetilde{\pi}_c(v_c, v_{-c}) = \frac{\left[2\psi(v_c)\psi''(v_c) - 3\psi'^2(v_c)\right]\pi_{-c}(v_{-c}, v_c)}{4\psi^2(v_c)} < 0,$$

$$\mathbb{D}_{12}\widetilde{\pi}_c(v_c, v_{-c}) = \frac{\psi'(v_c)\psi'(v_{-c})}{16\psi^2(v_c)\pi_{-c}(v_{-c}, v_c)} > 0.$$

- For $\widetilde{\rho}_c(v_c, v_{-c}) = 1 - \rho_{-c}(v_{-c}, v_c)$:

$$\mathbb{D}_1\widetilde{\rho}_c(v_c, v_{-c}) = \frac{\psi(v_{-c})\psi'(v_c)}{2\psi^2(v_c)} > 0,$$
$$\mathbb{D}_{11}\widetilde{\rho}_c(v_c, v_{-c}) = \frac{\psi(v_c)\psi(v_{-c})\psi''(v_c) - 2\psi(v_{-c})\psi'^2(v_c)}{\psi^3(v_c)} < 0,$$
$$\mathbb{D}_{12}\widetilde{\rho}_c(v_c, v_{-c}) = \frac{\psi'(v_c)\psi'(v_{-c})}{2\psi^2(v_c)} > 0.$$

I then show that $p_c(v_c, v_{-c}, e)$ is differentiable at the point $v_c = v_{-c} = v > 0$. Notice that in this case $\mu_c(v, v, 2) = \widetilde{\mu}_{-c}(v, v, 2) = \pi_c(v, v) = \widetilde{\pi}_{-c}(v, v) = \rho_c(v, v) = \widetilde{\rho}_{-c}(v, v) = 1/2$. Hence, for $e \in [2, \infty)$,

$$\mathbb{D}_1\mu_c(v, v, 2) = \mathbb{D}_1\widetilde{\pi}_c(v, v, 2) = \mathbb{D}_1\pi_c(v, v) = \mathbb{D}_1\widetilde{\pi}_c(v, v) = \frac{\psi'(v)}{4\psi(v)},$$

and for $e \in (0, 2)$,

$$\mathbb{D}_1\rho_c(v, v) = \mathbb{D}_1\widetilde{\rho}_c(v, v) = \frac{\psi'(v)}{2\psi(v)}.$$

Furthermore, by Corollary 3, the following inequalities are easy to see: for $e \in (2, \infty)$,

$$\mathbb{D}_1\mu_c\left(L(v_{-c}, e), v_{-c}, e\right) > \mathbb{D}_1\pi_c\left(L(v_{-c}, e), v_{-c}\right),$$
$$\mathbb{D}_1\widetilde{\mu}_c\left(R(v_{-c}, e), v_{-c}, e\right) > \mathbb{D}_1\widetilde{\pi}_c\left(R(v_{-c}, e), v_{-c}\right),$$

and for $e \in (0, 2)$,

$$\mathbb{D}_1\rho_c\left(\ell(v_{-c}, e), v_{-c}\right) > \mathbb{D}_1\mu_c\left(\ell(v_{-c}, e), v_{-c}, e\right),$$
$$\mathbb{D}_1\widetilde{\rho}_c\left(r(v_{-c}, e), v_{-c}\right) > \mathbb{D}_1\widetilde{\mu}_c\left(r(v_{-c}, e), v_{-c}, e\right).$$

Then Lemma 4 follows from the above inequalities and the specification of $p_c(v_c, v_{-c}, e)$ in Lemma 2 immediately. □

PROOF OF LEMMA 5. Notice that for candidate $c \in \{A, B\}$, her expected payoff is

$$U_c(v_c, v_{-c}, e) = p(v_c, v_{-c}, e) - \beta_c v_c.$$

Since $p(v_c, v_{-c}, e) \leqslant 1$, we know that candidate $c$ will never choose a valence $v_c > 1/\beta_c$. □

Proof of Proposition 6. For candidate $c \in \{A, B\}$, if her opponent choses $v_{-c}$, then the expected utility for candidate $c$ from choosing $v_c$ is

$$U_c(v_c, v_{-c}, e) = \begin{cases} 1 - \beta_c v_c & \text{if } v_c > v_{-c} \\ -\beta_c v_c & \text{if } v_c < v_{-c} \\ 1/2 - \beta_c v_c & \text{if } v_c = v_{-c}. \end{cases}$$

First, there cannot be an equilibrium with $v_c = v_{-c}$: by choosing $v_{-c} + \varepsilon$, where $\varepsilon$ is sufficiently small but positive, results in victory for candidate $c$. Since there always exists $\varepsilon > 0$ such that $[1 - \beta_c(v_{-c} + \varepsilon)] - (1/2 - \beta_c v_{-c}) = 1/2 - \beta_c \varepsilon > 0$, it is benefit for $c$ to chose some $v_c \in (v_{-c}, v_{-c} + 1/(2\beta_c))$. However, if $v_c > v_{-c}$, then a slight reduction in $v_c$ is a profitable deviation. Finally, if $v_c < v_{-c}$, then candidate $c$ should chose $v_c = 0$, and so either $v_{-c} = 0$ or a slight decrease in $v_{-c}$ is desirable. But if $v_{-c} = 0$, then a deviation to $v_c \in (0, 1/\beta_c)$ is desirable for candidate $c$. □

Proof of Lemma 7. I first show that $R(v_{-c}, e) \notin \mathrm{BR}_c(v_{-c}, e)$ for every $e \in (2, \infty)$ and every $v_{-c} > 0$; that is, candidate $c$'s optimal valence choice cannot be at the indifferentiable point $R(v_{-c}, e)$.

If $\mathbb{D}_1 \tilde{\pi}_c(R(v_{-c}, e), v_{-c}) < \beta_c$, then

$$\alpha := \operatorname*{argmax}_{v_c \in (v_{-c}, R(v_{-c}, e))} \left\{ \tilde{\pi}_c\left(v_c, v_{-c}\right) - \beta_c v_c \right\} < R(v_{-c}, e),$$

and so $R(v_{-c}, e)$ is not optimal: at least candidate $c$ is better off with the point $\alpha$ than with the point $R(v_{-c}, e)$ (of course $\alpha$ is not necessarily candidate $c$'s optimal choice).

