General Topology


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Preface

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Acknowledgements
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Acronyms

$\mathbb{R}$  the set of real numbers
$\mathbb{I}$  $[0, 1]$
$\mathbb{P}$  $\mathbb{R} \sim \mathbb{Q}$
SET THEORY AND METRIC SPACES

1.1 Set Theory

1A. Russell's Paradox

Exercise 1. The phenomenon to be presented here was first exhibited by Russell in 1901, and consequently is known as Russell's Paradox.

Suppose we allow as sets things \( A \) for which \( A \in A \). Let \( P \) be the set of all sets. Then \( P \) can be divided into two nonempty subsets, \( P_1 = \{ A \in P : A \notin A \} \) and \( P_2 = \{ A \in P : A \in A \} \). Show that this results in the contradiction: \( P_1 \in P_1 \iff P_1 \notin P_1 \). Does our (naive) restriction on sets given in 1.1 eliminate the contradiction?

Proof. If \( P_1 \in P_1 \), then \( P_1 \in P_2 \), i.e., \( P_1 \notin P_1 \). But if \( P_1 \notin P_1 \), then \( P_1 \in P_1 \). A contradiction.

1B. De Morgan's laws and the distributive laws

Exercise 2. a. \( A \cap (\bigcap_{\lambda \in A} B_{\lambda}) = \bigcup_{\lambda \in A} (A \setminus B_{\lambda}) \).

b. \( B \cup (\bigcap_{\lambda \in A} B_{\lambda}) = \bigcap_{\lambda \in A} (B \cup B_{\lambda}) \).

c. If \( A_{nm} \) is a subset of \( A \) for \( n = 1, 2, \ldots \) and \( m = 1, 2, \ldots \), is it necessarily true that

\[
\bigcup_{n=1}^{\infty} \left[ \bigcap_{m=1}^{\infty} A_{nm} \right] = \bigcap_{m=1}^{\infty} \left[ \bigcup_{n=1}^{\infty} A_{nm} \right] ?
\]

Proof. (a) If \( x \in A \setminus (\bigcap_{\lambda \in A} B_{\lambda}) \), then \( x \in A \) and \( x \notin \bigcap_{\lambda \in A} B_{\lambda} \); thus, \( x \in A \) and \( x \notin B_{\lambda} \) for some \( \lambda \), so \( x \in (A \setminus B_{\lambda}) \) for some \( \lambda \); hence \( x \in \bigcup_{\lambda \in A} (A \setminus B_{\lambda}) \). On the other hand, if \( x \in \bigcup_{\lambda \in A} (A \setminus B_{\lambda}) \), then \( x \in A \setminus B_{\lambda} \) for some \( \lambda \in A \), i.e., \( x \in A \) and \( x \notin B_{\lambda} \) for some \( \lambda \in A \). Thus, \( x \in A \) and \( x \notin \bigcap_{\lambda \in A} B_{\lambda} \); that is, \( x \in A \setminus (\bigcap_{\lambda \in A} B_{\lambda}) \).
(b) If \( x \in B \cup (\bigcap_{\lambda \in A} B_\lambda) \), then \( x \in B_\lambda \) for all \( \lambda \), then \( x \in (B \cup B_\lambda) \) for all \( \lambda \), i.e., \( x \in \bigcap_{\lambda \in A}(B \cup B_\lambda) \). On the other hand, if \( x \in \bigcap_{\lambda \in A}(B \cup B_\lambda) \), then \( x \in (B \cup B_\lambda) \) for all \( \lambda \), i.e., \( x \in B \) or \( x \in B_\lambda \) for all \( \lambda \); that is, \( x \in B \cup (\bigcap_{\lambda \in A} B_\lambda) \).

(c) They are one and the same set. \( \square \)

1C. Ordered pairs

> **Exercise 3.** Show that, if \((x_1, x_2)\) is defined to be \(\{\{x_1\}, \{x_1, x_2\}\}\), then \((x_1, x_2) = (y_1, y_2)\) iff \(x_1 = y_1\) and \(x_2 = y_2\).

**Proof.** If \( x_1 = y_1 \) and \( x_2 = y_2 \), then, clearly, \((x_1, x_2) = \{\{x_1\}, \{x_1, x_2\}\} = \{\{y_1\}, \{y_1, y_2\}\} = (y_1, y_2)\). Now assume that \(\{\{x_1\}, \{x_1, x_2\}\} = \{\{y_1\}, \{y_1, y_2\}\}\). If \( x_1 \neq x_2 \), then \(\{x_1\} = \{y_1\}\) and \(\{x_1, x_2\} = \{y_1, y_2\}\). So, first, \(x_1 = y_1\) and then \(\{x_1, x_2\} = \{y_1, y_2\}\) implies that \(x_2 = y_2\). If \(x_1 = x_2\), then \(\{x_1\} = \{x_1, x_1\}\) equals \(\{x_1\}\). So \(\{y_1\} = \{y_1, y_2\} = \{x_1\}\), and we get \(y_1 = y_2 = x_1\), so \(x_1 = y_1\) and \(x_2 = y_2\) holds in this case, too. \( \square \)

1D. Cartesian products

> **Exercise 4.** Provide an inductive definition of “the ordered \( n \)-tuple \((x_1, \ldots, x_n)\) of elements \(x_1, \ldots, x_n\) of a set” so that \((x_1, \ldots, x_n)\) and \((y_1, \ldots, y_n)\) are equal iff their coordinates are equal in order, i.e., iff \(x_1 = y_1, \ldots, x_n = y_n\).

