General Topology

A Solution Manual for Willard (2004)

Jianfei Shen

School of Economics, The University of New South Wales

Sydney, Australia October 15, 2011

Preface

Sydney, *Jianfei Shen*
October 15, 2011

Acknowledgements

Contents

Acronyms

$\mathbb{R}$	the set of real numbers
$\mathbb{I}$	$[0, 1]$
$\mathbb{P}$	$\mathbb{R} \setminus \mathbb{Q}$

1

SET THEORY AND METRIC SPACES

1.1 Set Theory

1A. Russell's Paradox

► Exercise 1. *The phenomenon to be presented here was first exhibited by Russell in 1901, and consequently is known as* Russell's Paradox*.*

Suppose we allow as sets things A for which $A \in A$. Let $\mathcal{P}$ be the set of all sets. Then $\mathcal{P}$ can be divided into two nonempty subsets, $\mathcal{P}_1 = \{A \in \mathcal{P} : A \notin A\}$ and $\mathcal{P}_2 = \{A \in \mathcal{P} : A \in A\}$. Show that this results in the contradiction: $\mathcal{P}_1 \in \mathcal{P}_1 \iff \mathcal{P}_1 \notin \mathcal{P}_1$. Does our (naive) restriction on sets given in 1.1 eliminate the contradiction?

Proof. If $\mathcal{P}_1 \in \mathcal{P}_1$, then $\mathcal{P}_1 \in \mathcal{P}_2$, i.e., $\mathcal{P}_1 \notin \mathcal{P}_1$. But if $\mathcal{P}_1 \notin \mathcal{P}_1$, then $\mathcal{P}_1 \in \mathcal{P}_1$. A contradiction. □

1B. De Morgan's laws and the distributive laws

► Exercise 2. a. $A \smallsetminus \left(\bigcap_{\lambda \in \Lambda} B_\lambda\right) = \bigcup_{\lambda \in \Lambda} (A \smallsetminus B_\lambda)$.

b. $B \cup \left(\bigcap_{\lambda \in \Lambda} B_\lambda\right) = \bigcap_{\lambda \in \Lambda} (B \cup B_\lambda)$.

c. *If A_{nm} is a subset of A for $n = 1, 2, \ldots$ and $m = 1, 2, \ldots$, is it necessarily true that*

$$\bigcup_{n=1}^{\infty} \left[\bigcap_{m=1}^{\infty} A_{nm}\right] = \bigcap_{m=1}^{\infty} \left[\bigcup_{n=1}^{\infty} A_{nm}\right]?$$

Proof. **(a)** If $x \in A \smallsetminus \left(\bigcap_{\lambda \in \Lambda} B_\lambda\right)$, then $x \in A$ and $x \notin \bigcap_{\lambda \in \Lambda} B_\lambda$; thus, $x \in A$ and $x \notin B_\lambda$ for some λ, so $x \in (A \smallsetminus B_\lambda)$ for some λ; hence $x \in \bigcup_{\lambda \in \Lambda} (A \smallsetminus B_\lambda)$. On the other hand, if $x \in \bigcup_{\lambda \in \Lambda} (A \smallsetminus B_\lambda)$, then $x \in A \smallsetminus B_\lambda$ for some $\lambda \in \Lambda$, i.e., $x \in A$ and $x \notin B_\lambda$ for some $\lambda \in \Lambda$. Thus, $x \in A$ and $x \notin \bigcap_{\lambda \in \Lambda} B_\lambda$; that is, $x \in A \smallsetminus \left(\bigcap_{\lambda \in \Lambda} B_\lambda\right)$.

(b) If $x \in B \cup \left(\bigcap_{\lambda \in \Lambda} B_\lambda\right)$, then $x \in B_\lambda$ for all λ, then $x \in (B \cup B_\lambda)$ for all λ, i.e., $x \in \bigcap_{\lambda \in \Lambda}(B \cup B_\lambda)$. On the other hand, if $x \in \bigcap_{\lambda \in \Lambda}(B \cup B_\lambda)$, then $x \in (B \cup B_\lambda)$ for all λ, i.e., $x \in B$ or $x \in B_\lambda$ for all λ; that is, $x \in B \cup \left(\bigcap_{\lambda \in \Lambda} B_\lambda\right)$.

(c) They are one and the same set. □

1C. Ordered pairs

► EXERCISE 3. *Show that, if (x_1, x_2) is defined to be $\{\{x_1\}, \{x_1, x_2\}\}$, then $(x_1, x_2) = (y_1, y_2)$ iff $x_1 = y_1$ and $x_2 = y_2$.*

PROOF. If $x_1 = y_1$ and $x_2 = y_2$, then, clearly, $(x_1, x_2) = \{\{x_1\}, \{x_1, x_2\}\} = \{\{y_1\}, \{y_1, y_2\}\} = (y_1, y_2)$. Now assume that $\{\{x_1\}, \{x_1, x_2\}\} = \{\{y_1\}, \{y_1, y_2\}\}$. If $x_1 \neq x_2$, then $\{x_1\} = \{y_1\}$ and $\{x_1, x_2\} = \{y_1, y_2\}$. So, first, $x_1 = y_1$ and then $\{x_1, x_2\} = \{y_1, y_2\}$ implies that $x_2 = y_2$. If $x_1 = x_2$, then $\{\{x_1\}, \{x_1, x_1\}\} = \{\{x_1\}\}$. So $\{y_1\} = \{y_1, y_2\} = \{x_1\}$, and we get $y_1 = y_2 = x_1$, so $x_1 = y_1$ and $x_2 = y_2$ holds in this case, too. □

1D. Cartesian products

► EXERCISE 4. *Provide an inductive definition of "the ordered n-tuple $(x_1, \ldots, x_n)$ of elements $x_1, \ldots, x_n$ of a set" so that $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are equal iff their coordinates are equal in order, i.e., iff $x_1 = y_1, \ldots, x_n = y_n$.*

PROOF. Define $(x_1, \ldots, x_n) = \{(1, x_1), \ldots, (n, x_n)\}$ as a finite sequence. □

► EXERCISE 5. *Given sets $X_1, \ldots, X_n$ define the Cartesian product $X_1 \times \cdots \times X_n$*

a. *by using the definition of ordered n-tuple you gave in Exercise 4,*

b. *inductively from the definition of the Cartesian product of two sets,*

and show that the two approaches are the same.

PROOF. **(a)** $X_1 \times \cdots \times X_n = \{f \in (\bigcup_{i=1}^n X_i)^n \colon f(i) \in X_i\}$.

(b) From the definition of the Cartesian product of two sets, $X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n) \colon x_i \in X_i\}$, where $(x_1, \ldots, x_n) = ((x_1, \ldots, x_{n-1}), x_n)$.

These two definitions are equal essentially since there is a bijection between them. □

► EXERCISE 6. *Given sets $X_1, \ldots, X_n$ let $X = X_1 \times \cdots \times X_n$ and let X^* be the set of all functions f from $\{1, \ldots, n\}$ into $\bigcup_{k=1}^n X_k$ having the property that $f(k) \in X_k$ for each $k = 1, \ldots, n$. Show that X^* is the "same" set as X.*

PROOF. Each function f can be written as $\{(1, x_1), \ldots, (n, x_n)\}$. So define $F \colon X^* \to X$ as $F(f) = (x_1, \ldots, x_n)$. □

► EXERCISE 7. *Use what you learned in Exercise 6 to define the Cartesian product $X_1 \times X_2 \times \cdots$ of denumerably many sets as a collection of certain functions with domain $\mathbb{N}$.*

PROOF. $X_1 \times X_2 \times \cdots$ consists of functions $f : \mathbb{N} \to \bigcup_{n=1}^{\infty} X_n$ such that $f(n) \in X_n$ for all $n \in \mathbb{N}$. □

1.2 Metric Spaces

2A. Metrics on $\mathbb{R}^n$

► EXERCISE 8. *Verify that each of the following is a metric on $\mathbb{R}^n$:*

a. $\rho(\boldsymbol{x}, \boldsymbol{y}) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.

b. $\rho_1(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} |x_i - y_i|$.

c. $\rho_2(\boldsymbol{x}, \boldsymbol{y}) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$.

PROOF. Clearly, it suffices to verify the triangle inequalities for all of the three functions. Pick arbitrary $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathbb{R}^n$.

(a) By Minkowski's Inequality, we have

$$\begin{aligned}
\rho(\boldsymbol{x}, \boldsymbol{z}) = \sqrt{\sum_{i=1}^{n} (x_i - z_i)^2} &= \sqrt{\sum_{i=1}^{n} [(x_i - y_i) + (y_i - z_i)]^2} \\
&\leqslant \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n} (y_i - z_i)^2} \\
&= \rho(\boldsymbol{x}, \boldsymbol{y}) + \rho(\boldsymbol{y}, \boldsymbol{z}).
\end{aligned}$$

(b) We have

$$\rho_1(\boldsymbol{x}, \boldsymbol{z}) = \sum_{i=1}^{n} |x_i - z_i| = \sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) = \rho_1(\boldsymbol{x}, \boldsymbol{y}) + \rho_1(\boldsymbol{y}, \boldsymbol{z}).$$

(c) We have

$$\begin{aligned}
\rho_2(\boldsymbol{x}, \boldsymbol{z}) &= \max\{|x_1 - z_1|, \ldots, |x_n - z_n|\} \\
&\leqslant \max\{|x_1 - y_1| + |y_1 - z_1|, \ldots, |x_n - y_n| + |y_n - z_n|\} \\
&\leqslant \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\} + \max\{|y_1 - z_1|, \ldots, |y_n - z_n|\} \\
&= \rho_2(\boldsymbol{x}, \boldsymbol{y}) + \rho_2(\boldsymbol{y}, \boldsymbol{z}).
\end{aligned}$$

□

2B. Metrics on $\mathscr{C}(\mathbb{I})$

▶ EXERCISE 9. *Let $\mathscr{C}(\mathbb{I})$ denote the set of all continuous real-valued functions on the unit interval $\mathbb{I}$ and let x_0 be a fixed point of $\mathbb{I}$.*

a. $\rho(f,g) = \sup_{x\in\mathbb{I}} |f(x) - g(x)|$ *is a metric on* $\mathscr{C}(\mathbb{I})$.

b. $\sigma(f,g) = \int_0^1 |f(x) - g(x)|\,\mathrm{d}x$ *is a metric on* $\mathscr{C}(\mathbb{I})$.

c. $\eta(f,g) = |f(x_0) - g(x_0)|$ *is a pseudometric on* $\mathscr{C}(\mathbb{I})$.

PROOF. Let $f,g,h \in \mathscr{C}(\mathbb{I})$. It is clear that ρ, σ, and η are positive, symmetric; it is also clear that ρ and σ satisfy M-b.

(a) We have

$$\begin{aligned}\rho(f,h) = \sup_{x\in\mathbb{I}} |f(x) - h(x)| &\leqslant \sup_{x\in\mathbb{I}}(|f(x) - g(x)| + |g(x) - h(x)|)\\ &\leqslant \sup_{x\in\mathbb{I}} |f(x) - g(x)| + \sup_{x\in\mathbb{I}} |g(x) - h(x)|\\ &= \rho(f,g) + \rho(g,h).\end{aligned}$$

(b) We have

$$\begin{aligned}\sigma(f,h) = \int_0^1 |f(x) - h(x)| &\leqslant \int_0^1 |f(x) - g(x)| + \int_0^1 |g(x) - h(x)|\\ &= \sigma(f,g) + \sigma(g,h).\end{aligned}$$

(c) For arbitrary $f,g \in \mathscr{C}(\mathbb{I})$ with $f(x_0) = g(x_0)$ we have $\eta(f,g) = 0$, so $\eta(f,g) = 0$ does not imply that $f = g$. Further, $\eta(f,h) = |f(x_0) - h(x_0)| \leqslant |f(x_0) - g(x_0)| + |g(x_0) - h(x_0)| = \eta(f,g) + \eta(g,h)$. □

2C. Pseudometrics

▶ EXERCISE 10. *Let (M,ρ) be a pseudometric space. Define a relation $\sim$ on M by $x \sim y$ iff $\rho(x,y) = 0$. Then $\sim$ is an equivalence relation.*

PROOF. (i) $x \sim x$ since $\rho(x,x) = 0$ for all $x \in M$. (ii) $x \sim y$ iff $\rho(x,y) = 0$ iff $\rho(y,x) = 0$ iff $y \sim x$. (iii) Suppose $x \sim y$ and $y \sim z$. Then $\rho(x,z) \leqslant \rho(x,y) + \rho(y,z) = 0$; that is, $\rho(x,z) = 0$. So $x \sim z$. □

▶ EXERCISE 11. *If M^* is he set of equivalence classes in M under the equivalence relation $\sim$ and if ρ^* is defined on M^* by $\rho^*([x],[y]) = \rho(x,y)$, then ρ^* is a well-defined metric on M^*.*

PROOF. ρ^* is well-defined since it does not dependent on the representative of $[x]$: let $x' \in [x]$ and $y' \in [y]$. Then

$$\rho(x',y') \leqslant \rho(x',x) + \rho(x,y) + \rho(y,y') = \rho(x,y).$$

Symmetrically, $\rho(x,y) \leqslant \rho(x',y')$. To verify ρ^* is a metric on M^*, it suffices to show that ρ^* satisfies the triangle inequality. Let $[x],[y],[z] \in M^*$. Then

$$\rho^*([x],[z]) = \rho(x,z) \leqslant \rho(x,y) + \rho(y,z) = \rho^*([x],[y]) + \rho^*([y],[z]). \qquad \square$$

► Exercise 12. *If $h\colon M \to M^*$ is the mapping $h(x) = [x]$, then a set A in M is closed (open) iff $h(A)$ is closed (open) in M^*.*

Proof. Let A be open in M and $h(x) = [x] \in h(A)$ for some $x \in A$. Since A is open, there exist an ε-disk $U_\rho(x,\varepsilon)$ contained in A. For each $y \in U_\rho(x,\varepsilon)$, we have $h(y) = [y] \in h(A)$, and $\rho^*([x],[y]) = \rho(x,y) \leqslant \varepsilon$. Hence, for each $[x] \in h(A)$, there exists an ε-disk $U_{\rho^*}([x],\varepsilon) = h(U_\rho(x,\varepsilon))$ contained in $h(A)$; that is, $h(A)$ is open in M^*. Since h is surjective, it is now easy to see that $h(A)$ is closed in M^* whenever A is closed in M. $\square$

► Exercise 13. *If f is any real-valued function on a set M, then the distance function $\rho_f(x,y) = |f(x) - f(y)|$ is a pseudometric on M.*

Proof. Easy. $\square$

► Exercise 14. *If (M,ρ) is any pseudometric space, then a function $f: M \to \mathbb{R}$ is continuous iff each set open in (M,ρ_f) is open in (M,ρ).*

Proof. Suppose that f is continuous and G is open in (M,ρ_f). For each $x \in G$, there is an $\varepsilon > 0$ such that if $|f(y) - f(x)| < \varepsilon$ then $y \in G$. The continuity of f at x implies that there exists $\delta > 0$ such that if $\rho(y,x) < \delta$ then $|f(y) - f(x)| < \varepsilon$, and so $y \in U$. We thus proved that for each $x \in U$ there exists a δ-disk $U_\rho(x,\rho)$ contained in G; that is, G is open in (M,ρ).