If $\mathbb{D}_1 \tilde{\pi}_c(R(v_{-c}, e), v_{-c}) \geqslant \beta_c$, then

$$\mathbb{D}_1 \tilde{\mu}_c(R(v_{-c}, e), v_{-c}, e) > \mathbb{D}_1 \tilde{\pi}_c(R(v_{-c}, e), v_{-c}) \geqslant \beta_c.$$

Hence,

$$\begin{aligned} \max_{v_c \geqslant R(v_{-c}, e)} \left\{ \tilde{\mu}_c\left(v_c, v_{-c}, e\right) - \beta_c v_c \right\} &> \tilde{\mu}_c\left(R(v_{-c}, e), v_{-c}, e\right) - \beta_c R\left(v_{-c}, e\right) \\ &= \tilde{\pi}_c\left(R(v_{-c}, v_{-c}, e)\right) - \beta_c R(v_{-c}, e) \\ &= \max_{v_c \in [v_{-c}, R(v_{-c}, e))} \left\{ \tilde{\pi}_c\left(v_c, v_{-c}\right) - \beta_c v_c \right\}. \end{aligned}$$

Similarly, it can be shown that $L(v_{-c}, e) \notin \mathrm{BR}_c(v_{-c}, e)$ for every $e \in (0, 2)$ and every candidate $c \in \{A, B\}$.

Finally, suppose that $(v_A^*, v_B^*)$ is an equilibrium for an election $e \in (2, \infty)$. I will show in Lemma 8 shortly that $v_A^* \neq v_B^*$ for every equilibrium. So, without loss of generality, assume that $v_A^* < v_B^*$. Then $v_B^* \in \mathrm{BR}_B(v_A^*, e)$ and so $v_B^* \neq$

$R(v_A^*, e)$ by the previous argument; that is, $\mathbb{D}_1 p_B(v_B^*, v_A^*, e)$ exists. Now suppose that $v_A^* = L(v_B^*, e)$. But then

$$\begin{aligned}\mu_A(v_A^*, v_B^*, e) = \pi_A(v_A^*, v_B^*) &\iff 1 - \mu_A(v_A^*, v_B^*, e) = 1 - \pi_A(v_A^*, v_B^*)\\ &\iff \widetilde{\mu}_B(v_B^*, v_A^*) = \widetilde{\pi}_B(v_B^*, v_A^*)\\ &\iff v_B^* = R(v_A^*, e).\end{aligned}$$

This results in a contradiction. Hence, $\mathbb{D}_1 p_A(v_A^*, v_B^*, e)$ also exists. The same reasoning can be applied to elections $e \in (0, 2)$. $\square$

Proof of Lemma 8. If $(v_A, v_B) = (0, 0)$, then a candidate $c \in \{A, B\}$ can increase her valence from 0 to an arbitrary small but positive real number $\varepsilon > 0$, so that her victory probability is increased from 1/2 to 1. Thus, there always exists such an $\varepsilon > 0$ to make candidate $c$ better off.

To see $v_A \geqslant v_B$ cannot be an equilibrium, recall that

$$p_c(v_c, v_{-c}, e) = p_c\left(k_c(v_c, v_{-c}), e\right).$$

According to this equality and the chain rule, we have

$$\begin{aligned}\mathbb{D}_1 p_A(v_A, v_B, e) &= \left[\mathbb{D}_1 p_A(k_A, e)\right] \cdot \left[\mathbb{D}_1 k_A(v_A, v_B)\right]\\ &= \left[\mathbb{D}_1 p_A(k_A, e)\right] \frac{\psi'(v_A)\psi(v_B)}{\left[\psi(v_A) + \psi(v_B)\right]^2},\end{aligned}$$

and

$$\begin{aligned}\mathbb{D}_1 p_B(v_B, v_A, e) &= \mathbb{D}_2\left[1 - p_A(v_A, v_B, e)\right]\\ &= -\left[\mathbb{D}_1 p_A(k_A, e)\right] \cdot \left[\mathbb{D}_2 k_A(v_A, v_B)\right]\\ &= \left[\mathbb{D}_1 p_A(k_A, e)\right] \frac{\psi(v_A)\psi'(v_B)}{\left[\psi(v_A) + \psi(v_B)\right]^2}.\end{aligned}$$

It follows from (5) and the above two equations that

$$\frac{\mathbb{D}_1 p_A(v_A, v_B, e)}{\mathbb{D}_1 p_B(v_B, v_A, e)} = \frac{\Lambda(v_A)}{\Lambda(v_B)} = \frac{\beta_A}{\beta_B} > 1, \tag{B.5}$$

where the last inequality follows from the specification that $\beta_A > \beta_B$.

Suppose that $(v_A, v_B)$ is an equilibrium and $v_A > v_B$. Then the left-hand side of (B.5) is strictly less than 1 since $\Lambda$ is strictly decreasing, but the right-hand side of (B.5) is strictly larger than 1. A contradiction.