**Proof.** Define \((x_1, \ldots, x_n) = \{(1, x_1), \ldots, (n, x_n)\}\) as a finite sequence. \( \square \)

> **Exercise 5.** Given sets \(X_1, \ldots, X_n\) define the Cartesian product \(X_1 \times \cdots \times X_n\)

(a) by using the definition of ordered \( n \)-tuple you gave in **Exercise 4**, 

(b) inductively from the definition of the Cartesian product of two sets, 

and show that the two approaches are the same.

**Proof.** (a) \(X_1 \times \cdots \times X_n = \{f \in (\bigcup_{i=1}^n X_i)^n: f(i) \in X_i\}\).

(b) From the definition of the Cartesian product of two sets, \(X_1 \times \cdots \times X_n = \{(x_1, \ldots, x_n): x_i \in X_i\}\), where \((x_1, \ldots, x_n) = ((x_1, \ldots, x_{n-1}), x_n)\).

These two definitions are equal essentially since there is a bijection between them. \( \square \)

> **Exercise 6.** Given sets \(X_1, \ldots, X_n\) let \(X = X_1 \times \cdots \times X_n\) and let \(X^*\) be the set of all functions \( f \) from \(\{1, \ldots, n\}\) into \(\bigcup_{k=1}^n X_k\) having the property that \(f(k) \in X_k\) for each \(k = 1, \ldots, n\). Show that \(X^*\) is the “same” set as \(X\).

**Proof.** Each function \( f \) can be written as \(\{(1, x_1), \ldots, (n, x_n)\}\). So define \(F: X^* \to X\) as \(F(f) = (x_1, \ldots, x_n)\). \( \square \)
Exercise 7. Use what you learned in Exercise 6 to define the Cartesian product $X_1 \times X_2 \times \cdots$ of denumerably many sets as a collection of certain functions with domain $\mathbb{N}$.

Proof. $X_1 \times X_2 \times \cdots$ consists of functions $f : \mathbb{N} \to \bigcup_{n=1}^{\infty} X_n$ such that $f(n) \in X_n$ for all $n \in \mathbb{N}$. $\square$

1.2 Metric Spaces

2A. Metrics on $\mathbb{R}^n$

Exercise 8. Verify that each of the following is a metric on $\mathbb{R}^n$:

a. $\rho(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$.

b. $\rho_1(x, y) = \sum_{i=1}^{n} |x_i - y_i|$.

c. $\rho_2(x, y) = \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$.

Proof. Clearly, it suffices to verify the triangle inequalities for all of the three functions. Pick arbitrary $x, y, z \in \mathbb{R}^n$.

(a) By Minkowski's Inequality, we have

$$\rho(x, z) = \sqrt{\sum_{i=1}^{n} (x_i - z_i)^2} = \sqrt{\sum_{i=1}^{n} [(x_i - y_i) + (y_i - z_i)]^2}$$

$$\leq \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} + \sqrt{\sum_{i=1}^{n} (y_i - z_i)^2}$$

$$= \rho(x, y) + \rho(y, z).$$

(b) We have

$$\rho_1(x, z) = \sum_{i=1}^{n} |x_i - z_i| = \sum_{i=1}^{n} (|x_i - y_i| + |y_i - z_i|) = \rho_1(x, y) + \rho_1(y, z).$$

(c) We have

$$\rho_2(x, z) = \max\{|x_1 - z_1|, \ldots, |x_n - z_n|\}$$

$$\leq \max\{|x_1 - y_1| + |y_1 - z_1|, \ldots, |x_n - y_n| + |y_n - z_n|\}$$

$$\leq \max\{|x_1 - y_1|, \ldots, |x_n - y_n|\} + \max\{|y_1 - z_1|, \ldots, |y_n - z_n|\}$$

$$= \rho_2(x, y) + \rho_2(y, z).$$ $\square$
2B. Metrics on $\mathcal{C}(\mathbb{I})$

**Exercise 9.** Let $\mathcal{C}(\mathbb{I})$ denote the set of all continuous real-valued functions on the unit interval $\mathbb{I}$ and let $x_0$ be a fixed point of $\mathbb{I}$.

a. $\rho(f, g) = \sup_{x \in \mathbb{I}} |f(x) - g(x)|$ is a metric on $\mathcal{C}(\mathbb{I})$.

b. $\sigma(f, g) = \int_0^1 |f(x) - g(x)| \, dx$ is a metric on $\mathcal{C}(\mathbb{I})$.

c. $\eta(f, g) = |f(x_0) - g(x_0)|$ is a pseudometric on $\mathcal{C}(\mathbb{I})$.

**Proof.** Let $f, g, h \in \mathcal{C}(\mathbb{I})$. It is clear that $\rho$, $\sigma$, and $\eta$ are positive, symmetric; it is also clear that $\rho$ and $\sigma$ satisfy M-b.

(a) We have

$$\rho(f, h) = \sup_{x \in \mathbb{I}} |f(x) - h(x)| \leq \sup_{x \in \mathbb{I}} (|f(x) - g(x)| + |g(x) - h(x)|)$$

$$\leq \sup_{x \in \mathbb{I}} |f(x) - g(x)| + \sup_{x \in \mathbb{I}} |g(x) - h(x)|$$

$$= \rho(f, g) + \rho(g, h).$$

(b) We have

$$\sigma(f, h) = \int_0^1 |f(x) - h(x)| \, dx \leq \int_0^1 |f(x) - g(x)| + \int_0^1 |g(x) - h(x)|$$

$$= \sigma(f, g) + \sigma(g, h).$$

(c) For arbitrary $f, g \in \mathcal{C}(\mathbb{I})$ with $f(x_0) = g(x_0)$ we have $\eta(f, g) = 0$, so $\eta(f, g) = 0$ does not imply that $f = g$. Further, $\eta(f, h) = |f(x_0) - h(x_0)| \leq |f(x_0) - g(x_0)| + |g(x_0) - h(x_0)| = \eta(f, g) + \eta(g, h)$. □

2C. Pseudometrics

**Exercise 10.** Let $(M, \rho)$ be a pseudometric space. Define a relation $\sim$ on $M$ by $x \sim y \iff \rho(x, y) = 0$. Then $\sim$ is an equivalence relation.