Conversely, suppose that each set is open in (M,ρ) whenever it is open in (M,ρ_f). For each $x \in (M,\rho_f)$, there is an ε-disk $U_{\rho_f}(x,\varepsilon)$ contained in M since M is open under ρ_f; then $U_{\rho_f}(x,\varepsilon)$ is open in (M,ρ) since $U_{\rho_f}(x,\varepsilon)$ is open in (M,ρ_f). Hence, there is an δ-disk $U_\rho(x,\delta)$ such that $U_\rho(x,\delta) \subset U_{\rho_f}(x,\varepsilon)$; that is, if $\rho(y,x) < \delta$, then $|f(y) - f(x)| < \varepsilon$. So f is continuous on M. $\square$

2D. Disks Are Open

► Exercise 15. *For any subset A of a metric space M and any $\varepsilon > 0$, the set $U(A,\varepsilon)$ is open.*

Proof. Let $A \subset M$ and $\varepsilon > 0$. Take an arbitrary point $x \in U(A,\varepsilon)$; take an arbitrary point $y \in A$ such that $\rho(x,y) < \varepsilon$. Observe that every ε-disk $U(y,\varepsilon)$ is contained in $U(A,\varepsilon)$. Since $x \in U(y,\varepsilon)$ and $U(y,\varepsilon)$ is open, there exists an δ-disk $U(x,\delta)$ contained in $U(y,\varepsilon)$. Therefore, $U(A,\varepsilon)$ is open. $\square$

2E. Bounded Metrics

► EXERCISE 16. *If ρ is any metric on M, the distance function $\rho^*(x,y) = \min\{\rho(x,y,),1\}$ is a metric also and is bounded.*

PROOF. To see ρ^* is a metric, it suffices to show the triangle inequality. Let $x,y,z \in M$. Then

$$\begin{aligned}\rho^*(x,z) = \min\{\rho(x,z),1\} &\leqslant \min\{\rho(x,y)+\rho(y,z),1\} \\ &\leqslant \min\{\rho(x,y),1\} + \min\{\rho(y,z),1\} \\ &= \rho^*(x,y) + \rho^*(y,z).\end{aligned}$$

It is clear that ρ^* is bounded above by 1. □

► EXERCISE 17. *A function f is continuous on (M,ρ) iff it is continuous on (M,ρ^*).*

PROOF. It suffices to show that ρ and ρ^* are equivalent. If G is open in (M,ρ), then for each $x \in G$ there is an ε-disk $U_\rho(x,\varepsilon) \subset G$. Since $U_{\rho^*}(x,\varepsilon) \subset U_\rho(x,\varepsilon)$, we know G is open in (M,ρ^*). Similarly, we can show that G is open in (M,ρ^*) whenever it is open in (M,ρ). □

2F. The Hausdorff Metric

Let ρ be a bounded metric on M; that is, for some constant A, $\rho(x,y) \leqslant A$ for all x and y in M.

► EXERCISE 18. *Show that the elevation of ρ to the power set $\mathcal{P}(M)$ as defined in 2.4 is not necessarily a pseudometric on $\mathcal{P}(M)$.*

PROOF. Let $M := \{(x_1,x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant 1\}$, and let ρ be the usual metric. Then ρ is a bounded metric on M. We show that the function $\rho^*\colon (E,F) \mapsto \inf_{x\in E, y\in F} \rho(x,y)$, for all $E,F \in \mathcal{P}(M)$, is not a pseudometric on $\mathcal{P}(M)$ by showing that the triangle inequality fails. Let $E,F,G \in \mathcal{P}(M)$, where $E = U_\rho((-1/4,0),1/4)$, $G = U_\rho((1/4,0),1/4)$, and F meets both E and G. Then $\rho^*(E,G) > 0$, but $\rho^*(E,F) = \rho^*(F,G) = 0$. □

► EXERCISE 19. *Let $\mathcal{F}(M)$ be all nonempty closed subsets of M and for $A,B \in \mathcal{F}(M)$ define*

$$\begin{aligned}d_A(B) &= \sup\{\rho(A,x) : x \in B\}\\ d(A,B) &= \max\{d_A(B), d_B(A)\}.\end{aligned}$$

Then d is a metric on $\mathcal{F}(M)$ with the property that $d(\{x\},\{y\}) = \rho(x,y)$. It is called the Hausdorff metric *on $\mathcal{F}(M)$.*

PROOF. Clearly, d is nonnegative and symmetric. If $d(A,B)=0$, then $d_A(B) = d_B(A) = 0$, i.e., $\sup_{y\in B}\rho(A,y) = \sup_{x\in A}\rho(B,x) = 0$. But then $\rho(A,y)=0$ for all $y \in B$ and $\rho(B,x)=0$ for all $x \in A$. Since A is closed, we have $y \in A$ for all $y \in B$; that is, $B \subset A$. Similarly, $A \subset B$. Hence, $A = B$.

We next show the triangle inequality of d. Let $A,B,C \in \mathcal{F}(M)$. For an arbitrary point $a \in A$, take a point $b \in C$ such that $\rho(a,b) = \rho(B,a)$ (since B is closed, such a point exists). Then

$$\rho(a,b) \leqslant \sup_{x\in A}\rho(B,x) = d_B(A) \leqslant d(A,B).$$

For this $b \in B$, we take a point $c \in C$ such that $\rho(b,c) \leqslant d(B,C)$. Therefore,

$$\rho(a,c) \leqslant \rho(a,b) + \rho(b,c) \leqslant d(A,B) + d(B,C).$$

We thus proved that for every $a \in A$, there exists $c \in C$ (depends on a), such that $\rho(a,c) \leqslant d(A,B) + d(B,C)$. In particular, we have

$$\rho(a,C) = \inf_{z\in C}\rho(a,z) \leqslant d(A,B) + d(B,C).$$

Since the above inequality holds for all $a \in A$, we obtain

$$d_C(A) = \sup_{x\in A}\rho(a,C) \leqslant d(A,B) + d(B,C). \tag{1.1}$$

Similarly, for each $c \in C$ there exists $b \in B$ with $\rho(c,b) \leqslant d(B,C)$; for this b, there exists $a \in A$ with $\rho(a,b) \leqslant d(A,B)$. Hence $\rho(a,c) \leqslant d(A,B) + d(B,C)$ for all $c \in C$. The same argument shows that

$$d_A(C) \leqslant d(A,B) + d(B,C). \tag{1.2}$$

Combining (1.1) and (1.2) we get the desired result.

Finally, notice that $d_{\{x\}}(\{y\}) = d_{\{y\}}(\{x\}) = \rho(x,y)$; hence, $d(\{x\},\{y\}) = \rho(x,y)$. □

► EXERCISE 20. *Prove that closed sets A and B are "close" in the Hausdorff metric iff they are "uniformly close"; that is, $d(A,B) < \varepsilon$ iff $A \subset U_\rho(B,\varepsilon)$ and $B \subset U_\rho(A,\varepsilon)$.*

PROOF. If $d(A,B) < \varepsilon$, then $\sup_{y\in B}\rho(A,y) = \rho_A(B) < \varepsilon$; that is, $\rho(A,y) < \varepsilon$ for all $y \in B$, so $B \subset U_\rho(A,\varepsilon)$. Similarly, $A \subset U_\rho(B,\varepsilon)$.

Conversely, if $A \subset U_\rho(B,\varepsilon)$, then $\rho(B,x) < \varepsilon$ for all $x \in A$. Since A is closed, we have $d_B(A) < \varepsilon$; similarly, $B \subset U_\rho(A,\varepsilon)$ implies that $d_A(B) < \varepsilon$. Hence, $d(A,B) < \varepsilon$. □

2G. Isometry

Metric spaces (M, ρ) and (N, σ) are *isometric* iff there is a one-one function f from M onto N such that $\rho(x, y) = \sigma(f(x), f(y))$ for all x and y in M; f is called an *isometry*.

► EXERCISE 21. *If f is an isometry from M to N, then both f and f^{-1} are continuous functions.*

PROOF. By definition, f is (uniformly) continuous on M: for every $\varepsilon > 0$, let $\delta = \varepsilon$; then $\rho(x, y) < \delta$ implies that $\sigma(f(x), f(y)) = \rho(x, y) < \varepsilon$.

On the other hand, for every $\varepsilon > 0$ and $y \in N$, pick the unique $f^{-1}(y) \in M$ (since f is bijective). For each $z \in N$ with $\sigma(y, z) < \varepsilon$, we must have $\rho(f^{-1}(y), f^{-1}(z)) = \sigma(f(f^{-1}(y)), f(f^{-1}(z))) = \sigma(y, z) < \varepsilon$; that is, f^{-1} is continuous. □

► EXERCISE 22. *$\mathbb{R}$ is not isometric to $\mathbb{R}^2$ (each with its usual metric).*

PROOF. Consider $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Notice that there are only two points around $f^{-1}(0, 0)$ with distance 1. □

► EXERCISE 23. *$\mathbb{I}$ is isometric to any other closed interval in $\mathbb{R}$ of the same length.*

PROOF. Consider the function $f : \mathbb{I} \to [a, a + 1]$ defined by $f(x) = a + x$ for all $x \in \mathbb{I}$. □

2

TOPOLOGICAL SPACES

2.1 Fundamental Concepts

3A. Examples of Topologies

► Exercise 24. *If $\mathcal{F}$ is the collection of all closed, bounded subset of $\mathbb{R}$ (in its usual topology), together with $\mathbb{R}$ itself, then $\mathcal{F}$ is the family of closed sets for a topology on $\mathbb{R}$ strictly weaker than the usual topology.*

Proof. It is easy to see that $\mathcal{F}$ is a topology. Further, for instance, $(-\infty, 0]$ is a closed set of $\mathbb{R}$, but it is not in $\mathcal{F}$. □

► Exercise 25. *If $A \subset X$, show that the family of all subsets of X which contain A, together with the empty set $\varnothing$, is a topology on X. Describe the closure and interior operations. What topology results when $A = \varnothing$? when $A = X$?*

Proof. Let

$$\mathcal{E} = \{E \subset X : A \subset E\} \cup \{\varnothing\}.$$

Now suppose that $E_\lambda \in \mathcal{E}$ for each $\lambda \in \Lambda$. Then $A \subset \bigcup_\lambda E_\lambda \subset X$ and so $\bigcup E_\lambda \in \mathcal{E}$. The other postulates are easy to check.

For any set $B \subset X$, if $A \subset B$, then $B \in \mathcal{E}$ and so $B^\circ = B$; if not, then $B^\circ = \varnothing$.

If $A = \varnothing$, then $\mathcal{E}$ is the discrete topology; if $A = X$, then $\mathcal{E} = \{\varnothing, X\}$. □

3D. Regularly Open and Regularly Closed Sets

An open subset G in a topological space is *regular open* iff G is the interior of its closure. A closed subset is *regularly closed* iff it is the closure of its interior.

► Exercise 26. *The complement of a regularly open set is regularly closed and vice versa.*

Proof. Suppose G is regular open; that is, $G = (\overline{G})^\circ$. Then

$$X \setminus G = X \setminus (\overline{G})^\circ = \overline{X \setminus \overline{G}} = \overline{(X \setminus G)^\circ}.$$

Hence, $X \setminus G$ is regularly closed. If F is regular closed, i.e., $F = \overline{F^\circ}$, then

$$X \setminus F = X \setminus \overline{F^\circ} = (X \setminus F^\circ)^\circ = (\overline{X \setminus F})^\circ;$$

that is, $X \setminus F$ is regularly open. □

► Exercise 27. *There are open sets in $\mathbb{R}$ which are not regularly open.*

Proof. Consider $\mathbb{Q}$. We have $(\overline{\mathbb{Q}})^\circ = \mathbb{R}^\circ = \mathbb{R} \neq \mathbb{Q}$. So $\mathbb{Q}$ is not regularly open. □

► Exercise 28. *If A is any subset of a topological space, then* $\operatorname{int}(\operatorname{cl}(A))$ *is regularly open.*

Proof. Let A be a subset of a topological space X. We then have

$$\operatorname{int}(\operatorname{cl}(A)) \subset \operatorname{cl}(\operatorname{int}(\operatorname{cl}(A))) \implies \operatorname{int}(\operatorname{cl}(A)) = \operatorname{int}(\operatorname{int}(\operatorname{cl}(A))) \subset \operatorname{int}(\operatorname{cl}(\operatorname{int}(\operatorname{cl}(A)))),$$

and

$$\begin{aligned}\operatorname{int}(\operatorname{cl}(A)) \subset \operatorname{cl}(A) &\implies \operatorname{cl}(\operatorname{int}(\operatorname{cl}(A))) \subset \operatorname{cl}(\operatorname{cl}(A)) = \operatorname{cl}(A)\\ &\implies \operatorname{int}(\operatorname{cl}(\operatorname{int}(\operatorname{cl}(A)))) \subset \operatorname{int}(\operatorname{cl}(A)).\end{aligned}$$

Therefore, $\operatorname{int}(\operatorname{cl}(A)) = \operatorname{int}(\operatorname{cl}(\operatorname{int}(\operatorname{cl}(A))))$; that is, $\operatorname{int}(\operatorname{cl}(A))$ is regularly open. □

► Exercise 29. *The intersection, but not necessarily the union, of two regularly open sets is regularly open.*

Proof. Let A and B be two regularly open sets in a topological space X. Then

$$(\overline{A \cap B})^\circ \subset (\overline{A} \cap \overline{B})^\circ = (\overline{A})^\circ \cap (\overline{B})^\circ = A \cap B,$$

and

$$\begin{aligned}(\overline{A} \cap \overline{B})^\circ = (\overline{A})^\circ \cap (\overline{B})^\circ = A \cap B \subset \overline{A \cap B}&\\ \implies A \cap B = (\overline{A} \cap \overline{B})^\circ = \left[(\overline{A} \cap \overline{B})^\circ\right]^\circ \subset (\overline{A \cap B})^\circ&.\end{aligned}$$

Hence, $A \cap B = (\overline{A \cap B})^\circ$.

To see that the union of two regularly open sets is not necessarily regularly open, consider $A = (0,1)$ and $B = (1,2)$ in $\mathbb{R}$ with its usual topology. Then

$$(\overline{A \cup B})^\circ = [0,2]^\circ = (0,2) \neq A \cup B.$$ □

3E. Metrizable Spaces

Let X be a metrizable space whose topology is generated by a metric ρ.