Finally, notice that when $v_A = v_B = v > 0$,

$$\mathbb{D}_1 \pi_c(v, v) = \mathbb{D}_1 \widetilde{\pi}_c(v, v) = \frac{1}{4}\frac{\psi'(v)}{\psi(v)},$$

and

$$\mathbb{D}_1 \rho_c(v, v) = \mathbb{D}_1 \widetilde{\rho}_c(v, v) = \frac{1}{2}\frac{\psi'(v)}{\psi(v)},$$

i.e., $p_c(v_c, v_{-c}, e)$ is differentiable at $v_c = v_{-c} = v$ for each candidate $c \in \{A, B\}$. So the case that $v_A = v_B$ can be ruled out by applying the above results. Thus, in every equilibrium it must be the case that $v_A < v_B$. □

PROOF OF LEMMA 9. For ease of exposition, let

$$Y(e) := \left(\frac{e^2 - 1 + e\sqrt{e^2-4}}{24e}\right)\left[\frac{1}{e^3 - 3e + \left(e^2-1\right)\sqrt{e^2-4}}\right]^{4/3}.$$

Notice that $Y(2) = 1/4$. With some simple but tedious computation, we also know that $Y'(e) < 0$, that is, $Y(e)$ is strictly decreasing with $e$.[10] Hence, $Y(e) < 1/4$ for all $e \in (2, \infty)$. Since

$$\mathbb{D}_1\widetilde{\pi}_c(v_{-c}, v_{-c}) = \frac{\pi_{-c}(v_{-c}, v_{-c})\psi'(v_c)}{2\psi(v_c)} = \frac{1}{4}\Lambda\left(v_{-c}\right),$$

we have

$$\begin{aligned}
\mathbb{D}_1\widetilde{\mu}_c\left(R(v_{-c}, e), v_{-c}\right) &= \left.\frac{8\left[2\psi(v_c) + \psi(v_{-c})\right]\mu_{-c}^4(v_{-c}, v_c, e)\psi'(v_c)}{3e\psi^2(v_{-c})}\right|_{v_c = R(v_{-c}, e)} \\
&= Y(e)\frac{\psi'\left(R(v_{-c}, e)\right)}{\psi(v_{-c})} \\
&< \frac{1}{4}\frac{\psi'\left(R(v_{-c}, e)\right)}{\psi(v_{-c})} \\
&\leqslant \frac{1}{4}\frac{\psi'(v_{-c})}{\psi(v_{-c})} \\
&= \mathbb{D}_1\widetilde{\pi}_c(v_{-c}, v_{-c}),
\end{aligned}$$

where the last inequality holds because $R(v_{-c}, e) > v_{-c}$ and $\psi$ is concave. □

PROOF OF PROPOSITION 10. Define $\mathbf{v}_{-c}$ by letting $\Lambda\left(\mathbf{v}_{-c}\right) = 4\beta_c$. Such a number $\mathbf{v}_{-c}$ must exist and is unique since the function $\Lambda$ is strictly decreasing, $\lim_{x \to 0^+}\Lambda(x) \to \infty$, and $\lim_{x\to\infty}\Lambda(x) \to 0$. For this $\mathbf{v}_{-c}$ and $e = 2$, if candidate $c$ choses $v_c = \mathbf{v}_{-c}$, then

$$\begin{aligned}
\mathbb{D}_1\mu_c\left(\mathbf{v}_{-c}, \mathbf{v}_{-c}, 2\right) = \mathbb{D}_1\widetilde{\mu}_c\left(\mathbf{v}_{-c}, \mathbf{v}_{-c}, 2\right) &= \mathbb{D}_1\pi_c\left(\mathbf{v}_{-c}, \mathbf{v}_{-c}\right) \\
&= \mathbb{D}_1\widetilde{\pi}_c\left(\mathbf{v}_{-c}, \mathbf{v}_{-c}\right) \\
&= \frac{\Lambda\left(\mathbf{v}_{-c}\right)}{4} \\
&= \beta_c;
\end{aligned}$$

[10] Actually, $Y(e)$ is strictly convex in $e$.

that is, candidate $c$'s first-order condition (5) holds for all $e \in [2, \infty)$ by choosing $v_c = \mathbf{v}_{-c}$. Since $p_c(v_c, \mathbf{v}_{-c}, 2)$ is strictly concave, the point $v_c = \mathbf{v}_{-c}$ is candidate $c$'s unique best response.

Because $\beta_A > \beta_B$, it follows from that

$$\frac{\Lambda\left(\mathbf{v}_B\right)}{\Lambda\left(\mathbf{v}_A\right)} = \frac{\beta_A}{\beta_B} > 1,$$

which implies that $\mathbf{v}_A > \mathbf{v}_B$.

I now show that the best response function $\mathrm{BR}_c$ for candidate $c$ is strictly increasing on $(0, \mathbf{v}_{-c})$ and is strictly decreasing on $(\mathbf{v}_{-c}, \infty]$. Differentiating (5) with respect to $v_{-c}$ yields

$$\left[\mathbb{D}_{11} p_c(v_c, v_{-c}, 2)\right] \cdot \left[\mathbb{D}_1 \mathrm{BR}_c(v_{-c}, 2)\right] + \mathbb{D}_{12} p(v_c, v_{-c}, 2) = 0;$$

thus,

$$\text{(B.6)} \qquad \mathbb{D}_1 \mathrm{BR}_c(v_{-c}, 2) = -\frac{\mathbb{D}_{12} p_c(v_c, v_{-c}, 2)}{\mathbb{D}_{11} p_c(v_c, v_{-c}, 2)} \begin{cases} < 0 & \text{if } v_c < v_{-c} \\ > 0 & \text{if } v_c > v_{-c}, \end{cases}$$

according to Lemma 4.

If $\beta_A = \beta_B = \beta$ and $e \in (2, \infty)$, then the necessary condition (6) implies that $\Lambda(v_{A,e}) = \Lambda(v_{B,e})$; that is, $v_{A,e} = v_{B,e} = v$ since $\Lambda$ is monotonely decreasing. Therefore, candidates' victory probabilities are $\pi_A(v, v)$ and $\widetilde{\pi}_B(v, v)$, respectively. Then,

$$\mathbb{D}_1 \pi_A(v, v) = \frac{\Lambda(v)}{4} = \beta \Longrightarrow \Lambda(v) = 4\beta.$$

This gives the unique equilibrium campaign effort levels:

$$v_A^* = v_B^* = v = \Lambda^{-1}(4\beta).$$

Clearly, candidate $c$ has no incentive to decrease her campaign effort level, and it follows from (7) that she also has no incentive to increase her campaign effort level.