**Proof.** (i) $x \sim x$ since $\rho(x, x) = 0$ for all $x \in M$. (ii) $x \sim y \iff \rho(x, y) = 0$ iff $\rho(y, x) = 0$ iff $y \sim x$. (iii) Suppose $x \sim y$ and $y \sim z$. Then $\rho(x, z) \leq \rho(x, y) + \rho(y, z) = 0$; that is, $\rho(x, z) = 0$. So $x \sim z$. □

**Exercise 11.** If $M^*$ is the set of equivalence classes in $M$ under the equivalence relation $\sim$ and if $\rho^*$ is defined on $M^*$ by $\rho^*([x], [y]) = \rho(x, y)$, then $\rho^*$ is a well-defined metric on $M^*$.

**Proof.** $\rho^*$ is well-defined since it does not dependent on the representative of $[x]$; let $x' \in [x]$ and $y' \in [y]$. Then

$$\rho(x', y') \leq \rho(x', x) + \rho(x, y) + \rho(y, y') = \rho(x, y).$$
Symmetrically, \( \rho(x, y) \leq \rho(x', y') \). To verify \( \rho^* \) is a metric on \( M^* \), it suffices to show that \( \rho^* \) satisfies the triangle inequality. Let \([x], [y], [z] \in M^* \). Then

\[
\rho^*([x], [z]) = \rho(x, z) \leq \rho(x, y) + \rho(y, z) = \rho^*([x], [y]) + \rho^*([y], [z]).
\]

\[\blacksquare\]

**Exercise 12.** If \( h: M \to M^* \) is the mapping \( h(x) = [x] \), then a set \( A \) in \( M \) is closed (open) iff \( h(A) \) is closed (open) in \( M^* \).

**Proof.** Let \( A \) be open in \( M \) and \( h(x) = [x] \in h(A) \) for some \( x \in A \). Since \( A \) is open, there exist an \( \varepsilon \)-disk \( U_\rho(x, \varepsilon) \) contained in \( A \). For each \( y \in U_\rho(x, \varepsilon) \), we have \( h(y) = [y] \in h(A) \), and \( \rho^*([x], [y]) = \rho(x, y) \leq \varepsilon \). Hence, for each \( [x] \in h(A) \), there exists an \( \varepsilon \)-disk \( U_\rho^*([x], \varepsilon) = h(U_\rho(x, \varepsilon)) \) contained in \( h(A) \); that is, \( h(A) \) is open in \( M^* \). Since \( h \) is surjective, it is now easy to see that \( h(A) \) is closed in \( M^* \) whenever \( A \) is closed in \( M \). \[\blacksquare\]

**Exercise 13.** If \( f \) is any real-valued function on a set \( M \), then the distance function \( \rho_f(x, y) = |f(x) - f(y)| \) is a pseudometric on \( M \).

**Proof.** Easy. \[\blacksquare\]

**Exercise 14.** If \( (M, \rho) \) is any pseudometric space, then a function \( f: M \to \mathbb{R} \) is continuous iff each set open in \( (M, \rho_f) \) is open in \( (M, \rho) \).

**Proof.** Suppose that \( f \) is continuous and \( G \) is open in \( (M, \rho_f) \). For each \( x \in G \), there is an \( \varepsilon > 0 \) such that if \( |f(y) - f(x)| < \varepsilon \) then \( y \in G \). The continuity of \( f \) at \( x \) implies that there exists \( \delta > 0 \) such that if \( \rho(y, x) < \delta \) then \( |f(y) - f(x)| < \varepsilon \), and so \( y \in U \). We thus proved that for each \( x \in U \) there exists a \( \delta \)-disk \( U_\rho(x, \delta) \) contained in \( G \); that is, \( G \) is open in \( (M, \rho) \).

Conversely, suppose that each set is open in \( (M, \rho) \) whenever it is open in \( (M, \rho_f) \). For each \( x \in (M, \rho_f) \), there is an \( \varepsilon \)-disk \( U_\rho_f(x, \varepsilon) \) contained in \( M \) since \( M \) is open under \( \rho_f \); then \( U_\rho_f(x, \varepsilon) \) is open in \( (M, \rho) \) since \( U_\rho_f(x, \varepsilon) \) is open in \( (M, \rho_f) \). Hence, there is a \( \delta \)-disk \( U_\rho(x, \delta) \) such that \( U_\rho(x, \delta) \subset U_\rho_f(x, \varepsilon) \); that is, if \( \rho(y, x) < \delta \), then \( |f(y) - f(x)| < \varepsilon \). So \( f \) is continuous on \( M \). \[\blacksquare\]

2D. Disks Are Open

**Exercise 15.** For any subset \( A \) of a metric space \( M \) and any \( \varepsilon > 0 \), the set \( U(A, \varepsilon) \) is open.

**Proof.** Let \( A \subset M \) and \( \varepsilon > 0 \). Take an arbitrary point \( x \in U(A, \varepsilon) \); take an arbitrary point \( y \in A \) such that \( \rho(x, y) < \varepsilon \). Observe that every \( \varepsilon \)-disk \( U(y, \varepsilon) \) is contained in \( U(A, \varepsilon) \). Since \( x \in U(y, \varepsilon) \) and \( U(y, \varepsilon) \) is open, there exists a \( \delta \)-disk \( U(x, \delta) \) contained in \( U(y, \varepsilon) \). Therefore, \( U(A, \varepsilon) \) is open. \[\blacksquare\]
2E. Bounded Metrics

**Exercise 16.** If \( \rho \) is any metric on \( M \), the distance function \( \rho^*(x, y) = \min\{\rho(x, y), 1\} \) is a metric also and is bounded.