► EXERCISE 30. *The metric 2ρ defined by $2\rho(x, y) = 2 \cdot \rho(x, y)$ generates the same topology on X.*

PROOF. Let $\mathcal{O}_\rho$ be the collection of open sets in (X, ρ), and let $\mathcal{O}_{2\rho}$ be the collection of open sets in $(X, 2\rho)$. If $O \in \mathcal{O}_\rho$, then for every $x \in O$, there exists an open ball $\mathbb{B}_\rho(x, \varepsilon) \subseteq O$; but then $\mathbb{B}_{2\rho}(x, \varepsilon/2) \subset O$. Hence, $O \in \mathcal{O}_{2\rho}$. Similarly, we can show that $\mathcal{O}_{2\rho} \subset \mathcal{O}_\rho$. In fact, ρ and 2ρ are equivalent metrics. □

► EXERCISE 31. *The closure of a set $E \subset X$ is given by $\overline{E} = \{y \in X : \rho(E, y) = 0\}$.*

PROOF. Denote $\widetilde{E} := \{y \in X : \rho(E, y) = 0\}$. We first show that $\widetilde{E}$ is closed (see Definition 2.5, p. 17). Take an arbitrary $x \in X$ such that for every $n \in \mathbb{N}$, there exists $y_n \in \widetilde{E}$ with $\rho(x, y_n) < 1/2n$. For each $y_n \in \widetilde{E}$, take $z_n \in E$ with $\rho(y_n, z_n) < 1/2n$. Then

$$\rho(x, z_n) \leqslant \rho(x, y_n) + \rho(y_n, z_n) < 1/n, \quad \text{for all } n \in \mathbb{N}.$$

Thus, $\rho(x, E) = 0$, i.e., $x \in \widetilde{E}$. Therefore, $\widetilde{E}$ is closed. It is clear that $E \subseteq \widetilde{E}$, and so $\overline{E} \subset \widetilde{E}$.

We next show that $\widetilde{E} \subseteq \overline{E}$. Take an arbitrary $x \in \widetilde{E}$ and a closed set K containing E. If $x \in X \smallsetminus K$, then $\rho(x, K) > 0$ (see Exercise 35). But then $\rho(x, E) > 0$ since $E \subset K$ and so

$$\inf_{y \in E} \rho(x, y) \geqslant \inf_{z \in K} \rho(x, z).$$

Hence, $\widetilde{E} \subset \overline{E}$. □

► EXERCISE 32. *The closed disk $U(x, \overline{\varepsilon}) = \{y : \rho(x, y) \leqslant \varepsilon\}$ is closed in X, but may not be the closure of the open disk $U(x, \varepsilon)$.*

PROOF. Fix $x \in X$. We show that the function $\rho(x, \cdot) \colon X \to \mathbb{R}$ is (uniformly) continuous. For any $y, z \in X$, the triangle inequality yields

$$|\rho(x, y) - \rho(x, z)| \leqslant \rho(y, z).$$

Hence, for every $\varepsilon > 0$, take $\delta = \varepsilon$, and $\rho(x, \cdot)$ satisfies the ε-δ criterion. Therefore, $U(x, \overline{\varepsilon})$ is closed since $U(x, \overline{\varepsilon}) = \rho^{-1}(x, [0, \varepsilon])$ and $[0, \varepsilon]$ is closed in $\mathbb{R}$.

To see it is not necessary that $U(x, \overline{\varepsilon}) = \overline{U(x, \varepsilon)}$, consider $\varepsilon = 1$ and the usual metric on

$$\left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\right\} \cup \left\{(x, 0) \in \mathbb{R}^2 : 0 \leqslant x \leqslant 1\right\};$$

see Figure 2.1. Observe that $(0, 0) \notin U(x, 1)$, but $(0, 0) \in U(x, \overline{1})$. It follows from Exercise 31 that $(0, 0) \notin \overline{U(x, 1)}$. □

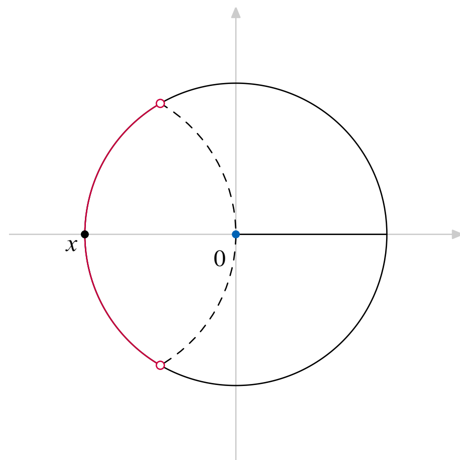

FIGURE 2.1. $U(x,\overline{1}) \neq \overline{U(x,1)}$.

3H. G_δ and F_σ Sets

► EXERCISE 33. *The complement of a G_δ is an F_σ, and vice versa.*

PROOF. If A is a G_δ set, then there exists a sequence of open sets $\{U_n\}$ such that $A = \bigcap_{n=1}^{\infty} U_n$. Then $A^c = \bigcup_{n=1}^{\infty} U_n^c$ is F_σ. Vice versa. □

► EXERCISE 34. *An F_σ can be written as the union of an* increasing *sequence $F_1 \subset F_2 \subset \cdots$ of closed sets.*

PROOF. Let $B = \bigcup_{n=1}^{\infty} E_n$, where E_n is closed for all $n \in \mathbb{N}$. Define $F_1 = E_1$ and $F_n = \bigcup_{i=1}^{n} E_i$ for $n \geqslant 2$. Then each F_n is closed, $F_1 \subset F_2 \subset \cdots$, and $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} = B$. □

► EXERCISE 35. *A closed set in a metric space is a G_δ.*

PROOF. For an arbitrary set $A \subset X$ and a point $x \in X$, define

$$\rho(x, A) = \inf_{y \in A} \{\rho(x, y)\}.$$

We first show that $\rho(\cdot, A)\colon X \to \mathbb{R}$ is (uniformly) continuous by showing

$$|\rho(x, A) - \rho(y, A)| \leqslant \rho(x, y), \quad \text{for all } x, y \in X. \tag{2.1}$$

For an arbitrary $z \in A$, we have

$$\rho(x, A) \leqslant \rho(x, z) \leqslant \rho(x, y) + \rho(y, z).$$

Take the infimum over $z \in A$ and we get

$$\rho(x, A) \leqslant \rho(x, y) + \rho(y, A). \tag{2.2}$$

Symmetrically, we have

$$\rho(y, A) \leqslant \rho(x, y) + \rho(x, A). \tag{2.3}$$

Hence, (2.1) follows from (2.2) and (2.3). We next show that if A is closed, then $\rho(x, A) = 0$ iff $x \in A$. The "if" part is trivial, so we do the "only if" part. If $\rho(x, A) = 0$, then for every $n \in \mathbb{N}$, there exists $y_n \in A$ such that $\rho(x, y_n) < 1/n$; that is, $y_n \to x$. Since $\{y_n\} \subset A$ and A is closed, we must have $x \in A$.

Therefore,

$$A = \bigcap_{n=1}^{\infty} \{x \in X : \rho(x, A) < 1/n\}.$$

The continuity of $\rho(\cdot, A)$ implies that $\{x \in X : \rho(x, A) < 1/n\}$ is open for all n. Thus, A is a G_δ set. □

► Exercise 36. *The rationals are an F_σ in $\mathbb{R}$.*

Proof. $\mathbb{Q}$ is countable, and every singleton set in $\mathbb{R}$ is closed; hence, $\mathbb{Q}$ is an F_σ. □

3I. Borel Sets

2.2 Neighborhoods

4A. The Sorgenfrey Line

► Exercise 37. *Verify that the set $[x, z)$, for $z > x$, do form a nhood base at x for a topology on the real line.*

Proof. We need only check that for each $x \in \mathbb{R}$, the family $\mathcal{B}_x := \{[x, z) : z > x\}$ satisfies V-a, V-b, and V-c in Theorem 4.5. V-a is trivial. If $[x, z_1) \in \mathcal{B}_x$ and $[x, z_2) \in \mathcal{B}_x$, then $[x, z_1) \cap [x, z_2) = [x, z_1 \wedge z_2) \in \mathcal{B}_x$ and is in $[x, z_1) \cap [x, z_2)$. For V-c, let $[x, z) \in \mathcal{B}_x$. Let $z' \in (x, z]$. Then $\left[x, z'\right) \in \mathcal{B}_x$, and if $y \in \left[x, z'\right)$, the right-open interval $\left[y, z'\right) \in \mathcal{B}_y$ and $\left[y, z'\right) \subset [x, z)$.

Then, define open sets using V-d: $G \subset \mathbb{R}$ is open if and only if G contains a set $[x, z)$ of each of its points x. □

► Exercise 38. *Which intervals on the real line are open sets in the Sorgenfrey topology?*

Solution.

- Sets of the form $(-\infty, x)$, $[x, z)$, or $[x, \infty)$ are both open and closed.
- Sets of the form (x, z) or $(x, +\infty)$ are open in $\mathbb{R}$, since

$$(x, z) = \bigcup \{[y, z) : x < y < z\}.$$ □

▶ EXERCISE 39. *Describe the closure of each of the following subset of the Sorgenfrey line: the rationals* $\mathbb{Q}$*, the set* $\{1/n : n = 1, 2, \ldots, \}$*, the set* $\{-1/n : n = 1, 2, \ldots\}$*, the integers* $\mathbb{Z}$*.*

SOLUTION. Recall that, by Theorem 4.7, for each $E \subset \mathbb{R}$, we have

$$\bar{E} = \left\{x \in \mathbb{R} : \text{each basic nhood of } x \text{ meets } E\right\}.$$

Then $\overline{\mathbb{Q}} = \mathbb{R}$ since for any $x \in \mathbb{R}$, we have $[x, z) \cap \mathbb{Q} \neq \varnothing$ for $z > x$. Similarly, $\overline{\{1/n : n = 1, 2, \ldots\}} = \{1/n : n = 1, 2, \ldots\}$, and $\overline{\mathbb{Z}} = \mathbb{Z}$. □

4B. The Moore Plane

▶ EXERCISE 40. *Verify that this gives a topology on* Γ*.*

PROOF. Verify (V-a)—(V-c). It is easy. □

4E. Topologies from nhoods

▶ EXERCISE 41. *Show that if each point* x *in a set* X *has assigned a collection* $\mathcal{U}_x$ *of subsets of* X *satisfying N-a through N-d of 4.2, then the collection*

$$\tau = \left\{G \subset X : \text{for each } x \text{ in } G, x \in U \subset G \text{ for some } U \in \mathcal{U}_x\right\}$$

is a topology for X*, in which the nhood system at each* x *is just* $\mathcal{U}_x$*.*

PROOF. We need to check G1—G3 in Definition 3.1. Since G1 and G3 are evident, we focus on G2. Let $E_1, E_2 \in \tau$. Take any $x \in E_1 \cap E_2$. Then there exist some $U_1, U_2 \in \mathcal{U}_x$ such that $x \in U_1 \subset E_1$ and $x \in U_2 \subset E_2$. By N-b, we know that $U_1 \cap U_2 \in \mathcal{U}_x$. Hence,

$$x \in U_1 \cap U_2 \subset E_1 \cap E_2,$$

and so $E_1 \cap E_2 \in \tau$. The induction principle then means that τ is closed under finite intersections. □

4F. Spaces of Functions

▶ EXERCISE 42. *For each* $f \in \mathbb{R}^{\mathbb{I}}$*, each finite subset* F *of* $\mathbb{I}$ *and each positive* δ*, let*

$$U(f, F, \delta) = \left\{g \in \mathbb{R}^{\mathbb{I}} : |g(x) - f(x)| < \delta, \text{ for each } x \in F\right\}.$$

Show that the sets $U(f, F, \delta)$ *form a nhood base at* f*, making* $\mathbb{R}^{\mathbb{I}}$ *a topological space.*

PROOF. Denote

$$\mathcal{B}_f = \left\{U(f, F, \delta) : F \subset \mathbb{I}, |F| < \infty, \delta > 0\right\}.$$

(V-a) For each $U(f, F, \delta) \in \mathcal{B}_f$, we have $|f(x) - f(x)| = 0 < \delta$ for all $x \in F$; hence, $f \in U(f, F, \delta)$.

(V-b) Let $U(f, F_1, \delta_1), U(f, F_2, \delta_2) \in \mathcal{B}_f$. Define $U(f, F_3, \delta_3)$ by letting

$$F_3 = F_1 \cup F_2, \quad \text{and} \quad \delta_3 = \min\{\delta_1, \delta_2\}.$$

Clearly, $U(f, F_3, \delta_3) \in \mathcal{B}_f$. If $g \in U(f, F_3, \delta_3)$, then

$$|g(x) - f(x)| < \min\{\delta_1, \delta_2\}, \quad \text{for all } x \in F_1 \cup F_2.$$

Hence, $|g(x) - f(x)| < \delta_1$ for all $x \in F_1$ and $|g(x) - f(x)| < \delta_2$ for all $x \in F_2$; that is, $g \in U(f, F_1, \delta_1) \cap U(f, F_2, \delta_2)$. Hence, there exists $U(f, F_3, \delta_3) \in \mathcal{B}_f$ such that $U(f, F_3, \delta_3) \subset U(f, F_1, \delta_1) \cap U(f, F_2, \delta_2)$.

(V-c) Pick $U(f, F, \delta) \in \mathcal{B}_f$. We must show that there exists some $U(f, F_0, \delta_0) \in \mathcal{B}_f$ such that if $g \in U(f, F_0, \delta_0)$, then there is some $U(g, F', \delta') \in \mathcal{B}_g$ with $U(g, F', \delta') \subset U(f, F, \delta)$.

Let $F_0 = F$, and $\delta_0 = \delta/2$. Then $U(f, F, \delta/2) \in \mathcal{B}_f$. For every $g \in U(f, F, \delta/2)$, we have

$$|g(x) - f(x)| < \delta/2, \quad \text{for all } x \in F.$$

Let $U(g, F', \delta') = U(g, F, \delta/2)$. If $h \in U(g, F, \delta/2)$, then

$$|h(x) - f(x)| < \delta/2, \quad \text{for all } x \in F.$$

Triangle inequality implies that

$$|h(x) - f(x)| \leqslant |h(x) - g(x)| + |g(x) - f(x)| < \delta/2 + \delta/2 = \delta, \quad \text{for all } x \in F;$$

that is, $h \in U(f, F, \delta)$. Hence, $U(g, F, \delta/2) \subset U(f, F, \delta)$.