Similarly, we can show that if $\beta_A = \beta_B = \beta$ and $e \in (0, 2)$, then the unique equilibriu is

$$v_A^* = v_B^* = \Lambda^{-1}(2\beta).$$

Since $\Lambda$ is strictly decreasing, we have $\Lambda^{-1}(4\beta) < \Lambda^{-1}(2\beta)$. □

Proof of Proposition 11. Evidently, if $(v_{A,e}^*, v_{B,e}^*)$ is an $\mathbf{E}_{\widetilde{\mu}e}$-equilibrium for an election $e \in (2, \infty)$, then conditions ($\mathbf{E}_{\widetilde{\mu}e}$-a) — ($\mathbf{E}_{\widetilde{\mu}e}$-c) must hold, and if $(v_A^*, v_B^*)$ is an $\mathbf{E}_\pi$-equilibrium, then conditions ($\mathbf{E}_\pi$-a) — ($\mathbf{E}_\pi$-c) must hold. So it suffices to do the *only if* part ion.

**(1-a).** In this step I first show that for every $e \in [2, \infty)$ there exists a unique pair $(v^*_{A,e}, v^*_{B,e})$ satisfying ($\mathbf{E}_{\tilde{\mu}e}$-a); then I show that both $v^*_{A,e}$ and $v^*_{B,e}$ are continuous and strictly increasing with $e$.

Observe that ($\mathbf{E}_{\tilde{\mu}e}$-a) implies that

$$v_{B,e} = \Lambda^{-1}\left[\frac{\beta_B}{\beta_A} \cdot \Lambda\left(v_{A,e}\right)\right] =: T\left(v_{A,e}\right). \tag{B.7}$$

This function $T\colon \mathbb{R}_+ \to \mathbb{R}_+$ is well-defined, continuous and differentiable since $\Lambda(\cdot)$ is strictly decreasing, continuous and differentiable. Furthermore, $T$ is strictly increasing: by the Inverse Function Theorem,

$$T'\left(v_{A,e}\right) = \frac{\beta_A}{\Lambda'\left(v_{B,e}\right) \cdot \beta_B \cdot \Lambda'\left(v_{A,e}\right)} > 0.$$

Thus, if a pair $(v_{A,e}, v_{B,e})$ satisfies ($\mathbf{E}_{\tilde{\mu}e}$-a), then $v_{A,e}$ and $v_{B,e}$ change in the same direction with respect to $e$. With this property, I show that there exists a unique $v^*_{A,e}$ such that $\mathbb{D}_1\mu_A(v^*_{A,e}, v^*_{B,e}) = \beta_A$, where $v^*_{B,e} = T(v^*_{A,e}) > v^*_{A,e}$.

Let $\Xi(v_{A,e}) := \psi(T(v_{A,e}))/\psi(v_{A,e})$. Then the partial derivative $\mathbb{D}_1\mu_A(\cdot,\cdot)$ on the trace $(v_{A,e}, T(v_{A,e}))$ can be written as follows:

$$\begin{aligned}\mathcal{D}\left(v_{A,e}\right) :=& \mathbb{D}_1\mu_A\left(v_{A,e}, T(v_{A,e})\right) \\ =& \left[\left(\frac{e^{1/3}}{6}\right)\frac{\Xi\left(v_{A,e}\right) + 2\Xi^2\left(v_{A,e}\right)}{\left[\Xi\left(v_{A,e}\right) + \Xi^2\left(v_{A,e}\right)\right]^{4/3}}\right]\Lambda\left(v_{A,e}\right).\end{aligned} \tag{B.8}$$

It follows from ($\mathbf{E}_{\tilde{\mu}e}$-a) that

$$1 > \frac{\psi\left(v_{A,e}\right)}{\psi\left(T(v_{A,e})\right)} = \frac{\beta_B}{\beta_A} \cdot \frac{\psi'\left(v_{A,e}\right)}{\psi'\left(T(v_{A,e})\right)} \geqslant \frac{\beta_B}{\beta_A}. \tag{B.9}$$

Notice that in the above display, $\psi'(T(v_{A,e}))$ is the first-order derivative of $\psi$ at the point $T(v_{A,e})$. Hence,

$$1 < \Xi\left(v_{A,e}\right) \leqslant \frac{\beta_A}{\beta_B}.$$

Therefore, the term in the brackets of (B.8) is bounded, which means that $\lim_{v_{A,e}\to+\infty} \mathcal{D}(v_{A,e}) = 0$, and $\lim_{v_{A,e}\to 0^+} \mathcal{D}(v_{A,e}) = +\infty$.

Next, I verify that $\mathcal{D}(v_{A,e})$ is strictly decreasing with $v_{A,e}$: Let $v'_{A,e} > v_{A,e}$; then $v_{A,e} < T(v_{A,e}) < T(v'_{A,e})$, and so

$$\begin{aligned}\mathcal{D}\left(v'_{A,e}\right) = \mathbb{D}_1\mu_A\left(v'_{A,e}, T(v'_{A,e})\right) &< \mathbb{D}_1\mu_A\left(v_{A,e}, T(v'_{A,e})\right) \\ &< \mathbb{D}_1\mu_A\left(v_{A,e}, T(v_{A,e})\right) \\ &= \mathcal{D}\left(v_{A,e}\right);\end{aligned}$$

where the first inequality follows because $\mathbb{D}_{11}\mu_A(\cdot, T(v'_{A,e})) < 0$, and the second inequality follows because $\mathbb{D}_{12}\mu_A(v_{A,e}, \cdot) < 0$.

We thus know that for every $e \in [2, \infty)$, there must exist a unique point $v^*_{A,e} > 0$ such that $\mathcal{D}(v^*_{A,e}) = \beta_A$.

Let

$$\Omega\left(v_{A,e}\right) := \frac{\Xi\left(v_{A,e}\right) + 2\Xi^2\left(v_{A,e}\right)}{\left[\Xi\left(v_{A,e}\right) + \Xi^2\left(v_{A,e}\right)\right]^{4/3}} \Lambda\left(v_{A,e}\right).$$

So $\mathcal{D}(v_{A,e}) = (e^{1/3}/6) \cdot \Omega(v_{A,e})$. Then $\Omega(\cdot)$ is continuous and strictly decreasing with $v_{A,e}$, and $\mathcal{D}(v^*_{A,e}) = \beta_A$ implies that

$$v^*_{A,e} = \Omega^{-1}\left(\frac{6\beta_A}{e^{1/3}}\right).$$

Hence, $v^*_{A,e}$ is continuous and strictly increasing with $e$.