**Proof.** To see \( \rho^* \) is a metric, it suffices to show the triangle inequality. Let \( x, y, z \in M \). Then
\[
\rho^*(x, z) = \min\{\rho(x, z), 1\} \leq \min\{\rho(x, y) + \rho(y, z), 1\}
\leq \min\{\rho(x, y), 1\} + \min\{\rho(y, z), 1\}
= \rho^*(x, y) + \rho^*(y, z).
\]
It is clear that \( \rho^* \) is bounded above by 1. \( \square \)

**Exercise 17.** A function \( f \) is continuous on \( (M, \rho) \) iff it is continuous on \( (M, \rho^*) \).

**Proof.** It suffices to show that \( \rho \) and \( \rho^* \) are equivalent. If \( G \) is open in \( (M, \rho) \), then for each \( x \in G \) there is an \( \varepsilon \)-disk \( U_\rho(x, \varepsilon) \subset G \). Since \( U_{\rho^*}(x, \varepsilon) \subset U_\rho(x, \varepsilon) \), we know \( G \) is open in \( (M, \rho^*) \). Similarly, we can show that \( G \) is open in \( (M, \rho^*) \) whenever it is open in \( (M, \rho) \). \( \square \)

2F. The Hausdorff Metric

Let \( \rho \) be a bounded metric on \( M \); that is, for some constant \( A \), \( \rho(x, y) \leq A \) for all \( x \) and \( y \) in \( M \).

**Exercise 18.** Show that the elevation of \( \rho \) to the power set \( \mathcal{P}(M) \) as defined in 2.4 is not necessarily a pseudometric on \( \mathcal{P}(M) \).

**Proof.** Let \( M := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 1\} \), and let \( \rho \) be the usual metric. Then \( \rho \) is a bounded metric on \( M \). We show that the function \( \rho^* : (E, F) \mapsto \inf_{x \in E, y \in F} \rho(x, y) \), for all \( E, F \in \mathcal{P}(M) \), is not a pseudometric on \( \mathcal{P}(M) \) by showing that the triangle inequality fails. Let \( E, F, G \in \mathcal{P}(M) \), where \( E = U_\rho((-1/4, 0), 1/4) \), \( G = U_\rho((1/4, 0), 1/4) \), and \( F \) meets both \( E \) and \( G \). Then \( \rho^*(E, G) > 0 \), but \( \rho^*(E, F) = \rho^*(F, G) = 0 \). \( \square \)

**Exercise 19.** Let \( \mathcal{F}(M) \) be all nonempty closed subsets of \( M \) and for \( A, B \in \mathcal{F}(M) \) define
\[
d_A(B) = \sup\{\rho(A, x) : x \in B\}
d(A, B) = \max\{d_A(B), d_B(A)\}.
\]
Then \( d \) is a metric on \( \mathcal{F}(M) \) with the property that \( d(\{x\}, \{y\}) = \rho(x, y) \). It is called the Hausdorff metric on \( \mathcal{F}(M) \).
Proof. Clearly, $d$ is nonnegative and symmetric. If $d(A, B) = 0$, then $d_A(B) = d_B(A) = 0$, i.e., $\sup_{y \in B} \rho(A, y) = \sup_{x \in A} \rho(B, x) = 0$. But then $\rho(A, y) = 0$ for all $y \in B$ and $\rho(B, x) = 0$ for all $x \in A$. Since $A$ is closed, we have $y \in A$ for all $y \in B$; that is, $B \subseteq A$. Similarly, $A \subseteq B$. Hence, $A = B$.

We next show the triangle inequality of $d$. Let $A, B, C \in \mathcal{F}(M)$. For an arbitrary point $a \in A$, take a point $b \in C$ such that $\rho(a, b) = \rho(B, a)$ (since $B$ is closed, such a point exists). Then

$$
\rho(a, b) \leq \sup_{x \in A} \rho(B, x) = d_B(A) \leq d(A, B).
$$

For this $b \in B$, we take a point $c \in C$ such that $\rho(b, c) \leq \rho(B, C)$. Therefore,

$$
\rho(a, c) \leq \rho(a, b) + \rho(b, c) \leq d(A, B) + d(B, C).
$$

We thus proved that for every $a \in A$, there exists $c \in C$ (depends on $a$), such that $\rho(a, c) \leq d(A, B) + d(B, C)$. In particular, we have

$$
\rho(a, C) = \inf_{z \in C} \rho(a, z) \leq d(A, B) + d(B, C).
$$

Since the above inequality holds for all $a \in A$, we obtain

$$
d_C(A) = \sup_{x \in A} \rho(a, C) \leq d(A, B) + d(B, C). \quad (1.1)
$$

Similarly, for each $c \in C$ there exists $b \in B$ with $\rho(c, b) \leq d(B, C)$; for this $b$, there exists $a \in A$ with $\rho(a, b) \leq d(A, B)$. Hence $\rho(a, c) \leq d(A, B) + d(B, C)$ for all $c \in C$. The same argument shows that

$$
d_A(C) \leq d(A, B) + d(B, C). \quad (1.2)
$$

Combining (1.1) and (1.2) we get the desired result.