Now, $G \subset \mathbb{R}^{\mathbb{I}}$ is open iff G is contains a $U(f, F, \delta)$ of each $f \in G$. This defines a topology on $\mathbb{R}^{\mathbb{I}}$. □

► EXERCISE 43. *For each $f \in \mathbb{R}^{\mathbb{I}}$, the closure of the one-point set $\{f\}$ is just $\{f\}$.*

PROOF. For every $g \in \mathbb{R}^{\mathbb{I}} \smallsetminus \{f\}$, pick $x \in \mathbb{I}$ with $g(x) \neq f(x)$. Define $U(g, F, \delta)$ with $F = \{x\}$ and $\delta < |g(x) - f(x)|$. Then $f \notin U(g, \{x\}, \delta)$; that is, $U(g, \{x\}, \delta) \in \mathbb{R}^{\mathbb{I}} \smallsetminus \{f\}$. Hence, $\mathbb{R}^{\mathbb{I}} \smallsetminus \{f\}$ is open, and so $\{f\}$ is closed. This proves that $\overline{\{f\}} = \{f\}$. □

► EXERCISE 44. *For $f \in \mathbb{R}^{\mathbb{I}}$ and $\varepsilon > 0$, let*

$$V(f, \varepsilon) = \left\{g \in \mathbb{R}^{\mathbb{I}} : |g(x) - f(x)| < \varepsilon, \text{ for each } x \in \mathbb{I}\right\}.$$

Verify that the sets $V(f, \varepsilon)$ form a nhood base at f, making $\mathbb{R}^{\mathbb{I}}$ a topological space.

PROOF. Denote $\mathcal{V}_f = \{V(f,\varepsilon) : \varepsilon > 0\}$. We verify the following properties.

(V-a) If $V(f,\varepsilon) \in \mathcal{V}_f$, then $|f(x) - f(x)| = 0 < \varepsilon$; that is, $f \in V(f,\varepsilon)$.

(V-b) Let $V(f,\varepsilon_1), V(f,\varepsilon_2) \in \mathcal{V}_f$. Let $\varepsilon_3 = \min\{\varepsilon_1,\varepsilon_2\}$. If $g \in V(f,\varepsilon_3)$, then

$$|g(x) - f(x)| < \varepsilon_3 = \min\{\varepsilon_1,\varepsilon_2\}, \quad \text{for all } x \in \mathbb{I}.$$

Hence, $V(f,\varepsilon_3) \subset V(f,\varepsilon_1) \cap V(f,\varepsilon_2)$.

(V-c) For an arbitrary $V(f,\varepsilon) \in \mathcal{V}_f$, pick $V(f,\varepsilon/2) \in \mathcal{V}_f$. For each $g \in V(f,\varepsilon/2)$, pick $V(g,\varepsilon/2) \in \mathcal{V}_g$. If $h \in V(g,\varepsilon/2)$, then $|h(x) - g(x)| < \varepsilon/2$ for all $x \in \mathbb{I}$. Hence

$$|h(x) - f(x)| \leqslant |h(x) - g(x)| + |g(x) - f(x)| < \varepsilon;$$

that is, $V(g,\varepsilon/2) \subset V(f,\varepsilon)$. □

► EXERCISE 45. *Compare the topologies defined in 1 and 3.*

PROOF. It is evident that for every $U(f,F,\delta) \in \mathcal{B}_f$, there exists $V(f,\delta) \in \mathcal{V}_f$ such that $V(f,\delta) \subset U(f,F,\delta)$. Hence, the topology in 1 is weaker than in 3 by Hausdorff criterion. □

2.3 Bases and Subbases

5D. No Axioms for Subbase

► EXERCISE 46. *Any family of subsets of a set X is a subbase for some topology on X and the topology which results is the smallest topology containing the given collection of sets.*

PROOF. Let $\mathcal{S}$ be a family of subsets of X. Let $\tau(\mathcal{S})$ be the intersection of all topologies containing $\mathcal{S}$. Such topologies exist, since 2^X is one such. Also $\tau(\mathcal{S})$ is a topology. It evidently satisfies the requirements "unique" and "smallest."

The topology $\tau(\mathcal{S})$ can be described as follows: It consists of $\varnothing$, X, all finite intersections of the $\mathcal{S}$-sets, and all arbitrary unions of these finite intersections. To verify this, note that since $\mathcal{S} \subset \tau(\mathcal{S})$, then $\tau(\mathcal{S})$ must contain all the sets listed. Conversely, because $\bigcup$ distributes over $\bigcap$, the sets listed actually do from a topology containing $\mathcal{S}$, and which therefore contains $\tau(\mathcal{S})$. □

5E. Bases for the Closed Sets

► EXERCISE 47. *$\mathcal{F}$ is a base for the closed sets in X iff the family of complements of members of $\mathcal{F}$ is a base for the open sets.*

PROOF. Let G be an open set in X. Then $G = X \smallsetminus E$ for some closed subset E. Since $E = \bigcap_{F \in \mathcal{G} \subset \mathcal{F}} F$, we obtain

$$G = X \smallsetminus \left(\bigcap_{F \in \mathcal{G} \subset \mathcal{F}} F \right) = \bigcup_{F \in \mathcal{G} \subset \mathcal{F}} F^c.$$

Thus, $\{F^c : F \in \mathcal{F}\}$ forms a base for the open sets. The converse direction is similar. □

► EXERCISE 48. *$\mathcal{F}$ is a base for the closed sets for some topology on X iff (a) whenever F_1 and F_2 belong to $\mathcal{F}$, $F_1 \cup F_2$ is an intersection of elements of $\mathcal{F}$, and (b) $\bigcap_{F \in \mathcal{F}} F = \varnothing$.*

PROOF. If $\mathcal{F}$ is a base for the closed sets for some topology on X, then (a) and (b) are clear. Suppose, on the other hand, X is a set and $\mathcal{F}$ a collection of subsets of X with (a) and (b). Let $\mathcal{T}$ be all intersections of subcollections from $\mathcal{F}$. Then any intersection of members of $\mathcal{T}$ certainly belongs to $\mathcal{T}$, so $\mathcal{T}$ satisfies (F-a). Moreover, if $\mathcal{F}_1 \subset \mathcal{F}$ and $\mathcal{F}_2 \subset \mathcal{F}$, so that $\bigcap_{E \in \mathcal{F}_1} E$ and $\bigcap_{F \in \mathcal{F}_2} F$ are elements of $\mathcal{T}$, then

$$\left(\bigcap_{E \in \mathcal{F}_1} E \right) \cup \left(\bigcap_{F \in \mathcal{F}_2} F \right) = \bigcap_{E \in \mathcal{F}_1} \bigcap_{F \in \mathcal{F}_2} (E \cup F).$$

But by property (a), the union of two elements of $\mathcal{F}$ is an intersection of elements of $\mathcal{F}$, so $(\bigcap_{E \in \mathcal{F}_1} E) \cup (\bigcap_{F \in \mathcal{F}_2} F)$ is an intersection of elements of $\mathcal{F}$, and hence belongs to $\mathcal{T}$. Thus $\mathcal{T}$ satisfies (F-b). Finally, $\varnothing \in \mathcal{T}$ by (b) and $X \in \mathcal{T}$ since X is the intersection of the empty subcollection from $\mathcal{F}$. Hence $\mathcal{T}$ satisfies (F-c). This completes the proof that $\mathcal{T}$ is the collection of closed sets of X. □

3

NEW SPACES FROM OLD

3.1 Subspaces

3.2 Continuous Functions

7A. Characterization of Spaces Using Functions

▶ Exercise 49. *The characteristic function of A is continuous iff A is both open and closed in X.*

Proof. Let $\mathbb{1}_A \colon X \to \mathbb{R}$ be the characteristic function of A, which is defined by

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

First suppose that $\mathbb{1}_A$ is continuous. Then, say, $\mathbb{1}_A^{-1}\big((1/2, 2)\big) = A$ is open, and $\mathbb{1}_A^{-1}\big((-1, 1/2)\big) = X \smallsetminus A$ is open. Hence, A is both open and closed in X.

Conversely, suppose that A is both open and closed in X. For any open set $U \subset \mathbb{R}$, we have

$$\mathbb{1}_A^{-1}(U) = \begin{cases} A & \text{if } 1 \in U \text{ and } 0 \notin U \\ X \smallsetminus A & \text{if } 1 \notin U \text{ and } 0 \in U \\ \varnothing & \text{if } 1 \notin U \text{ and } 0 \notin U \\ X & \text{if } 1 \in U \text{ and } 0 \in U. \end{cases}$$

Then $\mathbb{1}_A$ is continuous. □

▶ Exercise 50. *X has the discrete topology iff whenever Y is a topological space and $f \colon X \to Y$, then f is continuous.*

Proof. Let Y be a topological space and $f \colon X \to Y$. It is easy to see that f is continuous if X has the discrete topology, so we focus on the sufficiency direction. For any $A \subset X$, let $Y = \mathbb{R}$ and $f = \mathbb{1}_A$. Then by Exercise 49 A is open. □

7C. Functions Agreeing on A Dense Subset

► EXERCISE 51. *If f and g are continuous functions from X to $\mathbb{R}$, the set of points x for which $f(x) = g(x)$ is a closed subset of X. Thus two continuous maps on X to $\mathbb{R}$ which agree on a dense subset must agree on all of X.*

PROOF. Denote $A = \{x \in X : f(x) \neq g(x)\}$. Take a point $y \in A$ such that $f(y) > g(y)$ (if it is not true then let $g(y) > f(y)$). Take an $\varepsilon > 0$ such that $f(y) - \varepsilon \geqslant g(y) + \varepsilon$. Since f and g are continuous, there exist nhoods U_1 and U_2 of y such that $f[U_1] \subset (-\varepsilon + f(y), \varepsilon + f(y))$ and $g[U_2] \subset (-\varepsilon + g(y), \varepsilon + g(y))$. Let $U = U_1 \cap U_2$. Then U is a nhood of x and for every $z \in U$ we have

$$f(z) - g(z) > [f(x) - \varepsilon] - [g(x) + \varepsilon] \geqslant 0.$$

Hence, $U \subset A$; that is, U is open, and so $\{x \in X : f(x) = g(x)\} = X \smallsetminus U$ is closed.

Now suppose that $D := \{x \in X : f(x) = g(x)\}$ is dense. Take an arbitrary $x \in X$. Since f and g are continuous, for each $n \in \mathbb{N}$, there exist nhoods V_f and V_g such that $|f(y) - f(x)| < 1/n$ for all $y \in V_f$ and $|g(y) - g(x)| < 1/n$ for all $y \in V_g$. Let $V_n = V_f \cap V_g$. Then there exists $x_n \in V_n \cap D$ with $|f(x_n) - f(x)| < 1/2n$ and $|g(x_n) - g(x)| < 1/2n$. Since $f(x_n) = g(x_n)$, we have

$$\begin{aligned}|f(x) - g(x)| \leqslant |f(x) - f(x_n)| + |f(x_n) - g(x)| &= |f(x) - f(x_n)| + |g(x_n) - g(x)| \\ &< 1/n.\end{aligned}$$

Therefore, $f(x) = g(x)$. □

7E. Range Immaterial

► EXERCISE 52. *If $Y \subset Z$ and $f : X \to Y$, then f is continuous as a map from X to Y iff f is continuous as a map from X to Z.*

PROOF. Let $f : X \to Z$ be continuous. Let U be open in Y. Then $U = Y \cap V$ for some V which is open in Z. Therefore,

$$f^{-1}(U) = f^{-1}(Y \cap V) = f^{-1}(Y) \cap f^{-1}(V) = X \cap f^{-1}(V) = f^{-1}(V)$$

is open in X, and so f is continuous as a map from X to Y.

Conversely, let $f : X \to Y$ be continuous and V be open in Z. Then $f^{-1}(V) = f^{-1}(Y \cap V)$. Since $Y \cap V$ is open in Y and f is continuous from X to Y, the set $f^{-1}(Y \cap V)$ is open in X and so f is continuous as a map from X to Z. □

7G. Homeomorphisms within the Line

► EXERCISE 53. *Show that all open intervals in* $\mathbb{R}$ *are homeomorphic.*

PROOF. We have

- $(a,b) \sim (0,1)$ by $f_1(x) = (x-a)/(b-a)$.
- $(a,\infty) \sim (1,\infty)$ by $f_2(x) = x - a + 1$.
- $(1,\infty) \sim (0,1)$ by $f_3(x) = 1/x$.
- $(-\infty,-a) \sim (a,\infty)$ by $f_4(x) = -x$.
- $(-\infty,\infty) \sim (-\pi/2,\pi/2)$ by $f_5(x) = \arctan x$.

Therefore, by compositing, every open interval is homeomorphic to $(0,1)$. □

► EXERCISE 54. *All bounded closed intervals in* $\mathbb{R}$ *are homeomorphic.*

PROOF. $[a,b] \sim [0,1]$ by $f(x) = (x-a)/(b-a)$. □

► EXERCISE 55. *The property that every real-valued continuous function on X assumes its maximum is a topological property. Thus,* $\mathbb{I} := [0,1]$ *is not homeomorphic to* $\mathbb{R}$.

PROOF. Every continuous function assumes its maximum on $[0,1]$; however, x^2 has no maximum on $\mathbb{R}$. Therefore, $\mathbb{I} \not\sim \mathbb{R}$. □

7K. Semicontinuous Functions

► EXERCISE 56. *If f_α is a lower semicontinuous real-valued function on X for each $\alpha \in A$, and if $\sup_\alpha f_\alpha(x)$ exists at each $x \in X$, then the function $f(x) = \sup_\alpha f_\alpha(x)$ is lower semicontinuous on X.*

PROOF. For an arbitrary $a \in \mathbb{R}$, we have $f(x) \leqslant a$ iff $f_\alpha(x) \leqslant a$ for all $\alpha \in A$. Hence,

$$\{x \in X : f(x) \leqslant \alpha\} = \bigcap_{\alpha \in A} \{x \in X : f_\alpha(x) \leqslant a\},$$

and so $f^{-1}(-\infty,a]$ is closed; that is, f is lower semicontinuous. □

► EXERCISE 57. *Every continuous function from X to $\mathbb{R}$ is lower semicontinuous. Thus the supremum of a family of continuous functions, if it exists, is lower semicontinuous. Show by an example that "lower semicontinuous" cannot be replaced by "continuous" in the previous sentence.*

PROOF. Suppose that $f: X \to \mathbb{R}$ is continuous. Since $(-\infty,x]$ is closed in $\mathbb{R}$, the set $f^{-1}(-\infty,x]$ is closed in X; that is, f is lower semicontinuous.