**(1-b).** Suppose that there exists an election $e \in (2, \infty)$ and a pair of valences $(v^*_{A,e}, v^*_{B,e})$ satisfying conditions ($\mathbf{E}_{\widetilde{\mu}e}$-a) — ($\mathbf{E}_{\widetilde{\mu}e}$-c). I show that $(v^*_{A,e}, v^*_{B,e})$ is then an $\mathbf{E}_{\widetilde{\mu}e}$-equilibrium for this $e$ by verifying that there is no candidate who wants to deviate.

I first verify that candidate $A$ has no incentive to deviate if candidate $B$ chooses $v^*_{B,e}$. The following argument is based on Lemma 4. Given $v^*_{B,e}$, ($\mathbf{E}_{\widetilde{\mu}e}$-a), and ($\mathbf{E}_{\widetilde{\mu}e}$-b):

*(i).* Candidate $A$ has no incentive to choose a valence from the set $[0, L(v^*_{B,e})] \setminus \{v^*_{A,e}\}$ because of the first-order condition in ($\mathbf{E}_{\widetilde{\mu}e}$-a).

*(ii).* Candidate $A$ has no incentive to choose a valence $v'_A \in (L(v^*_{B,e}), v^*_{B,e}]$. This is because

$$\begin{aligned}
\mathbb{D}_1\pi_A\left(v'_A, v^*_{B,e}\right) < \mathbb{D}_1\pi_A\left(L(v^*_{B,e}), v^*_{B,e}\right) &< \mathbb{D}_1\mu_A\left(L(v^*_{B,e}), v^*_{B,e}\right) \\
&< \mathbb{D}_1\mu_A\left(v^*_{A,e}, v^*_{B,e}\right) \\
&= \beta_A,
\end{aligned}$$

and so

$$\begin{aligned}
\pi_A\left(v'_A, v^*_{B,e}\right) - \beta_A v'_A &< \pi_A\left(L(v^*_{B,e}), v^*_{B,e}\right) - \beta_A L(v^*_{B,e}) \\
&= \mu_A\left(L(v^*_{B,e}), v^*_{B,e}\right) - \beta_A L(v^*_{B,e}) \\
&< \mu_A\left(v^*_{A,e}, v^*_{B,e}\right) - \beta_A v^*_{A,e}.
\end{aligned}$$

*(iii).* Candidate $A$ also has no incentive to increase her valence to some point in the interval $(v^*_{B,e}, R(v^*_{B,e})]$ since

$$\mathbb{D}_1\pi_A\left(v^*_{B,e}, v^*_{B,e}\right) = \mathbb{D}_1\widetilde{\pi}_A\left(v^*_{B,e}, v^*_{B,e}\right),$$

and $\mathbb{D}_1\widetilde{\pi}_A(v_A, v^*_{B,e})$ is strictly decreasing with $v_A$ on $(v^*_{B,e}, R(v^*_{B,e})]$.

*(iv).* Therefore, candidate $A$'s only possible profitable deviation is to increase her valence to some $v_A' > R(v_{B,e}^*)$ such that $p_A(v_A', v_{B,e}^*) = \widetilde{\mu}_A(v_A', v_{B,e}^*)$. However, this deviation is also undesirable for candidate $A$ because

$$\begin{aligned}\mathbb{D}_1\widetilde{\mu}_A\left(v_A', v_{B,e}^*\right) < \mathbb{D}_1\widetilde{\mu}_A\left(R(v_{B,e}^*), v_{B,e}^*\right) &< \mathbb{D}_1\widetilde{\pi}_A\left(v_{B,e}^*, v_{B,e}^*\right)\\ &= \mathbb{D}_1\pi_A\left(v_{B,e}^*, v_{B,e}^*\right)\\ &< \mathbb{D}_1\mu_A\left(v_{A,e}^*, v_{B,e}^*\right)\\ &= \beta_A,\end{aligned}$$

where the first inequality holds since the partial derivative is strictly decreasing and $v_A' > R(v_{B,e}^*)$ by assumption, and the second inequality follows from (7) in Lemma 9.

I then consider candidate $B$'s incentive compatability problem. It follows from (3) that if ($\mathbf{E}_{\widetilde{\mu}e}$-b) holds then $R(v_{A,e}^*) < v_{B,e}^*$ automatically holds; that is, $v_{B,e}^*$ locates at the right side of the indifferentiable point $R(v_{A,e}^*)$. It is clear that candidate $B$ never wants to increase her valence or to decrease her valence to some point in the interval $[R(v_{A,e}^*), v_{B,e}^*)$. Also, the deviation to a point $v_B' \in [0, v_{A,e}^*]$ is undesirable for candidate $B$ because

$$\begin{aligned}\mathbb{D}_1 p_B\left(v_B', v_{A,e}^*\right) \geqslant \mathbb{D}_1\pi_B\left(v_{A,e}^*, v_{A,e}^*\right) &= \mathbb{D}_1\widetilde{\pi}_B\left(v_{A,e}^*, v_{A,e}^*\right)\\ &> \mathbb{D}_1\widetilde{\mu}_B\left(R(v_{A,e}^*), v_{A,e}^*\right)\\ &> \mathbb{D}_1\widetilde{\mu}_B\left(v_{B,e}^*, v_{A,e}^*\right)\\ &= \beta_B,\end{aligned}$$

where, in an abuse of notation, $\mathbb{D}_1 p_B(v_B', v_{A,e}^*)$ is either the left or right derivative if $v_B' = L(v_{A,e}^*)$; the last inequality follows from (7). Hence, if $B$ wants to deviate, she will only deviate from $v_{B,e}^*$ to some point $v_B' \in (v_{A,e}^*, R(v_{A,e}^*))$ such that $p_B(v_B', v_{A,e}^*) = \widetilde{\pi}_B(v_B', v_{A,e}^*)$. This deviation is prevented by ($\mathbf{E}_{\widetilde{\mu}e}$-c).