Finally, notice that $d_{\{x\}}(\{y\}) = d_{\{y\}}(\{x\}) = \rho(x, y)$; hence, $d(\{x\}, \{y\}) = \rho(x, y)$. □

Exercise 20. Prove that closed sets $A$ and $B$ are "close" in the Hausdorff metric iff they are "uniformly close"; that is, $d(A, B) < \varepsilon$ iff $A \subseteq U_\rho(B, \varepsilon)$ and $B \subseteq U_\rho(A, \varepsilon)$.

Proof. If $d(A, B) < \varepsilon$, then $\sup_{y \in B} \rho(A, y) = \rho_A(B) < \varepsilon$; that is, $\rho(A, y) < \varepsilon$ for all $y \in B$, so $B \subseteq U_\rho(A, \varepsilon)$. Similarly, $A \subseteq U_\rho(B, \varepsilon)$.

Conversely, if $A \subseteq U_\rho(B, \varepsilon)$, then $\rho(B, x) < \varepsilon$ for all $x \in A$. Since $A$ is closed, we have $d_B(A) < \varepsilon$; similarly, $B \subseteq U_\rho(A, \varepsilon)$ implies that $d_A(B) < \varepsilon$. Hence, $d(A, B) < \varepsilon$. □
2G. Isometry

Metric spaces \((M, \rho)\) and \((N, \sigma)\) are isometric iff there is a one-one function \(f\) from \(M\) onto \(N\) such that \(\rho(x, y) = \sigma(f(x), f(y))\) for all \(x\) and \(y\) in \(M\); \(f\) is called an isometry.

**Exercise 21.** If \(f\) is an isometry from \(M\) to \(N\), then both \(f\) and \(f^{-1}\) are continuous functions.

**Proof.** By definition, \(f\) is (uniformly) continuous on \(M\): for every \(\varepsilon > 0\), let \(\delta = \varepsilon\); then \(\rho(x, y) < \delta\) implies that \(\sigma(f(x), f(y)) = \rho(x, y) < \varepsilon\).

On the other hand, for every \(\varepsilon > 0\) and \(y \in N\), pick the unique \(f^{-1}(y) \in M\) (since \(f\) is bijective). For each \(z \in N\) with \(\sigma(y, z) < \varepsilon\), we must have \(\rho(f^{-1}(y), f^{-1}(z)) = \sigma(f(f^{-1}(y)), f(f^{-1}(z))) = \sigma(y, z) < \varepsilon\); that is, \(f^{-1}\) is continuous.

**Exercise 22.** \(\mathbb{R}\) is not isometric to \(\mathbb{R}^2\) (each with its usual metric).

**Proof.** Consider \(S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}\). Notice that there are only two points around \(f^{-1}(0, 0)\) with distance 1.

**Exercise 23.** \(I\) is isometric to any other closed interval in \(\mathbb{R}\) of the same length.

**Proof.** Consider the function \(f : I \rightarrow [a, a + 1]\) defined by \(f(x) = a + x\) for all \(x \in I\).
2.1 Fundamental Concepts

3A. Examples of Topologies

**Exercise 24.** If $\mathcal{F}$ is the collection of all closed, bounded subset of $\mathbb{R}$ (in its usual topology), together with $\mathbb{R}$ itself, then $\mathcal{F}$ is the family of closed sets for a topology on $\mathbb{R}$ strictly weaker than the usual topology.

**Proof.** It is easy to see that $\mathcal{F}$ is a topology. Further, for instance, $(-\infty, 0]$ is a closed set of $\mathbb{R}$, but it is not in $\mathcal{F}$. □

**Exercise 25.** If $A \subseteq X$, show that the family of all subsets of $X$ which contain $A$, together with the empty set $\emptyset$, is a topology on $X$. Describe the closure and interior operations. What topology results when $A = \emptyset$? when $A = X$?

**Proof.** Let

$$\mathcal{E} = \{ E \subseteq X : A \subseteq E \} \cup \{ \emptyset \}.$$  

Now suppose that $E_\lambda \in \mathcal{E}$ for each $\lambda \in \Lambda$. Then $A \subseteq \bigcup_{\lambda} E_\lambda \subseteq X$ and so $\bigcup E_\lambda \in \mathcal{E}$. The other postulates are easy to check.

For any set $B \subseteq X$, if $A \subseteq B$, then $B \in \mathcal{E}$ and so $B^* = B$; if not, then $B^* = \emptyset$. If $A = \emptyset$, then $\mathcal{E}$ is the discrete topology; if $A = X$, then $\mathcal{E} = \{ \emptyset, X \}$. □

3D. Regularly Open and Regularly Closed Sets

An open subset $G$ in a topological space is regular open iff $G$ is the interior of its closure. A closed subset is regularly closed iff it is the closure of its interior.

**Exercise 26.** The complement of a regularly open set is regularly closed and vice versa.

**Proof.** Suppose $G$ is regular open; that is, $G = (\overline{G})^\circ$. Then
\[ X \sim G = X \setminus (\overline{G})^o = \overline{X \setminus G} = (X \sim G)^o. \]