To construct an example, let $f: [0,\infty) \to \mathbb{R}$ be defined as follows:

$$f_n(x) = \begin{cases} nx & \text{if } 0 \leqslant x \leqslant 1/n \\ 1 & \text{if } x > 1/n. \end{cases}$$

Then

$$f(x) = \sup_n f_n(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0, \end{cases}$$

and f is not continuous. □

► Exercise 58. *The characteristic function of a set A in X is lower semicontinuous iff A is open, upper semicontinuous iff A is closed.*

Proof. Observe that

$$\mathbb{1}_A^{-1}(-\infty, a] = \begin{cases} \varnothing & \text{if } a < 0 \\ X \smallsetminus A & \text{if } 0 \leqslant a < 1 \\ X & \text{if } a \geqslant 1. \end{cases}$$

Therefore, $\mathbb{1}_A$ is LSC iff A is open. Similarly for the USC case. □

► Exercise 59. *If X is metrizable and f is a lower semicontinuous function from X to $\mathbb{I}$, then f is the supremum of an increasing sequence of continuous functions on X to $\mathbb{I}$.*

Proof. Let d be the metric on X. First assume f is nonnegative. Define

$$f_n(x) = \inf_{z \in X} \{f(z) + nd(x,z)\}.$$

If $x, y \in X$, then $f(z) + nd(x,z) \leqslant f(z) + nd(y,z) + nd(x,y)$. Take the inf over z (first on the left side, then on the right side) to obtain $f_n(x) \leqslant f_n(y) + nd(x,y)$. By symmetry,

$$|f_n(x) - f_n(y)| \leqslant nd(x,y);$$

hence, f_n is uniformly continuous on X. Furthermore, since $f \geqslant 0$, we have $0 \leqslant f_n(x) \leqslant f(x) + nd(x,x) = f(x)$. By definition, f_n increases with n; we must show that $\lim_n f_n$ is actually f.

Given $\varepsilon > 0$, by definition of $f_n(x)$ there is a point $z_n \in X$ such that

$$f_n(x) + \varepsilon > f(z_n) + nd(x,z_n) \geqslant nd(x,z_n) \tag{3.1}$$

since $f \geqslant 0$. But $f_n(x) + \varepsilon \leqslant f(x) + \varepsilon$; hence $d(x,z_n) \to 0$. Since f is LSC, we have $\liminf_n f(z_n) \geqslant f(x)$ (Ash, 2009, Theorem 8.4.2); hence

$$f(z_n) > f(x) - \varepsilon \quad \text{ev.} \tag{3.2}$$

By (3.1) and (3.2),

$$f_n(x) > f(z_n) - \varepsilon + nd(x,z_n) \geqslant f(z_n) - \varepsilon > f(x) - 2\varepsilon$$

for all sufficiently large n. Thus, $f_n(x) \to f(x)$.

If $|f| \leqslant M < \infty$, then $f + M$ is LSC, finite-valued, and nonnegative. If $0 \leqslant g_n \uparrow (f + M)$, then $f_n = (g_n - M) \uparrow f$ and $|f_n| \geqslant M$. □

7M. $C(X)$ and $C^*(X)$

► EXERCISE 60. *If f and g belong to $C(X)$, then so do $f + g$, $f \cdot g$ and $a \cdot f$, for $a \in \mathbb{R}$. If, in addition, f and g are bounded, then so are $f + g$, $f \cdot g$ and $a \cdot f$.*

PROOF. We first do $f + g$. Since $f, g \in C(X)$, for each $x \in X$ and each $\varepsilon > 0$, there exist nhoods U_1 and U_2 of x such that $f[U_1] \subset (-\varepsilon/2 + f(x), \varepsilon/2 + f(x))$ and $g[U_2] \subset (-\varepsilon/2 + g(x), \varepsilon/2 + g(x))$. Let $U = U_1 \cap U_2$. Then U is a nhood of x, and for every $y \in U$, we have

$$|[f(y) + g(y)] - [f(x) + g(x)]| \leqslant |f(y) - f(x)| + |g(y) - g(x)| < \varepsilon;$$

that is, $f + g$ is continuous.

We then do $a \cdot f$. We suppose that $a > 0$ (all other cases are similar). For each $x \in X$ and $\varepsilon > 0$, there exists a nhood U of x such that $f[U] \subset (-\varepsilon/a + f(x), \varepsilon/a + f(x))$. Then $(a \cdot f)[U] \in (-\varepsilon + a \cdot f(x), \varepsilon + a \cdot f(x))$. So $a \cdot f \in C(X)$.

Finally, to do $f \cdot g$, we first show that $f^2 \in C(X)$ whenever $f \in C(X)$. For each $x \in X$ and $\varepsilon > 0$, there is a nhood U of x such that $f[U] \subset (-\sqrt{\varepsilon} + f(x), \sqrt{\varepsilon} + f(x))$. Then $f^2[U] \subset (-\varepsilon + f^2(x), \varepsilon + f^2(x))$, i.e., $f^2 \in C(X)$. Since

$$f(x) \cdot g(x) = \frac{1}{4}\left[\big(f(x) + g(x)\big)^2 - \big(f(x) - g(x)\big)^2\right],$$

we know that $f \cdot g \in C(X)$ from the previous arguments. □

► EXERCISE 61. *$C(X)$ and $C^*(X)$ are algebras over the real numbers.*

PROOF. It follows from the previous exercise that $C(X)$ is a vector space on $\mathbb{R}$. So everything is easy now. □

► EXERCISE 62. *$C^*(X)$ is a normed linear space with the operations of addition and scalar multiplication given above and the norm $\|f\| = \sup_{x \in X} |f(x)|$.*

PROOF. It is easy to see that $C^*(X)$ is a linear space. So it suffices to show that $\|\cdot\|$ is a norm on $C^*(X)$. We focus on the triangle inequality. Let $f, g \in C^*(X)$. Then for every $x \in X$, we have $|f(x) + g(x)| \leqslant |f(x)| + |g(x)| \leqslant \|f\| + \|g\|$; hence, $\|f + g\| \leqslant \|f\| + \|g\|$. □

3.3 Product Spaces, Weak Topologies

8A. Projection Maps

▶ Exercise 63. *The βth projection map π_β is continuous and open. The projection $\pi_1\colon \mathbb{R}^2 \to \mathbb{R}$ is not closed.*

Proof. Let U_β be open in X_β. Then $\pi_\beta^{-1}(U_\beta)$ is a subbasis open set of the Tychonoff topology on $\bigtimes_\alpha X_\alpha$, and so is open. Hence, π_β is continuous.

Take an arbitrary basis open set U in the Tychonoff topology. Denote $I := \{1,\dots,n\}$. Then

$$U = \bigtimes_\alpha U_\alpha,$$

where U_α is open in X_α for every $\alpha \in A$, and $U_{\alpha_j} = X_{\alpha_j}$ for all $j \notin I$. Hence,

$$\pi_\beta(U) = \begin{cases} U_\beta & \text{if } \beta = \alpha_i \text{ for some } i \in I \\ X_\beta & \text{otherwise.} \end{cases}$$

That is, $\pi_\beta(U)$ is open in X_β in both case. Since any open set is a union of basis open sets, and since functions preserve unions, the image of any open set under π_β is open.

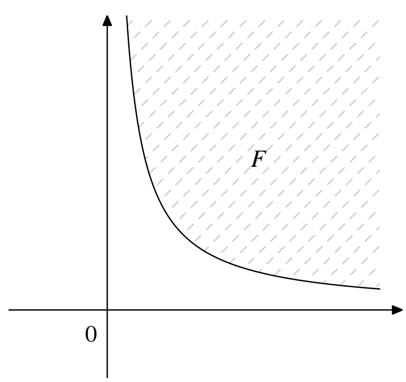

Figure 3.1. $\pi_1(F) = (0,\infty)$

Finally, let $F = \operatorname{epi}(1/x)$. Then F is closed in $\mathbb{R}^2$, but $\pi_1(F) = (0,\infty)$ is open in $\mathbb{R}$; that is, π_1 is not closed. See Figure 3.1. □

▶ Exercise 64. *Show that the projection of $\mathbb{I} \times \mathbb{R}$ onto $\mathbb{R}$ is a closed map.*

Proof. Let $\pi\colon \mathbb{I} \times \mathbb{R} \to \mathbb{R}$ be the projection. Suppose $A \subset \mathbb{I} \times \mathbb{R}$ is closed, and suppose $y_0 \in \mathbb{R} \smallsetminus \pi[A]$. For every $x \in \mathbb{I}$, since $(x, y_0) \notin A$ and A is closed, we find a basis open subset $U(x) \times V(x)$ of $\mathbb{I} \times \mathbb{R}$ that contains (x, y_0), and $[U(x) \times V(x)] \cap A = \varnothing$. The collection $\{U(x) : x \in \mathbb{I}\}$ covers $\mathbb{I}$, so finitely many of them cover $\mathbb{I}$ by compactness, say $U(x_1),\dots,U(x_n)$ do. Now define $V =$

$\bigcap_{i=1}^{n} V(x_i)$, and note that V is an open nhood of y_0, and $V \cap \pi[A] = \varnothing$. So $\pi[A]$ is closed; that is, π is closed. See Lee (2011, Lemma 4.35, p. 95) for the Tube Lemma.

Generally, if $\pi\colon X \times Y \to X$ is a projection may where Y is compact, then π is a closed map. □

8B. Separating Points from Closed Sets

► EXERCISE 65. *If f_α is a map (continuous function) of X to X_α for each $\alpha \in A$, then $\{f_\alpha : \alpha \in A\}$ separates points from closed sets in X iff $\{f_\alpha^{-1}[V] : \alpha \in A, V$ open in $X_\alpha\}$ is a base for the topology on X.*

PROOF. Suppose that $\{f_\alpha^{-1}[V] : \alpha \in A, V$ open in $X_\alpha\}$ consists of a base for the topology on X. Let B be closed in X and $x \notin B$. Then $x \in X \smallsetminus B$ and $X \smallsetminus B$ is open in X. Hence there exists $f_\alpha^{-1}[V]$ such that $x \in f_\alpha^{-1}[V] \subset X \smallsetminus B$; that is, $f_\alpha(x) \in V$. Since $V \cap f_\alpha[B] = \varnothing$, i.e., $f_\alpha[B] \subset X_\alpha \smallsetminus V$, and $X_\alpha \smallsetminus V$ is closed, we get $\overline{f_\alpha[B]} \subset X_\alpha \smallsetminus V$. Thus, $f_\alpha(x) \notin \overline{f_\alpha[B]}$.

Next assume that $\{f_\alpha : \alpha \in A\}$ separates points from closed sets in X. Take an arbitrary open subset $U \subset X$ and $x \in U$. Then $B := X \smallsetminus U$ is closed in X, and hence there exists $\alpha \in A$ such that $f_\alpha(x) \notin \overline{f_\alpha[B]}$. Then $f_\alpha(x) \in X_\alpha \smallsetminus \overline{f_\alpha[B]}$ and, since $X_\alpha \smallsetminus \overline{f_\alpha[B]}$ is open in X_α, there exists an open set V of X_α such that $f_\alpha(x) \in V \subset X_\alpha \smallsetminus \overline{f_\alpha[B]}$. Therefore,

$$\begin{aligned} x \in f_\alpha^{-1}[V] \subset f_\alpha^{-1}\left[X_\alpha \smallsetminus \overline{f_\alpha[B]}\right] &= X \smallsetminus f_\alpha^{-1}\left[\overline{f_\alpha[B]}\right] \\ &\subset X \smallsetminus f_\alpha^{-1}[f_\alpha[B]] \\ &\subset X \smallsetminus B \\ &= U. \end{aligned}$$

Hence, $\{f_\alpha^{-1}[V] : \alpha \in A, V$ open in $X_\alpha\}$ is a base for the topology on X. □

8D. Closure and Interior in Products

Let X and Y be topological spaces containing subsets A and B, respectively. In the product space $X \times Y$:

► EXERCISE 66. $(A \times B)^\circ = A^\circ \times B^\circ$.

PROOF. Since $A^\circ \subset A$ is open in A and $B^\circ \subset B$ is open in B, the set $A^\circ \times B^\circ \subset A \times B$ is open in $A \times B$; hence, $A^\circ \times B^\circ \subset (A \times B)^\circ$.

For the converse inclusion, let $\boldsymbol{x} = (a, b) \in (A \times B)^\circ$. Then there is an basis open set $U_1 \times U_2$ such that $\boldsymbol{x} \in U_1 \times U_2 \subset A \times B$, where U_1 is open in A and U_2 is open in B. Hence, $a \in U_1 \subset A$ and $b \in U_2 \subset B$; that is, $a \in A^\circ$ and $b \in B^\circ$. Then $\boldsymbol{x} \in A^\circ \times B^\circ$. □

► EXERCISE 67. $\overline{A \times B} = \overline{A} \times \overline{B}$.

PROOF. See Exercise 68. □

► EXERCISE 68. *Part 2 can be extended to infinite products, while part 1 can be extended only to finite products.*

PROOF. Assume that $\boldsymbol{y} = (y_\alpha) \in \overline{\bigtimes A_\alpha}$; we show that $y_\alpha \in \overline{A}_\alpha$ for each α; that is, $\boldsymbol{y} \in \bigtimes \overline{A}_\alpha$. Let $y_\alpha \in U_\alpha$, where U_α is open in Y_α; since $\boldsymbol{y} \in \pi_\alpha^{-1}(U_\alpha)$, we must have

$$\varnothing \neq \pi_\alpha^{-1}(U_\alpha) \cap \bigtimes A_\alpha = (U_\alpha \cap A_\alpha) \times \left(\bigtimes_{\beta \neq \alpha} A_\beta \right),$$

and so $U_\alpha \cap A_\alpha \neq \varnothing$. This proves $y_\alpha \in \overline{A}_\alpha$. The converse inclusion is established by reversing these steps: If $\boldsymbol{y} \in \bigtimes \overline{A}_\alpha$, then for any open nhood

$$B := U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \left(\bigtimes \left\{ Y_\beta : \beta \neq \alpha_1, \ldots, \alpha_n \right\} \right),$$

each $U_{\alpha_i} \cap A_{\alpha_i} \neq \varnothing$ so that $B \cap \bigtimes A_\alpha \neq \varnothing$. □

► EXERCISE 69. $\operatorname{Fr}(A \times B) = [\overline{A} \times \operatorname{Fr}(B)] \cup [\operatorname{Fr}(A) \times \overline{B}]$.

PROOF. We have

$$\begin{aligned}
\operatorname{Fr}(A \times B) &= \overline{A \times B} \cap \overline{(X \times Y) \smallsetminus (A \times B)} \\
&= (\overline{A} \times \overline{B}) \cap \left[(X \times Y) \smallsetminus (A^\circ \times B^\circ) \right] \\
&= (\overline{A} \times \overline{B}) \cap \Big[\big(X \times (Y \smallsetminus B^\circ) \big) \cup \big((X \smallsetminus A^\circ) \times Y \big) \Big] \\
&= [\overline{A} \times \operatorname{Fr}(B)] \cup [\operatorname{Fr}(A) \times \overline{B}].
\end{aligned}$$ □

► EXERCISE 70. *If X_α is a nonempty topological space and $A_\alpha \subset X_\alpha$, for each $\alpha \in A$, then $\bigtimes A_\alpha$ is dense in $\bigtimes X_\alpha$ iff A_α is dense in X_α, for each α.*

PROOF. It follows from Exercise 68 that

$$\overline{\bigtimes A_\alpha} = \bigtimes \overline{A}_\alpha;$$

that is, $\bigtimes A_\alpha$ is dense in $\bigtimes X_\alpha$ iff A_α is dense in X_α, for each α. □

8E. Miscellaneous Facts about Product Spaces

Let X_α be a nonempty topological space for each $\alpha \in A$, and let $X = \bigtimes X_\alpha$.