In sum, given a pair of valences $(v_{A,e}^*, v_{B,e}^*)$ satisfying ($\mathbf{E}_{\widetilde{\mu}e}$-a) and ($\mathbf{E}_{\widetilde{\mu}e}$-b), candidate $A$ never wants to deviate since to make a deviation profitable, she has to increase her valence to a very high point — larger than $R(v_{B,e}^*)$ — this increase of valence cannot complement the cost. However, candidate $B$ may find it is desirable to reduce her valence to a point less that $R(v_{A,e}^*)$. But ($\mathbf{E}_{\widetilde{\mu}e}$-c) makes this deviation impossible.

**(2).** It is now easy to show that there exists a unique pair $(v_A^*, v_B^*)$ that satisfies ($\mathbf{E}_\pi$-a) and is independent of $e$. So I just verify that if there exists a pair of valences $(v_A^*, v_B^*)$ satisfying ($\mathbf{E}_\pi$-a) — ($\mathbf{E}_\pi$-c), then $(v_A^*, v_B^*)$ is an $\mathbf{E}_\pi$-equilibrium.

I first verify that when ($\mathbf{E}_\pi$-a) and ($\mathbf{E}_\pi$-b) hold, candidate $A$ has no incentive to deviate. It is true because if candidate $A$ reduces her valence to some point

$v'_A \in (0, L(v^*_{B,e})]$, then

$$\begin{aligned}\mathbb{D}_1\mu_A\left(v'_A, v^*_{B,e}\right) \geqslant \mathbb{D}_1\mu_A\left(L(v^*_{B,e}), v^*_{B,e}\right) &> \mathbb{D}_1\pi_A\left(L(v^*_{B,e}), v^*_{B,e}\right)\\ &> \mathbb{D}_1\pi_A\left(v^*_{A,e}, v^*_{B,e}\right)\\ &= \beta_A.\end{aligned}$$

Similar to the cases of the $\mathbf{E}_{\widetilde{\mu}e}$-equilibrium, candidate $A$ also has no incentive to increase her valence.

Finally, when ($\mathbf{E}_\pi$-a) and ($\mathbf{E}_\pi$-b) hold, we have $v^*_{B,e} < R(v^*_{A,e})$ by (3). Also, it is now easy to see that candidate $B$ has no incentive either to reduce her valence or increase her valence to some point in the interval $(v^*_{B,e}, R(v^*_{A,e})]$. Therefore, candidate $B$ has no incentive to deviate if and only if choosing some valence $v'_{B,e} > R(v^*_{A,e})$ is undesirable for her; that is, if and only if condition ($\mathbf{E}_\pi$-c) holds. ◻

Proof of Proposition 13. I first show the properties in the first claim step by step.

**(1: the point $e_{\widetilde{\mu}}$).** In this step I show that there exists a cutoff point $e_{\widetilde{\mu}}$ such that ($\mathbf{E}_{\widetilde{\mu}e}$-b) and ($\mathbf{E}_{\widetilde{\mu}e}$-c) hold for all $e \in (2, e_{\widetilde{\mu}})$.

Let $(v^*_{A,e}, v^*_{B,e})$ satisfies condition ($\mathbf{E}_{\widetilde{\mu}e}$-a). Then by (B.9),

$$1 < \frac{\psi\left(v^*_{B,e}\right)}{\psi\left(v^*_{A,e}\right)} \leqslant \frac{\beta_A}{\beta_B}. \tag{B.9$'$}$$

Since $\psi(v^*_{B,e})/\psi(v^*_{A,e})$ is continuous with $e$, and $X_R$ is strictly increasing with $e$, there must exist a point $e_{\widetilde{\mu}1} > 2$ such that for all $e \in (2, e_{\widetilde{\mu}1})$,

$$\frac{\psi\left(v^*_{B,e}\right)}{\psi\left(v^*_{A,e}\right)} > X_R(e);$$

that is, $v^*_{B,e} > R(v^*_{A,e})$ for all $e \in (2, e_{\widetilde{\mu}1})$. This proves that ($\mathbf{E}_{\widetilde{\mu}e}$-b) holds for all $e \in (0, e_{\widetilde{\mu}1})$.

Now define a correspondence $G\colon [2,\infty) \twoheadrightarrow \mathbb{R}_{++}$ by letting

$$G(e) = \left[v^*_{A,e}, R\left(v^*_{A,e}\right)\right],$$

for every $e \in [2,\infty)$. Since $v^*_{A,e}$ is continuous with $e$ by Proposition 13, and so $R(v^*_{A,e})$ is also continuous with $e$, we know that $G(e)$ is continuous with $e$; since $v^*_{A,e}$, and consequently, $R(v^*_{A,e})$, is bounded,[11] we know that $G(e)$ is

[11] It is clear that candidate $A$ will never let $v^*_{A,e} \to \infty$.

compact-valued. Remember that $U_B^{\widetilde{\pi}}(v_B, v_A) := \pi_B(v_B, v_A) - \beta_B v_B$. Let

$$v_B^{\widetilde{\pi}}(e) := \underset{v_B \in G(e)}{\operatorname{argmax}} \left\{ U_B^{\widetilde{\pi}} \left( v_B, v_{A,e}^* \right) \right\},$$

and

$$W_B^{\widetilde{\pi}}(e) := \max_{v_B \in G(e)} \left\{ U_B^{\widetilde{\pi}} \left( v_B, v_{A,e}^* \right) \right\}.$$

Since $U_B^{\widetilde{\pi}}(v_B, v_{A,e}^*)$ is strictly concave with respect to $v_B$, and $G(e)$ is compact, there is a unique maximum of $U_B^{\widetilde{\pi}}(v_B, v_{A,e}^*)$ on $G(e)$. It follows from the Maximum Theorem (e.g., Aliprantis and Border 2006, Section 17.5, or Ok 2007, Section E.3), that both $v_B^{\widetilde{\pi}}$ and $W_B^{\widetilde{\pi}}$ are continuous at $e$ for all $e \in [2, \infty)$. Since $W_B^{\widetilde{\pi}}(2) < U_B^{\widetilde{\mu}}(v_{B,2}^*, v_{A,2}^*, 2)$, and $U_B^{\widetilde{\mu}}(v_{B,e}^*, v_{A,e}^*)$ is continuous with respect to $e$, there must exist an $e_{\widetilde{\mu}2} > 2$ such that $W_B^{\widetilde{\pi}(e)} \leqslant U_B^{\widetilde{\mu}}(v_{B,e}^*, v_{A,e}^*, e)$ for all $e \in [2, e_{\widetilde{\mu}2}]$. This proves that ($\mathbf{E}_{\widetilde{\mu}e}$-c) holds for all $e \in [2, e_{\widetilde{\mu}2}]$.