Hence, \( X \sim G \) is regularly closed. If \( F \) is regular closed, i.e., \( F = \overline{F^o} \), then
\[ X \sim F = X \setminus \overline{F^o} = (X \sim F^o)^o = (\overline{X \setminus F})^o; \]
that is, \( X \sim F \) is regularly open. \( \square \)

\begin{exercise}
There are open sets in \( \mathbb{R} \) which are not regularly open.
\end{exercise}

\begin{proof}
Consider \( \mathbb{Q} \). We have \((\overline{\mathbb{Q}})^o = \mathbb{R}^o = \mathbb{R} \neq \mathbb{Q} \). So \( \mathbb{Q} \) is not regularly open. \( \square \)
\end{proof}

\begin{exercise}
If \( A \) is any subset of a topological space, then \( \text{int}(\text{cl}(A)) \) is regularly open.
\end{exercise}

\begin{proof}
Let \( A \) be a subset of a topological space \( X \). We then have
\[ \text{int}(\text{cl}(A)) \subset \text{cl}(\text{int}(\text{cl}(A))) \implies \text{int}(\text{cl}(A)) = \text{int}(\text{int}(\text{cl}(A))) \subset \text{int}(\text{cl}(\text{cl}(A))). \]
and
\[ \text{int}(\text{cl}(A)) \subset \text{cl}(A) \implies \text{cl}(\text{int}(\text{cl}(A))) \subset \text{cl}(\text{cl}(A)) = \text{cl}(A) \]
\[ \implies \text{int}(\text{cl}(\text{cl}(A))) \subset \text{int}(\text{cl}(A)). \]
Therefore, \( \text{int}(\text{cl}(A)) = \text{int}(\text{cl}(\text{cl}(A))) \); that is, \( \text{int}(\text{cl}(A)) \) is regularly open. \( \square \)

\begin{exercise}
The intersection, but not necessarily the union, of two regularly open sets is regularly open.
\end{exercise}

\begin{proof}
Let \( A \) and \( B \) be two regularly open sets in a topological space \( X \). Then
\[ (A \cap B)^o \subset (\overline{A \cap B})^o = (\overline{A})^o \cap (\overline{B})^o = A \cap B, \]
and
\[ (\overline{A \cap B})^o = (\overline{A})^o \cap (\overline{B})^o = A \cap B \subset \overline{A \cap B} \]
\[ \implies A \cap B = (\overline{A \cap B})^o = \left[ (\overline{A \cap B})^o \right]^o \subset (\overline{A \cap B})^o. \]
Hence, \( A \cap B = (\overline{A \cap B})^o \).

To see that the union of two regularly open sets is not necessarily regularly open, consider \( A = (0, 1) \) and \( B = (1, 2) \) in \( \mathbb{R} \) with its usual topology. Then
\[ (\overline{A \cup B})^o = [0, 2]^o = (0, 2) \neq A \cup B. \] \( \square \)
3E. Metrizable Spaces

Let $X$ be a metrizable space whose topology is generated by a metric $\rho$.

**Exercise 30.** The metric $2\rho$ defined by $2\rho(x, y) = 2 \cdot \rho(x, y)$ generates the same topology on $X$.

**Proof.** Let $\mathcal{O}_{\rho}$ be the collection of open sets in $(X, \rho)$, and let $\mathcal{O}_{2\rho}$ be the collection of open sets in $(X, 2\rho)$. If $O \in \mathcal{O}_{\rho}$, then for every $x \in O$, there exists an open ball $B_\rho(x, \epsilon) \subseteq O$; but then $B_{2\rho}(x, \epsilon/2) \subseteq O$. Hence, $O \in \mathcal{O}_{2\rho}$. Similarly, we can show that $\mathcal{O}_{2\rho} \subseteq \mathcal{O}_\rho$. In fact, $\rho$ and $2\rho$ are equivalent metrics. □

**Exercise 31.** The closure of a set $E \subset X$ is given by $\overline{E} = \{ y \in X : \rho(E, y) = 0 \}$.

**Proof.** Denote $\overline{E} := \{ y \in X : \rho(E, y) = 0 \}$. We first show that $\overline{E}$ is closed (see Definition 2.5, p. 17). Take an arbitrary $x \in \overline{E}$ such that for every $n \in \mathbb{N}$, there exists $y_n \in \overline{E}$ with $\rho(x, y_n) < 1/2n$. For each $y_n \in \overline{E}$, take $z_n \in E$ with $\rho(y_n, z_n) < 1/2n$. Then

$$\rho(x, z_n) \leq \rho(x, y_n) + \rho(y_n, z_n) < 1/n.$$

Thus, $\rho(x, E) = 0$, i.e., $x \in \overline{E}$. Therefore, $\overline{E}$ is closed. It is clear that $E \subseteq \overline{E}$, and so $\overline{E} \subseteq E$.

We next show that $\overline{E} \subseteq \overline{E}$. Take an arbitrary $x \in \overline{E}$ and a closed set $K$ containing $E$. If $x \in X \setminus K$, then $\rho(x, K) > 0$ (see Exercise 35). But then $\rho(x, E) > 0$ since $E \subseteq K$ and so

$$\inf_{y \in \overline{E}} \rho(x, y) \geq \inf_{z \in K} \rho(x, z).$$

Hence, $\overline{E} \subseteq \overline{E}$. □

**Exercise 32.** The closed disk $U(x, \epsilon) = \{ y : \rho(x, y) \leq \epsilon \}$ is closed in $X$, but may not be the closure of the open disk $U(x, \epsilon)$.

**Proof.** Fix $x \in X$. We show that the function $\rho(x, \cdot) : X \to \mathbb{R}$ is (uniformly) continuous. For any $y, z \in X$, the triangle inequality yields

$$|\rho(x, y) - \rho(x, z)| \leq \rho(y, z).$$

Hence, for every $\epsilon > 0$, take $\delta = \epsilon$, and $\rho(x, \cdot)$ satisfies the $\epsilon$-$\delta$ criterion. Therefore, $U(x, \epsilon) = \rho^{-1}(\epsilon, [0, \epsilon])$ and $[0, \epsilon]$ is closed in $\mathbb{R}$.

To see it is not necessary that $U(x, \epsilon) = \overline{U}(x, \epsilon)$, consider $\epsilon = 1$ and the usual metric on

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(x, 0) \in \mathbb{R}^2 : 0 \leq x \leq 1\};$$

see Figure 2.1. Observe that $(0, 0) \notin U(x, 1)$, but $(0, 0) \in U(x, 1)$. It follows from Exercise 31 that $(0, 0) \notin \overline{U}(x, 1)$. □
3H. $G_δ$ and $F_α$ Sets

► Exercise 33. The complement of a $G_δ$ is an $F_α$, and vice versa.