► EXERCISE 71. *If V is a nonempty open set in X, then $\pi_\alpha(V) = X_\alpha$ for all but finitely many $\alpha \in A$.*

PROOF. Let $\mathcal{T}_\alpha$ be the topology on X_α for each $\alpha \in A$. Let V be an arbitrary open set in X. Then $V = \bigcup_{k \in K} B_k$, where for each $k \in K$ we have $B_k = \bigtimes_{\alpha \in A} E_{\alpha k}$, and for each $\alpha \in A$ we have $E_{\alpha k} \in \mathcal{T}_\alpha$ while

$$A_k := \{\alpha \in A : E_{\alpha k} \neq X_\alpha\}$$

is finite. Then $\bigcap_{k \in K} A_k$ is finite. If $\alpha_0 \notin \bigcap_{k \in K} A_k$, then there exists $k_0 \in K$ such that $E_{\alpha_0 k_0} = X_{\alpha_0}$. Then

$$\pi_{\alpha_0}^{-1}(B_{k_0}) = \pi_{\alpha_0}^{-1}\left(\bigtimes_{\alpha \in A} E_{\alpha k_0}\right) = X_{\alpha_0},$$

and so $X_{\alpha_0} = \pi_{\alpha_0}^{-1}(B_{k_0}) \subset \pi_{\alpha_0}^{-1}(V)$ implies that $\pi_{\alpha_0}^{-1}(V) = X_{\alpha_0}$. □

► EXERCISE 72. *If b_α is a fixed point in X_α, for each $\alpha \in A$, then $X'_{\alpha_0} = \{x \in X : x_\alpha = b_\alpha$ whenever $\alpha \neq \alpha_0\}$ is homeomorphic to X_{α_0}.*

PROOF. Write an element in X'_{α_0} as $(x_{\alpha_0}, \boldsymbol{b}_{-\alpha_0})$. Then consider the mapping $(x_{\alpha_0}, \boldsymbol{b}_{-\alpha_0}) \mapsto x_{\alpha_0}$. □

8G. The Box Topology

Let X_α be a topological space for each $\alpha \in A$.

► EXERCISE 73. *In $\bigtimes X_\alpha$, the sets of the form $\bigtimes U_\alpha$, where U_α is open in X_α for each $\alpha \in A$, form a base for a topology.*

PROOF. Let $\mathcal{B} := \left\{\bigtimes U_\alpha : \alpha \in A, U_\alpha \text{ open in } X_\alpha\right\}$. Then it is clear that $\bigtimes X_\alpha \in \mathcal{B}$ since X_α is open for each $\alpha \in A$. Now take any $B_1, B_2 \in \mathcal{B}$, with $B_1 = \bigtimes U_\alpha^1$ and $B_2 = \bigtimes U_\alpha^2$. Let

$$p = \left(p_1, p_2, \dots\right) \in B_1 \cap B_2 = \bigtimes \left(U_\alpha^1 \cap U_\alpha^2\right).$$

Then $p_\alpha \in U_\alpha^1 \cap U_\alpha^2$, and so there exists an open set $B_\alpha \subset X_\alpha$ such that $p_\alpha \in B_\alpha \subset U_\alpha^1 \cap U_\alpha^2$. Hence, $\bigtimes B_\alpha \in \mathcal{B}$ and $p \in B \subset B_1 \cap B_2$. □

8H. Weak Topologies on Subspaces

Let X have the weak topology induced by a collection of maps $f_\alpha \colon X \to X_\alpha$, for $\alpha \in A$.

► EXERCISE 74. *If each X_α has the weak topology given by a collection of maps $g_{\alpha\lambda} \colon X_\alpha \to Y_{\alpha\lambda}$, for $\lambda \in \Lambda_\alpha$, then X has the weak topology given by the maps $g_{\alpha\lambda} \circ f_\alpha \colon X \to Y_{\alpha\lambda}$ for $\alpha \in A$ and $\lambda \in \Lambda_\alpha$.*

PROOF. A subbase for the weak topology on X_α induced by $\{g_{\alpha\lambda} : \lambda \in \Lambda_\alpha\}$ is

$$\left\{g_{\alpha\lambda}^{-1}(U_{\alpha\lambda}) : \lambda \in \Lambda_\alpha, U_{\alpha\lambda} \text{ open in } Y_{\alpha\lambda}\right\}.$$

Then a subbasic open set in X for the weak topology on X induced by $\{f_\alpha : \alpha \in A\}$ is

$$\left\{ f_\alpha^{-1}[g_{\alpha\lambda}^{-1}(U_{\alpha\lambda})] : \alpha \in A, \lambda \in \Lambda_\alpha, U_{\alpha\lambda} \text{ open in } Y_{\alpha\lambda} \right\}.$$

Since $f_\alpha^{-1}(g_{\alpha\lambda}^{-1}(U_{\alpha\lambda})) = (g_{\alpha\lambda} \circ f_\alpha)^{-1}(U_{\alpha\lambda})$, we get the result. $\square$

► EXERCISE 75. *Any $B \subset X$ has the weak topology induced by the maps $f_\alpha \restriction B$.*

PROOF. As a subspace of X, the subbase on B is

$$\left\{ B \cap f_\alpha^{-1}(U_\alpha) : \alpha \in A, U_\alpha \text{ open in } X_\alpha \right\}.$$

On the other hand, $(f_\alpha \restriction B)^{-1}(U_\alpha) = B \cap f_\alpha^{-1}(U_\alpha)$ for every $\alpha \in A$ and U_α open in X_α. Hence, the above set is also the subbase for the weak topology induced by $\{f_\alpha \restriction B : \alpha \in A\}$. $\square$

3.4 QUOTIENT SPACES

9B. Quotients versus Decompositions

► EXERCISE 76. *The process given in 9.5 for forming the topology on a decomposition space does define a topology.*

PROOF. Let $(X, \mathcal{T})$ be a topological space; let $\mathcal{D}$ be a decomposition of X. Define

$$\mathcal{F} \subset \mathcal{D} \text{ is open in } \mathcal{D} \iff \bigcup\{F : F \in \mathcal{F}\} \text{ is open in } X. \tag{3.3}$$

Let $\mathfrak{T}$ be the collection of open sets defined by (3.3). We show that $(\mathcal{D}, \mathfrak{T})$ is a topological space.

- Take an arbitrary collection $\{\mathcal{F}_i\}_{i \in I} \subset \mathfrak{T}$; then $\bigcup\{F : F \in \mathcal{F}_i\}$ is open in X for each $i \in I$. Hence, $\bigcup_{i \in I} \mathcal{F}_i \in \mathfrak{T}$ since

$$\bigcup_{F \in \bigcup_{i \in I} \mathcal{F}_i} F = \bigcup_{i \in I} \left(\bigcup_{F \in \mathcal{F}_i} F \right)$$

  is open in X.

- Let $\mathcal{F}_1, \mathcal{F}_2 \in \mathfrak{T}$; then $\bigcup_{E \in \mathcal{F}_1} E$ and $\bigcup_{F \in \mathcal{F}_2} F$ are open in X. Therefore, $\mathcal{F}_1 \cap \mathcal{F}_2 \in \mathfrak{T}$ since

$$\bigcup_{F \in \mathcal{F}_1 \cap \mathcal{F}_2} F = \left(\bigcup_{E \in \mathcal{F}_1} E \right) \cap \left(\bigcup_{F \in \mathcal{F}_2} F \right)$$

  is open in X.

- $\varnothing \in \mathfrak{T}$ since $\bigcup \varnothing = \varnothing$ is open in X; finally, $\mathcal{D} \in \mathfrak{T}$ since $\bigcup \mathcal{D} = X$. $\square$

▶ EXERCISE 77. *The topology on a decomposition space $\mathscr{D}$ of X is the quotient topology induced by the natural map $P\colon X \to \mathscr{D}$. (See 9.6.)*

PROOF. Let $\mathfrak{T}$ be the decomposition topology of $\mathscr{D}$, and let $\mathfrak{T}_P$ be the quotient topology induced by P. Take an open set $\mathscr{F} \in \mathfrak{T}$; then $\bigcup_{F\in\mathscr{F}} F$ is open in X. Hence,

$$P^{-1}(\mathscr{F}) = P^{-1}\left(\bigcup_{F\in\mathscr{F}} F\right) = \bigcup_{F\in\mathscr{F}} P^{-1}(F) = \bigcup_{F\in\mathscr{F}} F$$

is open in X, and so $\mathscr{F} \in \mathfrak{T}_P$. We thus proved that $\mathfrak{T} \subset \mathfrak{T}_P$.

Next take an arbitrary $\mathscr{F} \in \mathfrak{T}_P$. By definition, we have $P^{-1}(\mathscr{F}) = \bigcup_{F\in\mathscr{F}} F$ is open in X. But then $\mathscr{F} \in \mathfrak{T}$.

We finally prove Theorem 9.7 (McCleary, 2006, Theorem 4.18): *Suppose $f\colon X \to Y$ is a quotient map. Suppose $\sim$ is the equivalence relation defined on X by $x \sim x'$ if $f(x) = f(x')$. Then the quotient space $X/\sim$ is homeomorphic to Y.*

By the definition of the equivalence relation, we have the diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "P"'] \arrow[dr, "h\circ P = f"] & Y \arrow[d, equal] \\
X/\sim \arrow[r, "h"'] & Y
\end{tikzcd}$$

Define $h\colon X/\sim \to Y$ by letting $h([x]) = f(x)$. It is well-defined. Notice that $h \circ P = f$ since for each $x \in X$ we obtain

$$(h \circ P)(x) = h(P(x)) = h([x]) = f(x).$$

Both f and P are quotient maps so h is continuous by Theorem 9.4. We show that h is injective, subjective and h^{-1} is continuous, which implies that h is a homeomorphism. If $h([x]) = h([x'])$, then $f(x) = f(x')$ and so $x \sim x'$; that is, $[x] = [x']$, and h is injective. If $y \in Y$, then $y = f(x)$ since f is surjective and $h([x]) = f(x) = y$ so h is surjective. To see that h^{-1} is continuous, observe that since f is a quotient map and P is a quotient map, this shows $P = h^{-1} \circ f$ and Theorem 9.4 implies that h^{-1} is continuous. ◻

4

CONVERGENCE

4.1 Inadequacy of Sequences

10B. Sequential Convergence and Continuity

► Exercise 78. *Find spaces X and Y and a function $F\colon X \to Y$ which is not continuous, but which has the property that $F(x_n) \to F(x)$ in Y whenever $x_n \to x$ in X.*

Proof. Let $X = \mathbb{R}^{\mathbb{R}}$ and $Y = \mathbb{R}$. Define $F\colon \mathbb{R}^{\mathbb{R}} \to \mathbb{R}$ by letting $F(f) = \sup_{x\in\mathbb{R}} |f(x)|$. Then F is not continuous: Let

$$E = \left\{ f \in \mathbb{R}^{\mathbb{R}} : f(x) = 0 \text{ or } 1 \text{ and } f(x) = 0 \text{ only finitely often} \right\},$$

and let $g \in \mathbb{R}^{\mathbb{R}}$ be the function which is 0 everywhere. Then $g \in \overline{E}$. However, $0 \in F[\overline{E}]$ since $F(g) = 0$, and $\overline{F[E]} = \{1\}$. □

10C. Topology of First-Countable Spaces

Let X and Y be first-countable spaces.

► Exercise 79. *$U \subset X$ is open iff whenever $x_n \to x \in U$, then (x_n) is eventually in U.*

Proof. If U is open and $x_n \to x \in U$, then x has a nhood V such that $x \in V \subset U$. By definition of convergence, there is some positive integer n_0 such that $n \geqslant n_0$ implies $x_n \in V \subset U$; hence, (x_n) is eventually in U.

Conversely, suppose that whenever $x_n \to x \in U$, then (x_n) is eventually in U. If U is not open, then there exists $x \in U$ such that for every nhood of V of x we have $V \cap (X \smallsetminus U) \neq \varnothing$. Since X is first-countable, we can pick a countable nhood base $\{V_n : n \in \mathbb{N}\}$ at x. Replacing $V_n = \bigcap_{i=1}^{n} V_i$ where necessary, we may assume that $V_1 \supset V_2 \supset \cdots$. Now $V_n \cap (X \smallsetminus U) \neq \varnothing$ for each n, so we can pick $x_n \in V_n \cap (X \smallsetminus U)$. The result is a sequence (x_n) contained in $X \smallsetminus U$

which converges to $x \in U$; that is, $x_n \to x$ but (x_n) is not eventually in U. A contradiction. □

► Exercise 80. *$F \subset X$ is closed iff whenever (x_n) is contained in F and $x_n \to x$, then $x \in F$.*

Proof. Let F be closed; let (x_n) be contained in F and $x_n \to x$. Then $x \in \overline{F} = F$.

Conversely, assume that whenever (x_n) is contained in F and $x_n \to x$, then $x \in F$. It follows from Theorem 10.4 that $x \in \overline{F}$ with the hypothesis; therefore, $\overline{F} \subset F$, i.e., $\overline{F} = F$ and so F is closed. □

► Exercise 81. *$f\colon X \to Y$ is continuous iff whenever $x_n \to x$ in X, then $f(x_n) \to f(x)$ in Y.*

Proof. Suppose f is continuous and $x_n \to x$. Since f is continuous at x, for every nhood V of $f(x)$ in Y, there exists a nhood U of x in X such that $f(U) \subset V$. Since $x_n \to x$, there exists n_0 such that $n \geqslant n_0$ implies that $x_n \in U$. Hence, for every nhood V of $f(x)$, there exists n_0 such that $n \geqslant n_0$ implies that $f(x_n) \in V$; that is, $f(x_n) \to f(x)$.

Conversely, let the criterion hold. Suppose that f is not continuous. Then there exists $x \in X$ and a nhood V of $f(x)$, such that for every nhood base U_n, $n \in \mathbb{N}$, of x, there is $x_n \in U_n$ with $f(x_n) \notin V$. By letting $U_1 \supset U_2 \supset \cdots$, we have $x_n \to x$ and so $f(x_n) \to f(x)$; that is, eventually, $f(x_n)$ is in V. A contradiction. □

4.2 Nets

11A. Examples of Net Converence

► Exercise 82. *In $\mathbb{R}^{\mathbb{R}}$, let*

$$E = \left\{ f \in \mathbb{R}^{\mathbb{R}} : f(x) = 0 \textit{ or } 1, \textit{ and } f(x) = 0 \textit{ only finitely often} \right\},$$

and g be the function in $\mathbb{R}^{\mathbb{R}}$ which is identically 0. Then, in the product topology on $\mathbb{R}^{\mathbb{R}}$, $g \in \overline{E}$. Find a net (f_λ) in E which converges to g.