Let $e_{\widetilde{\mu}} = \min\{e_{\widetilde{\mu}1}, e_{\mu 2}\}$ and this completes the proof of the first part. So I turn to show the properties in the second claim.

**(2-a: the point $e_\pi$).** I first show that there exists a unique cutoff point $e_\pi$ such that $(v_A^*, v_B^*)$ identified in the Proposition 13(2) is an equilibrium for all elections $e \in (e_\pi, \infty)$. Let $(v_A^*, v_B^*)$ satisfies ($\mathbf{E}_\pi$-a).

Let $e_{\pi 1} > 2$ be defined by letting

$$L\big(v_B^*, e_{\pi 1}\big) = \psi^{-1}\left[X_L(e_{\pi 1}) \cdot \psi(v_B^*)\right] = v_A^*;$$

that is,

$$X_L(e_{\pi 1}) = \frac{\psi(v_A^*)}{\psi(v_B^*)}.$$

Since $\psi(v_A^*)/\psi(v_B^*) \in (0, 1)$, and $X_L(e)$ is strictly decreasing with $e$, $X_L(2) = 1$, and $\lim_{e \to \infty} X_L(e) = 0$ (see Figure A.1), such a point $e_{\pi 1}$ must exist and is unique. Moreover, $v_A^* > L(v_B^*, e)$ for all $e > e_{\pi 1}$. Therefore, ($\mathbf{E}_\pi$-b) holds for all $e \in (e_{\pi 1}, \infty)$.

Next, let $e_{\pi 2}$ be defined by letting

$$\widetilde{\pi}_B(v_A^*, v_B^*) - \beta_B v_B^* = \max_{v_B > R(v_A^*, e_{\pi 2})} \left\{ \widetilde{\mu}_B \left( v_B, v_A^*, e_{\pi 2} \right) - \beta_B v_B \right\}.$$

Now, if there does not exist such a point $e_{\pi 2}$, let $e_{\pi 2} = 0$; if such a number $e_{\pi 2}$ does exist, it is unique since the function $\widetilde{\mu}_B(v_B, v_A^*, e) - \beta_B v_B$ is strictly decreasing with $e$ for every $v_B$. So ($\mathbf{E}_\pi$-c) holds whenever $e \in [e_{\pi 2}, \infty)$.

Let $e_\pi = \max\{e_{\pi 1}, e_{\pi 2}\}$. Then $\mathbf{E}_\pi$ is an equilibrium for every election $e \in (e_\pi, \infty)$.

**(2-b: the point $e'_\pi$).** For this property, it suffices to show there exists a unique $e''_\pi$ such that $\mathbf{E}_{\widetilde{\mu}e}$ is not an equilibrium for every $e \in [e''_\pi, \infty)$.

Notice that (B.9) can be reformulated as follows:

$$\frac{\beta_B}{\beta_A} \leqslant \frac{\psi\left(v^*_{A,e}\right)}{\psi\left(v^*_{B,e}\right)} < 1. \tag{B.10}$$

Pick an $e''_\pi$ such that

$$X_L\left(e''_\pi\right) = \frac{\beta_B}{\beta_A}.$$

Such an $e''_\pi$ must exist, unique and if $e \in [e''_\pi, \infty)$ then $X_L(e) < \beta_B/\beta_A$, and so

$$\psi\left(L\left(v^*_{B,e}\right)\right) = X_L(e) \cdot \psi\left(v^*_{B,e}\right) < \frac{\beta_B}{\beta_A} \cdot \psi\left(v^*_{B,e}\right) < \psi\left(v^*_{A,e}\right),$$

where the last inequality follows from (B.10). Hence, if $e \in [e''_\pi, \infty)$, then $L(v^*_{B,e}) < v^*_{A,e}$; that is, ($\mathbf{E}_{\tilde{\mu}e}$-b) fails.

Let $e'_\pi = \max\{e_\pi, e''_\pi\}$. Then, combining claim 1 in this proposition, we know that $\mathbf{E}_\pi$ is the unique pure strategy equilibrium for all $e \in (e'_\pi, \infty)$. □

Proof of Corollary 14. If $\psi(x) = x^\gamma$, where $\gamma \in (0, 1]$, then $\Lambda(x) = \gamma/x$. So the necessary condition (6) implies that $v^*_{B,e} = (\beta_A/\beta_B)v^*_{A,e} =: \hat{\beta} v^*_{A,e}$. Hence,

$$\mu_A\left(v^*_{A,e}, v^*_{B,e}\right) = \frac{1}{2}\left(\frac{e}{\hat{\beta} + \hat{\beta}^2}\right)^{1/3},$$

and

$$v^*_{A,e} = \frac{\left(\hat{\beta}^\gamma + 2\hat{\beta}^{2\gamma}\right)\gamma e^{1/3}}{6\left(\hat{\beta} + \hat{\beta}^2\right)^{4/3}\beta_A}, \quad v^*_{B,e} = \frac{\left(\hat{\beta}^\gamma + 2\hat{\beta}^{2\gamma}\right)\gamma e^{1/3}}{6\left(\hat{\beta} + \hat{\beta}^2\right)^{4/3}\beta_B}.$$