Proof. If $A$ is a $G_δ$ set, then there exists a sequence of open sets $\{U_n\}$ such that $A = \bigcap_{n=1}^{\infty} U_n$. Then $A^c = \bigcup_{n=1}^{\infty} U_n^c$ is $F_α$. Vice versa. □

► Exercise 34. An $F_α$ can be written as the union of an increasing sequence $F_1 \subset F_2 \subset \cdots$ of closed sets.

Proof. Let $B = \bigcup_{n=1}^{\infty} E_n$, where $E_n$ is closed for all $n \in \mathbb{N}$. Define $F_1 = E_1$ and $F_n = \bigcup_{i=1}^{n} E_i$ for $n \geq 2$. Then each $F_n$ is closed, $F_1 \subset F_2 \subset \cdots$, and $\bigcup_{n=1}^{\infty} F_n = \bigcup_{n=1}^{\infty} B$. □

► Exercise 35. A closed set in a metric space is a $G_δ$.

Proof. For an arbitrary set $A \subset X$ and a point $x \in X$, define

$$\rho(x, A) = \inf_{y \in A} \{\rho(x, y)\}.$$ 

We first show that $\rho(\cdot, A): X \to \mathbb{R}$ is (uniformly) continuous by showing

$$|\rho(x, A) - \rho(y, A)| \leq \rho(x, y), \quad \text{for all } x, y \in X. \quad (2.1)$$

For an arbitrary $z \in A$, we have

$$\rho(x, A) \leq \rho(x, z) \leq \rho(x, y) + \rho(y, z).$$

Take the infimum over $z \in A$ and we get

$$\rho(x, A) \leq \rho(x, y) + \rho(y, A). \quad (2.2)$$
Symmetrically, we have
\[ \rho(y, A) \leq \rho(x, y) + \rho(x, A). \]  
(2.3)

Hence, (2.1) follows from (2.2) and (2.3). We next show that if \( A \) is closed, then \( \rho(x, A) = 0 \) iff \( x \in A \). The “if” part is trivial, so we do the “only if” part. If \( \rho(x, A) = 0 \), then for every \( n \in \mathbb{N} \), there exists \( y_n \in A \) such that \( \rho(x, y_n) < 1/n \); that is, \( y_n \to x \). Since \( \{y_n\} \subset A \) and \( A \) is closed, we must have \( x \in A \).

Therefore,
\[ A = \bigcap_{n=1}^{\infty} \{x \in X : \rho(x, A) < 1/n\}. \]

The continuity of \( \rho(\cdot, A) \) implies that \( \{x \in X : \rho(x, A) < 1/n\} \) is open for all \( n \). Thus, \( A \) is a \( G_\delta \) set.

**Exercise 36.** The rationals are an \( F_\sigma \) in \( \mathbb{R} \).

**Proof.** \( \mathbb{Q} \) is countable, and every singleton set in \( \mathbb{R} \) is closed; hence, \( \mathbb{Q} \) is an \( F_\sigma \).

3.2 Borel Sets

2.2 Neighborhoods

4. The Sorgenfrey Line

**Exercise 37.** Verify that the set \( [x, z] \), for \( z > x \), do form a nhhood base at \( x \) for a topology on the real line.

**Proof.** We need only check that for each \( x \in \mathbb{R} \), the family \( \mathcal{B}_x := \{[x, z) : z > x\} \) satisfies V-a, V-b, and V-c in Theorem 4.5. V-a is trivial. If \( [x, z_1) \in \mathcal{B}_x \) and \( [x, z_2) \in \mathcal{B}_x \), then \( [x, z_1) \cap [x, z_2) = [x, z_1 \land z_2) \in \mathcal{B}_x \) and is in \( [x, z_1) \cap [x, z_2) \).

For V-c, let \( [x, z) \in \mathcal{B}_x \). Let \( z' \in (x, z] \). Then \( [x, z') \in \mathcal{B}_x \), and if \( y \in [x, z') \), the right-open interval \( [y, z') \in \mathcal{B}_y \) and \( [y, z') \subset [x, z) \).

Then, define open sets using V-d: \( G \subset \mathbb{R} \) is open if and only if \( G \) contains a set \( [x, z) \) of each of its points \( x \).

**Exercise 38.** Which intervals on the real line are open sets in the Sorgenfrey topology?

**Solution.**

- Sets of the form \((\infty, x), [x, z), \) or \([x, \infty) \) are both open and closed.
- Sets of the form \((x, z) \) or \((x, \infty) \) are open in \( \mathbb{R} \), since
\[ (x, z) = \bigcup \{[y, z) : x < y < z\}. \]
EXERCISE 39. Describe the closure of each of the following subset of the Sorgenfrey line: the rationals \( \mathbb{Q} \), the set \( \{1/n : n = 1, 2, \ldots\} \), the set \( \{-1/n : n = 1, 2, \ldots\} \), the integers \( \mathbb{Z} \).

SOLUTION. Recall that, by Theorem 4.7, for each \( E \subset \mathbb{R} \), we have

\[
\bar{E} = \{ x \in \mathbb{R} : \text{each basic nhood of } x \text{ meets } E \}.
\]

Then \( \bar{\mathbb{Q}} = \mathbb{R} \) since for any \( x \in \mathbb{R} \), we have \( \{x, z\} \cap \mathbb{Q} \neq \emptyset \) for \( z > x \). Similarly, \( \{1/n : n = 1, 2, \ldots\} = \{1/n : n = 1, 2, \ldots\} \), and \( \bar{\mathbb{Z}} = \mathbb{Z} \).