Proof. Let $\mathcal{U}_g = \{U(g, F, \varepsilon) : \varepsilon > 0, F \subset \mathbb{R} \text{ a finite set}\}$ be the nhood base of g. Order $\mathcal{U}_g$ as follows:

$$\begin{aligned} U(g, F_1, \varepsilon_1) \leqslant U(g, F_2, \varepsilon_2) &\iff U(g, F_2, \varepsilon_2) \subset U(g, F, \varepsilon_2) \\ &\iff F_1 \subset F_2 \text{ and } \varepsilon_2 \leqslant \varepsilon_1. \end{aligned}$$

Then $\mathcal{U}_g$ is a directed set. So we have a net $(f_{F,\varepsilon})$ converging to g. □

11B. Subnets and Cluster Points

► EXERCISE 83. *Every subnet of an ultranet is an ultranet.*

PROOF. Take an arbitrary subset $E \subset X$. Let (x_λ) be an ultranet in X, and suppose that (x_λ) is residually in E, i.e., there exists some $\lambda_0 \in \Lambda$ such that $\lambda \geqslant \lambda_0$ implies that $x_\lambda \in E$. If (x_{λ_μ}) is a subnet of (x_λ), then there exists some μ_0 such that $\lambda_{\mu_0} \geqslant \lambda_0$. Then for every $\mu \geqslant \mu_0$, we have $\lambda_\mu \geqslant \lambda_0$, and so $\mu \geqslant \mu_0$ implies that $x_{\lambda_\mu} \in E$; that is, (x_{λ_μ}) is residually in E. □

► EXERCISE 84. *Every net has a subnet which is an ultranet.*

PROOF. See Adamson (1996, Exercise 127, p. 40). □

► EXERCISE 85. *If an ultranet has x as a cluster point, then it converges to x.*

PROOF. Let (x_λ) be an ultranet, and x be a cluster point of (x_λ). Let U be a nhood of x. Then (x_λ) lies in U eventually since for any λ_0 there exists $\lambda \geqslant \lambda_0$ such that $x_\lambda \in U$. □

11D. Nets Describe Topologies

► EXERCISE 86. *Nets have the following four properties:*

a. *if $x_\lambda = x$ for each $\lambda \in \Lambda$, then $x_\lambda \to x$,*

b. *if $x_\lambda \to x$, then every subnet of (x_λ) converges to x,*

c. *if every subnet of (x_λ) has a subnet converging to x, then (x_λ) converges to x,*

d. *(Diagonal principal) if $x_\lambda \to x$ and, for each $\lambda \in \Lambda$, a net $(x_\mu^\lambda)_{u \in M_\lambda}$ converges to x_λ, then there is a diagonal net converging to x; i.e., the net $(x_\mu^\lambda)_{\lambda \in \Lambda, \mu \in M_\lambda}$, ordered lexicographically by Λ, then by M_λ, has a subnet which converges to x.*

PROOF. **(a)** If the net (x_λ) is trivial, then for each nhood U of x, we have $x_\lambda \in U$ for all $\lambda \in \Lambda$. Hence, $x_\lambda \to x$.

(b) Let $(x_{\varphi(\mu)})_{\mu \in M}$ be a subnet of (x_λ). Take any nhood U of x. Then there exists $\lambda_0 \in \Lambda$ such that $\lambda \geqslant \lambda_0$ implies that $x_\lambda \in U$ since $x_\lambda \to x$. Since φ is cofinal in Λ, there exists $\mu_0 \in M$ such that $\varphi(\mu_0) \geqslant \lambda_0$; since φ is increasing, $\mu \geqslant \mu_0$ implies that $\varphi(\mu) \geqslant \varphi(\mu_0) \geqslant \lambda_0$. Hence, there exists $\mu_0 \in M$ such that $\mu \geqslant \mu_0$ implies that $x_{\varphi(\mu)} \in U$; that is, $x_{\varphi(\mu)} \to x$.

(c) Suppose by way of contradiction that (x_λ) does not converge to x. Then there exists a nhood U of x such that for any $\lambda \in \Lambda$, there exists some $\varphi(\lambda) \geqslant \lambda$ with $x_{\varphi(\lambda)} \notin U$. Then $(x_{\varphi(\lambda)})$ is a subnet of (x_λ), but which has no converging subnets.

(d) Order $\{(\lambda,\mu) : \lambda \in \Lambda, \mu \in M_\lambda\}$ as follows:

$$(\lambda_1,\mu_1) \leqslant (\lambda_2,\mu_2) \iff \lambda_1 \leqslant \lambda_2, \text{ or } \lambda_1 = \lambda_2 \text{ and } \mu_1 \leqslant \mu_2.$$

Let $\mathcal{U}$ be the nhood system of x which is ordered by $U_1 \leqslant U_2$ iff $U_2 \subset U_1$ for all $U_1, U_2 \in \mathcal{U}$. Define

$$\Gamma = \left\{(\lambda, U) : \lambda \in \Lambda, U \in \mathcal{U} \text{ such that } x^\lambda \in U\right\}.$$

Order Γ as follows: $(\lambda_1, U_1) \leqslant (\lambda_2, U_2)$ iff $\lambda_1 \leqslant \lambda_2$ and $U_2 \subset U_1$. For each $(\lambda, U) \in \Gamma$ pick $\mu_\lambda \in M_\lambda$ so that $x^\lambda_\mu \in U$ for all $\mu \geqslant \mu_\lambda$ (such a μ_λ exists since $x^\lambda_\mu \to x^\lambda$ and $x^\lambda \in U$). Define $\varphi\colon (\lambda, U) \mapsto x^\lambda_{\mu_\lambda}$ for all $(\lambda, U) \in \Gamma$. It now easy to see that this subnet converges to x. ◻

4.3 Filters

12A. Examples of Filter Convergence

► Exercise 87. *Show that if a filter in a metric space converges, it must converge to a unique point.*

Proof. Suppose a filter $\mathcal{F}$ in a metric space (X, d) converges to $x, y \in X$. If $x \neq y$, then there exists $r > 0$ such that $\mathbb{B}(x,r) \cap \mathbb{B}(y,r) = \varnothing$. But since $\mathcal{F} \to x$ and $\mathcal{F} \to y$, we must have $\mathbb{B}(x,r) \in \mathcal{F}$ and $\mathbb{B}(y,r) \in \mathcal{F}$. This contradicts the fact that the intersection of every two elements in a filter is nonempty. Thus, $x = y$. ◻

12C. Ultrafilters: Uniqueness

► Exercise 88. *If a filter $\mathcal{F}$ is contained in a unique ultrafilter $\mathcal{F}'$, then $\mathcal{F} = \mathcal{F}'$.*

Proof. We first show: *Every filter $\mathcal{F}$ on a non-empty set X is the intersection of the family of ultrafilters which include $\mathcal{F}$.*

Let E be a set which does not belong to $\mathcal{F}$. Then for each set $F \in \mathcal{F}$ we cannot have $F \subset E$ and hence we must have $F \cap E^c \neq \varnothing$. So $\mathcal{F} \cup \{E^c\}$ generates a filter on X, which is included in some ultrafilter $\mathcal{F}_E$. Since $E^c \in \mathcal{F}_E$ we must have $E \notin \mathcal{F}_E$. Thus E does not belong to the intersection of the set of all ultrafilters which include $\mathcal{F}$. Hence this intersection is just the filter $\mathcal{F}$ itself.

Now, if $\mathcal{F}$ is contained in a unique ultrafilter $\mathcal{F}'$, we must have $\mathcal{F} = \mathcal{F}'$. ◻

12D. Nets and Filters: The Translation Process

► EXERCISE 89. *A net (x_λ) has x as a cluster point iff the filter generated by (x_λ) has x as a cluster point.*

PROOF. Suppose x is a cluster point of the net (x_λ). Then for every nhood U of x, we have $x_\lambda \in U$ i. o. But then U meets every $B_{\lambda_0} := \{x_\lambda : \lambda \geqslant \lambda_0\}$, the filter base of the filter $\mathscr{F}$ generated by (x_λ); that is, x is a cluster point of $\mathscr{F}$. The converse implication is obvious. □

► EXERCISE 90. *A filter $\mathscr{F}$ has x as a cluster point iff the net based on $\mathscr{F}$ has x as a cluster point.*

PROOF. Suppose x is a cluster point of $\mathscr{F}$. If U is a nhood of x, then U meets every $F \in \mathscr{F}$. Then for an arbitrary $(p, F) \in \Lambda_{\mathscr{F}}$, pick $q \in F \cap U$ so that $(q, F) \in \Lambda_{\mathscr{F}}$, $(q, F) \geqslant (p, F)$, and $P(p, F) = p \in U$; that is, x is a cluster point of the net based on $\mathscr{F}$.

Conversely, suppose the net based on $\mathscr{F}$ has x as a cluster point. Let U be a nhood of x. Then for every $(p_0, F_0) \in \Lambda_{\mathscr{F}}$, there exists $(p, F) \geqslant (p_0, F_0)$ such that $p \in U$. Then $F_0 \cap U \neq \varnothing$, and so x is a cluster point of $\mathscr{F}$. □

► EXERCISE 91. *If (x_{λ_μ}) is a subnet of (x_λ), then the filter generated by (x_{λ_μ}) is finer than the filter generated by (x_λ).*

PROOF. Suppose (x_{λ_μ}) is a subnet of (x_λ). Let $\mathscr{F}_{\lambda_\mu}$ is the filter generated by (x_{λ_μ}), and $\mathscr{F}_\lambda$ be the filter generated by (x_λ). Then the base generating $\mathscr{F}_{\lambda_\mu}$ is the sets $B_{\lambda_{\mu_0}} = \{x_{\lambda_\mu} : \mu \geqslant \mu_0\}$, and the base generating $\mathscr{F}_\lambda$ is the sets $B_{\lambda_0} = \{x_\lambda : \lambda \geqslant \lambda_0\}$. For each such a B_{λ_0}, there exists μ_0 such that $\lambda_{\mu_0} \geqslant \lambda_0$; that is, $B_{\lambda_{\mu_0}} \subset B_{\lambda_0}$. Therefore, $\mathscr{F}_\lambda \subset \mathscr{F}_{\lambda_\mu}$. □

► EXERCISE 92. *The net based on an ultrafilter is an ultranet and the filter generated by an ultranet is an ultrafilter.*

PROOF. Suppose $\mathscr{F}$ is an ultrafilter. Let $E \subset X$ and we assume that $E \in \mathscr{F}$. Pick $p \in E$. If $(q, F) \geqslant (p, E)$, then $q \in E$; that is, $P(p, F) \in E$ ev. Hence, the net based on $\mathscr{F}$ is an ultranet.

Conversely, suppose (x_λ) is an ultranet. Let $E \subset X$ and we assume that there exists λ_0 such that $x_\lambda \in E$ for all $\lambda \geqslant \lambda_0$. Then $B_{\lambda_0} = \{x_\lambda : \lambda \geqslant \lambda_0\} \subset E$ and so $E \in \mathscr{F}$, where $\mathscr{F}$ is the filter generated by (x_λ). Hence, $\mathscr{F}$ is an ultrafilter. □

► EXERCISE 93. *The net based on a free ultrafilter is a nontrivial ultranet. Hence, assuming the axiom of choice, there are nontrivial ultranets.*

PROOF. Let $\mathscr{F}$ be a free ultrafilter, and (x_λ) be the net based on $\mathscr{F}$. It follows from the previous exercise that (x_λ) is an ultranet. If (x_λ) is trivial, i.e., $x_\lambda = x$ for some $x \in X$ and all $\lambda \in \Lambda_{\mathscr{F}}$, then for all $F \in \mathscr{F}$, we must have $F = \{x\}$. But then $\bigcap \mathscr{F} = \{x\} \neq \varnothing$; that is, $\mathscr{F}$ is fixed. A contradiction.

Now, for instance, the Frechet filter $\mathcal{F}$ on $\mathbb{R}$ is contained in some free ultrafilter $\mathcal{G}$ by Example (b) when the Axiom of Choice is assumed. Hence, the net based on $\mathcal{G}$ is a nontrivial ultranet. $\square$

5

SEPARATION AND COUNTABILITY

5.1 The Separation Axioms

13B. T_0- and T_1-Spaces

► Exercise 94. *Any subspace of a T_0- or T_1-space is, respectively, T_0 or T_1.*

Proof. Let X be a T_0-space, and $A \subset X$. Let x and y be distinct points in A. Then, say, there exists an open nhood U of x such that $y \notin U$. Then $U \cap A$ is relatively open in A, contains x, and $y \notin A \cap U$. The T_1 case can be proved similarly. □

► Exercise 95. *Any nonempty product space is T_0 or T_1 iff each factor space is, respectively, T_0 or T_1.*

Proof. If X_α is a T_0-space, for each $\alpha \in A$, and $x \neq y$ in $\bigtimes X_\alpha$, then for some coordinate α we have $x_\alpha \neq y_\alpha$, so there exists an open set U_α containing, say, x_α but not y_α. Now $\pi_\alpha^{-1}(U_\alpha)$ is an open set in $\bigtimes X_\alpha$ containing x but not y. Thus, $\bigtimes X_\alpha$ is T_0.

Conversely, if $\bigtimes X_\alpha$ is a nonempty T_0-space, pick a fixed point $b_\alpha \in X_\alpha$, for each $\alpha \in A$. Then the subspace $B_\alpha := \{x \in \bigtimes X_\alpha : x_\beta = b_\beta \text{ unless } \beta = \alpha\}$ is T_0, by Exercise 94, and is homeomorphic to X_α under the restriction to B_α of the projection map. Thus X_α is T_0, for each $\alpha \in A$. The T_1 case is similar. □

13C. The T_0-Identification

For any topological space X, define $\sim$ by $x \sim y$ iff $\overline{\{x\}} = \overline{\{y\}}$.

► Exercise 96. *$\sim$ is an equivalence relation on X.*

Proof. Straightforward. □

► Exercise 97. *The resulting quotient space $X/\sim = \widetilde{X}$ is T_0.*

Proof. We first show that X is T_0 iff whenever $x \neq y$ then $\overline{\{x\}} \neq \overline{\{y\}}$. If X is T_0 and $x \neq y$, then there exists an open nhood U of x such that $y \notin U$; then $y \notin \overline{\{x\}}$. Since $y \in \overline{\{y\}}$, we have $\overline{\{x\}} \neq \overline{\{y\}}$. Conversely, suppose that $x \neq y$ implies that $\overline{\{x\}} \neq \overline{\{y\}}$. Take any $x \neq y$ in X and we show that there exists an open nhood of one of the two points such that the other point is not in U. If not, then $y \in \overline{\{x\}}$; since $\overline{\{x\}}$ is closed, we have $\overline{\{y\}} \subset \overline{\{x\}}$; similarly, $\overline{\{x\}} \subset \overline{\{y\}}$. A contradiction.