Let $e_{\mu 1}$ be defined by letting $L(v_{B,e_{\mu 1}}) = v^*_{A,e_{\mu 1}}$. Then

$$e_{\mu 1} = X_L^{-1}\left[\left(\frac{v^*_{B,e}}{v^*_{A,e}}\right)^\gamma\right] = X_L^{-1}\left(\frac{1}{\hat{\beta}^\gamma}\right).$$

Now for $e \in (2, e_{\mu 1})$, we have

$$\begin{aligned}\left[L\left(v^*_{B,e}\right)\right]^\gamma - \left(v^*_{A,e}\right)^\gamma &= X_L(e) \cdot \left(v^*_{B,e}\right)^\gamma - \left(v^*_{A,e}\right)^\gamma \\ &> X_L(e_{\mu 1}) \cdot \left(v^*_{B,e}\right)^\gamma - \left(v^*_{A,e}\right)^\gamma \\ &= \left[X_L(e_{\mu 1}) \cdot \hat{\beta}^\gamma - 1\right]\left(v^*_{A,e}\right)^\gamma \\ &= 0;\end{aligned}$$

that is, $v^*_{A,e} < L(v^*_{B,e})$ for all $e \in (2, e_{\mu 1})$.

To decide the last cutoff point $e_{\mu 2}$, let

$$\beta_B = \mathbb{D}_1\tilde{\pi}_B\left(v_{A,e_{\mu 2}}, R(v_{A,e_{\mu 2}})\right) = \pi_A\left(R(v_{A,e_{\mu 2}}), v_{A,e_{\mu 2}}\right)\frac{\gamma}{2R(v_{A,e_{\mu 2}})}$$
$$= \frac{1}{4}\left(\frac{1}{\hat{\beta}}\right)^{\gamma/2}\frac{\gamma}{X_R\left(e_{\mu 2}\right)\cdot v_{A,e_{\mu 2}}}.$$

Then, if $e \in (2, e_{\mu 2})$, we have $\mathbb{D}_1\tilde{\pi}_B\left(R(v_{A,e}), v_{A,e}\right) > \beta_B$, and so

$$\begin{aligned}\tilde{\mu}_B\left(v^*_{B,e}, v^*_{A,e}\right) - \beta_B v^*_{B,e} &> \tilde{\mu}_B\left(R(v^*_{A,e}), v^*_{A,e}\right) - \beta_B R(v^*_{A,e})\\ &= \max_{v_B}\left\{\tilde{\pi}_B\left(v_B, v^*_{A,e}\right) - \beta_B v_B\right\}\\ &= \tilde{\pi}_B\left(R(v^*_{A,e}), v^*_{A,e}\right) - \beta_B R(v^*_{A,e}).\end{aligned}$$

Let $e_{\tilde{\mu}} = \min\{e_{\mu 1}, e_{\mu 2}\}$ and we are done. □

Proof of Proposition 19. Fix an election $e$. The necessary condition (6) implies that for every equilibrium $(v_A, v_B)$,

$$\Lambda(v_B) = \frac{\beta_B}{\beta_A}\cdot\Lambda(v_A).$$

It follows from the above equation that for each $v_A$, if $\beta_B$ increases, then $v_B$ strictly decreases since $\Lambda$ is strictly decreasing. This result indicates that when the two candidates becomes similar ($\beta_A - \beta_B$ becomes smaller), then their equilibrium valence choices becomes similar, too.

Write $v_B(\beta_B) = T(v_A, \beta_B)$, where $T$ is defined in (B.7). Define

$$\Xi\left(v_A, \beta_B\right) := \frac{\psi\left(T(v_A), \beta_B\right)}{\psi(v_A)},$$

and

$$\Omega\left(v_A, \beta_B\right) := \frac{\Xi\left(v_A, \beta_B\right) + 2\Xi^2\left(v_A, \beta_B\right)}{\left[\Xi\left(v_A, \beta_B\right) + \Xi^2\left(v_A, \beta_B\right)\right]^{4/3}}\Lambda\left(v_A\right).$$

Then $\Xi(v_A, \beta_B)$ is strictly decreasing with $\beta_B$. Since the function

$$x \mapsto \frac{x + 2x^2}{\left(x + x^2\right)^{4/3}}$$

is strictly decreasing with $x$ when $x > 0$, we know that $\Omega(v_A, \beta_B)$ is strictly increasing with $\beta_B$ when $v_A$ is fixed.

The following equalities hold at equilibrium:

$$\beta_A = \mathbb{D}_1\mu_A\left(v_A, T(v_A, \beta_B)\right) = \left(\frac{e^{1/3}}{6}\right)\Omega\left(v_A, \beta_B\right).$$

Since $\Omega$ is strictly decreasing with $v_A$, we infer that if $\beta_B$ increases, then $v_{A,e}$ must increase, too. This proves that $v_A$ is strictly increasing with $\beta_B$ and $\Delta v$ is strictly decreasing with $\beta_B$ in the equilibrium $\mathbf{E}_{\widetilde{\mu}e}$.

Now consider the equilibrium $\mathbf{E}_\pi$. In equilibrium we have

$$\begin{aligned}\beta_A = \mathbb{D}_1\pi_A\left(v_A, T(v_A,\beta_B)\right) &= \frac{1}{4}\left[\frac{\psi(v_A)}{\psi\left(T(v_A,\beta_B)\right)}\right]^{1/2} \Lambda(v_A) \\ &= \frac{1}{4}\left[\frac{1}{\Xi(v_A,\beta_B)}\right]^{1/2} \Lambda(v_A).\end{aligned}$$

Since $\mathbb{D}_{11}\pi_A < 0$ and $\mathbb{D}_{12}\pi_A < 0$ by Lemma 4, we know that $\mathbb{D}_1\pi_A(v_A, T(v_A,\beta_B))$ is strictly decreasing with $v_A$ when $\beta_B$ is fixed. Because $\Xi(v_A,\beta_B)$ is strictly decreasing with $\beta_B$, we infer that when $\beta_B$ increases, $v_A$ must increase in the equilibrium $\mathbf{E}_\pi$.

□

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