Now take any $\overline{\{x\}} \neq \overline{\{y\}}$ in $X/\sim$. Then $\overline{\{x\}} = \overline{\overline{\{x\}}} \neq \overline{\overline{\{y\}}} = \overline{\{y\}}$. Hence, $X/\sim$ is T_0. □

13D. The Zariski Topology

For a polynomial P in n real variables, let $Z(P) = \{(x_1, \dots, x_n) \in \mathbb{R}^n : P(x_1, \dots, x_n) = 0\}$. Let $\mathcal{P}$ be the collection of all such polynomials.

► Exercise 98. *$\{Z(P) : P \in \mathcal{P}\}$ is a base for the closed sets of a topology (the* Zariski topology*) on $\mathbb{R}^n$.*

Proof. Denote $\mathcal{Z} := \{Z(P) : P \in \mathcal{P}\}$. If $Z(P_1)$ and $Z(P_2)$ belong to $\mathcal{Z}$, then $Z(P_1) \cup Z(P_2) = Z(P_1 \cdot P_2) \in \mathcal{Z}$ since $P_1 \cdot P_2 \in \mathcal{P}$. Further, $\bigcap_{P \in \mathcal{P}} Z(P) = \varnothing$ since there are $P \in \mathcal{P}$ with $Z(P) = \varnothing$ (for instance, $P = 1 + X_1^2 + \cdots + X_n^2$). It follows from Exercise 48 that $\mathcal{Z}$ is a base for the closed sets of the Zariski topology on $\mathbb{R}^n$. □

► Exercise 99. *The Zariski topology on $\mathbb{R}^n$ is T_1 but not T_2.*

Proof. To verify that the Zariski topology is T_1, we show that every singleton set in $\mathbb{R}^n$ is closed (by Theorem 13.4). For each $(x_1, \dots, x_n) \in \mathbb{R}^n$, define a polynomial $P \in \mathcal{P}$ as follows:

$$P = (X_1 - x_1)^2 + \cdots (X_n - x_n)^2.$$

Then $Z(P) = \{(x_1, \dots, x_n)\}$; that is, $\{(x_1, \dots, x_n)\}$ is closed.

To see the Zariski topology is not T_2, consider the $\mathbb{R}$ case. In $\mathbb{R}$, the Zariski topology coincides with the cofinite topology (see Exercise 100). It is well know that the cofinite topology is not Hausdorff (Example 13.5(a)). □

► Exercise 100. *On $\mathbb{R}$, the Zariski topology coincides with the cofinite topology; in $\mathbb{R}^n$, $n > 1$, they are different.*

Proof. On $\mathbb{R}$, every $Z(P)$ is finite. So on $\mathbb{R}$ every closed set in the Zariski topology is finite since every closed set is an intersection of some subfamily of $\mathcal{Z}$. However, if $n > 1$, then $Z(P)$ can be infinite: for example, consider the polynomial $X_1 X_2$ (let $X_1 = 0$, then all $X_2 \in \mathbb{R}$ is a solution). □

13H. Open Images of Hausdorff Spaces

► EXERCISE 101. *Given any set X, there is a Hausdorff space Y which is the union of a collection $\{Y_x : x \in X\}$ of disjoint subsets, each dense in Y.*

PROOF. □

5.2 Regularity and Complete Regularity

THEOREM 5.1 (Dugundji 1966). a. *Let $P: X \to Y$ be a closed map. Given any subset $S \subset Y$ and any open U containing $P^{-1}(S)$, there exists an open $V \supset S$ such that $P^{-1}(V) \subset U$.*

b. *Let $P: X \to Y$ be an open map. Given any subset $S \subset Y$, and any closed A containing $P^{-1}S$, there exists a closed $B \supset S$ such that $P^{-1}(B) \subset A$.*

PROOF. It is enough to prove (a). Let $V = Y \setminus P(X \setminus U)$. Then

$$\begin{aligned} P^{-1}(S) \subset U &\Longrightarrow X \setminus U \subset X \setminus P^{-1}(S) = P^{-1}(Y \setminus S) \\ &\Longrightarrow P(X \setminus U) \subset P[P^{-1}(Y \setminus S)] \\ &\Longrightarrow Y \setminus P[P^{-1}(Y \setminus S)] \subset V. \end{aligned}$$

Since $P[P^{-1}(Y \setminus S)] \subset Y \setminus S$, we obtain

$$S = Y \setminus (Y \setminus S) \subset Y \setminus P[P^{-1}(Y \setminus S)] \subset V;$$

that is, $S \subset V$. Because P is closed, V is open in Y. Observing that

$$P^{-1}(V) = X \setminus P^{-1}[P(X \setminus U)] \subset X \setminus (X \setminus U) = U$$

completes the proof. □

THEOREM 5.2 (Theorem 14.6). *If X is T_3 and f is a continuous, open and closed map of X onto Y, then Y is T_2.*

PROOF. By Theorem 13.11, it is sufficient to show that the set

$$A := \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}$$

is closed in $X \times X$. If $(x_1, x_2) \notin A$, then $x_1 \notin f^{-1}[f(x_2)]$. Since a T_3-space is T_1, the singleton set $\{x_2\}$ is closed in X; since f is closed, $\{f(x_2)\}$ is closed in Y; since f is continuous, $f^{-1}[f(x_2)]$ is closed in X. Because X is T_3, there are disjoint open sets U and V with

$$x_1 \in U, \quad \text{and} \quad f^{-1}[f(x_2)] \subset V.$$

Since f is closed, it follows from Theorem 5.1 that there exists open set $W \subset Y$ such that $\{f(x_2)\} \subset W$, and $f^{-1}(W) \subset V$; that is,

$$f^{-1}[f(x_2)] \subset f^{-1}(W) \subset V.$$

Then $U \times f^{-1}(W)$ is a nhood of (x_1, x_2). We finally show that $[U \times f^{-1}(W)] \cap A = \varnothing$. If there exists $(y_1, y_2) \in A$ such that $(y_1, y_2) \in U \times f^{-1}(W)$, then $y_1 \in f^{-1}[f(y_2)] \subset f^{-1}(W)$; that is, $y_1 \in U \times f^{-1}(W)$. However, $U \cap V = \varnothing$ and $f^{-1}(W) \subset V$ imply that $U \cap f^{-1}(W) = \varnothing$. A contradiction. □

DEFINITION 5.3. If X is a space and $A \subset X$, then X/A denotes the quotient space obtained via the equivalence relation whose equivalence classes are A and the single point sets $\{x\}$, $x \in X \smallsetminus A$.

THEOREM 5.4. *If X is T_3 and Y is obtained from X by identifying a single closed set A in X with a point, then Y is T_2.*

PROOF. Let A be a closed subset of a T_3-space X. Then $X \smallsetminus A$ is an open subset in both X and X/A and its two subspace topologies agree. Thus, points in $X \smallsetminus A \subset X/A$ are different from $[A]$ and have disjoint nhoods as X is Hausdorff. Finally, for $x \in X \smallsetminus A$, there exist disjoint open nhoods $V(x)$ and $W(A)$. Their images, $f(V)$ and $f(W)$, are disjoint open nhoods of x and $[A]$ in X/A, because $V = f^{-1}[f(V)]$ and $W = f^{-1}[f(W)]$ are disjoint open sets in X. □

5.3 Normal Spaces

15B. Completely Normal Spaces

► EXERCISE 102. *X is completely normal iff whenever A and B are subsets of X with $A \cap \overline{B} = \overline{A} \cap B = \varnothing$, then there are disjoint open sets $U \supset A$ and $V \supset B$.*

PROOF. Suppose that whenever A and B are subsets of X with $A \cap \overline{B} = \overline{A} \cap B = \varnothing$, then there are disjoint open sets $U \supset A$ and $V \supset B$. Let $Y \subset X$, and $C, D \subset Y$ be disjoint closed subsets of Y. Hence,

$$\varnothing = \mathrm{cl}_Y(C) \cap \mathrm{cl}_Y(D) = [\overline{C} \cap Y] \cap [\overline{D} \cap Y] = \overline{C} \cap [\overline{D} \cap Y].$$

Since $D \subset \mathrm{cl}_Y(D)$, we have $\overline{C} \cap D = \varnothing$. Similarly, $C \cap \overline{D} = \varnothing$. Hence there are disjoint open sets U' and V' in X such that $C \subset U'$ and $D \subset V'$. Let $U = U' \cap Y$ and $V = V' \cap Y$. Then U and V are open in Y, $C \subset U$, and $D \subset V$; that is, Y is normal, and so X is completely normal.

Now suppose that X is completely normal and consider the subspace $Y := X \smallsetminus (\overline{A} \cap \overline{B})$. We first show that $A, B \subset Y$. If $A \not\subset Y$, then there exists $x \in A$ with $x \notin Y$; that is, $x \in \overline{A} \cap \overline{B}$. But then $x \in A \cap \overline{B}$. A contradiction. Similarly for B. In the normal space Y, we have

$$\mathrm{cl}_Y(A) \cap \mathrm{cl}_Y(B) = [\bar{A} \cap Y] \cap [\bar{B} \cap Y] = (\bar{A} \cap \bar{B}) \cap [X \smallsetminus (\bar{A} \cap \bar{B})] = \varnothing.$$

Therefore, there exist disjoint open sets $U \supset \mathrm{cl}_Y(A)$ and $V \supset \mathrm{cl}_Y(B)$. Since $A \subset \mathrm{cl}_Y(A)$ and $B \subset \mathrm{cl}_Y(B)$, we get the desired result. □

► Exercise 103. *Why can't the method used to show every subspace of a regular space is regular be carried over to give a proof that every subspace of a normal space is normal?*

Proof. In the first proof, if $A \subset Y \subset X$ is closed in Y and $x \in Y \smallsetminus A$, then there must exists closed set B in X such that $x \notin B$. This property is not applied if $\{x\}$ is replaced a general closed set B in Y. □

► Exercise 104. *Every metric space is completely normal.*

Proof. Every subspace of a metric space is a metric space; every metric space is normal Royden and Fitzpatrick (2010, Proposition 11.7). □

5.4 Countability Properties

16A. First Countable Spaces

► Exercise 105. *Every subspace of a first-countable space is first countable.*

Proof. Let $A \subset X$. If $x \in A$, then V is a nhood of x in A iff $V = U \cap A$, where U is a nhood of $x \in X$ (Theorem 6.3(d)). □

► Exercise 106. *A product $\bigtimes X_\alpha$ of first-countable spaces is first countable iff each X_α is first countable, and all but countably many of the X_α are trivial spaces.*

Proof. If $\bigtimes X_\alpha$ is first-countable, then each X_α is first countable since it is homeomorphic to a subspace of $\bigtimes X_\alpha$. If the number of the family of untrivial sets $\{X_\alpha\}$ is uncountable, then for $x \in \bigtimes X_\alpha$ the number of nhood bases is uncountable. □

► Exercise 107. *The continuous image of a first-countable space need not be first countable; but the continuous open image of a first-countable space is first countable.*

Proof. Let X be a discrete topological space. Then any function defined on X is continuous.

Now suppose that X is first countable, and f is a continuous open map of X onto Y. Pick an arbitrary $y \in Y$. Let $x \in f^{-1}(y)$, and $\mathcal{U}_x$ be a countable nhood base of x. If W is a nhood of y, then there is a nhood V of x such that

$f(V) \subset W$ since f is continuous. So there exists $U \in \mathcal{U}_x$ with $f(U) \subset W$. This proves that $\{f(U) : U \in \mathcal{U}_x\}$ is a nhood base of y. Since $\{f(U) : U \in \mathcal{U}_x\}$ is $\square$

6

COMPACTNESS

6.1 Compact Spaces

17B. Compact Subsets

▶ Exercise 108. *A subset E of X is compact iff every cover of E by open subsets of X has a finite subcover.*

Remark (Lee 2011, p. 94). To say that a *subset* of a topological space is compact is to say that it is a compact space when endowed with the subspace topology. In this situation, it is often useful to extend our terminology in the following way. If X is a topological space and $A \subset X$, a collection of subsets of X whose union contains A is also called a *cover of* A; if the subsets are open in X we sometimes call it an *open cover of* A. We try to make clear in each specific situation which kind of open cover of A is meant: a collection of open subsets of A whose union is A, or a collection of open subsets of X whose union contains A.

Proof. The "only if" part is trivial. So we focus on the "if" part. Let $\mathcal{U}$ be an open cover of E, i.e., $U = \bigcup\{U : U \in \mathcal{U}\}$. For every $U \in \mathcal{U}$, there exists an open set V_U in X such that $U = V_U \cap E$. Then $\{V_U : U \in \mathcal{U}\}$ is an open cover of E, i.e., $U \subset \bigcup\{V_U : U \in \mathcal{U}\}$. Then there exists a finite subcover, say $V_{U_1}, \dots, V_{U_n}$ of $\{V_U : U \in \mathcal{U}\}$, such that $E \subset \bigcup_{i=1}^n V_{U_i}$. Hence, $E = \bigcup_{i=1}^n (V_{U_i} \cap E)$; that is, E is compact. □

▶ Exercise 109. *The union of a finite collection of compact subsets of X is compact.*

Proof. Let A and B be compact, and $\mathcal{U}$ be a family of open subsets of X which covers $A \cup B$. Then $\mathcal{U}$ covers A and there is a finite subcover, say, $U_1^A, \dots, U_m^A$ of A; similarly, there is a finite subcover, say, $U_1^B, \dots, U_n^B$ of B. But then $\{U_1^A, \dots, U_m^A, U_1^B, \dots, U_n^B\}$ is an open subcover of $A \cup B$, so $A \cup B$ is compact. □

References

[1] ADAMSON, IAIN T. (1996) *A General Topology Workbook*, Boston: Birkhäuser. [33]

[2] ASH, ROBERT B. (2009) *Real Variables with Basic Metric Space Topology*, New York: Dover Publications, Inc. [22]

[3] DUGUNDJI, JAMES (1966) *Topology*, Boston: Allyn and Bacon, Inc. [39]

[4] LEE, JOHN M. (2011) *Introduction to Topological Manifolds*, **202** of Graduate Texts in Mathematics, New York: Springer-Verlag, 2nd edition. [25, 43]

[5] MCCLEARY, JOHN (2006) *A First Course in Topology: Continuity and Dimension*, **31** of Student Mathematical Library, Providence, Rhode Island: American Mathematical Society. [29]

[6] ROYDEN, HALSEY AND PATRICK FITZPATRICK (2010) *Real Analysis*, New Jersey: Prentice Hall, 4th edition. [41]

[7] WILLARD, STEPHEN (2004) *General Topology*, New York: Dover Publications, Inc. [i]