4B. The Moore Plane

EXERCISE 40. Verify that this gives a topology on \( \Gamma \).

PROOF. Verify (V-a)—(V-c). It is easy. \( \Box \)

4E. Topologies from nhoods

EXERCISE 41. Show that if each point \( x \) in a set \( X \) has assigned a collection \( \mathcal{U}_x \) of subsets of \( X \) satisfying N-a through N-d of 4.2, then the collection

\[
\tau = \{ G \subset X : \text{for each } x \in G, x \in U \subset G \text{ for some } U \in \mathcal{U}_x \}
\]

is a topology for \( X \), in which the nhood system at each \( x \) is just \( \mathcal{U}_x \).

PROOF. We need to check G1—G3 in Definition 3.1. Since G1 and G3 are evident, we focus on G2. Let \( E_1, E_2 \in \tau \). Take any \( x \in E_1 \cap E_2 \). Then there exist some \( U_1, U_2 \in \mathcal{U}_x \) such that \( x \in U_1 \subset E_1 \) and \( x \in U_2 \subset E_2 \). By N-b, we know that \( U_1 \cap U_2 \in \mathcal{U}_x \). Hence,

\[
x \in U_1 \cap U_2 \subset E_1 \cap E_2,
\]

and so \( E_1 \cap E_2 \in \tau \). The induction principle then means that \( \tau \) is closed under finite intersections. \( \Box \)

4F. Spaces of Functions

EXERCISE 42. For each \( f \in \mathbb{R}^I \), each finite subset \( F \) of \( I \) and each positive \( \delta \), let

\[
U(f, F, \delta) = \{ g \in \mathbb{R}^I : |g(x) - f(x)| < \delta, \text{ for each } x \in F \}.
\]

Show that the sets \( U(f, F, \delta) \) form a nhood base at \( f \), making \( \mathbb{R}^I \) a topological space.

PROOF. Denote
\( \mathcal{B}_f = \{ U(f, F, \delta) : F \subseteq \mathbb{I}, |F| < \infty, \delta > 0 \} \).

**(V-a)** For each \( U(f, F, \delta) \in \mathcal{B}_f \), we have \( |f(x) - f(x)| = 0 < \delta \) for all \( x \in F \); hence, \( f \in U(f, F, \delta) \).

**(V-b)** Let \( U(f, F_1, \delta_1), U(f, F_2, \delta_2) \in \mathcal{B}_f \). Define \( U(f, F_3, \delta_3) \) by letting

\[
F_3 = F_1 \cup F_2, \quad \text{and} \quad \delta_3 = \min\{\delta_1, \delta_2\}.
\]

Clearly, \( U(f, F_3, \delta_3) \in \mathcal{B}_f \). If \( g \in U(f, F_3, \delta_3) \), then

\[
|g(x) - f(x)| < \min\{\delta_1, \delta_2\}, \quad \text{for all} \ x \in F_1 \cup F_2.
\]

Hence, \( |g(x) - f(x)| < \delta_1 \) for all \( x \in F_1 \) and \( |g(x) - f(x)| < \delta_2 \) for all \( x \in F_2 \); that is, \( g \in U(f, F_1, \delta_1) \cap U(f, F_2, \delta_2) \). Hence, there exists \( U(f, F_3, \delta_3) \in \mathcal{B}_f \) such that \( U(f, F_3, \delta_3) \subseteq U(f, F_1, \delta_1) \cap U(f, F_2, \delta_2) \).

**(V-c)** Pick \( U(f, F, \delta) \in \mathcal{B}_f \). We must show that there exists some \( U(f, G, \delta_0) \in \mathcal{B}_f \) such that if \( g \in U(f, G, \delta_0) \), then there is some \( U(g, F', \delta') \in \mathcal{B}_g \) with \( U(g, F', \delta') \subseteq U(f, F, \delta) \).

Let \( F_0 = F \), and \( \delta_0 = \delta/2 \). Then \( U(f, F, \delta/2) \in \mathcal{B}_f \). For every \( g \in U(f, F, \delta/2) \), we have

\[
|g(x) - f(x)| < \delta/2, \quad \text{for all} \ x \in F.
\]

Let \( U(g, F', \delta') = U(g, F, \delta/2) \). If \( h \in U(g, F, \delta/2) \), then

\[
|h(x) - f(x)| < \delta/2, \quad \text{for all} \ x \in F.
\]

Triangle inequality implies that

\[
|h(x) - f(x)| \leq |h(x) - g(x)| + |g(x) - f(x)| < \delta/2 + \delta/2 = \delta, \quad \text{for all} \ x \in F;
\]

that is, \( h \in U(f, F, \delta) \). Hence, \( U(g, F, \delta/2) \subseteq U(f, F, \delta) \).

Now, \( G \subseteq \mathbb{R}^1 \) is open iff \( G \) contains a \( U(f, F, \delta) \) of each \( f \in G \). This defines a topology on \( \mathbb{R}^1 \). \( \square \)

**Exercise 43.** For each \( f \in \mathbb{R}^1 \), the closure of the one-point set \( \{f\} \) is just \( \{f\} \).

**Proof.** For every \( g \in \mathbb{R}^1 \setminus \{f\} \), pick \( x \in \mathbb{I} \) with \( g(x) \neq f(x) \). Define \( U(g, F, \delta) \) with \( F = \{x\} \) and \( \delta < |g(x) - f(x)| \). Then \( f \notin U(g, \{x\}, \delta) \); that is, \( U(g, \{x\}, \delta) \in \mathbb{R}^1 \setminus \{f\} \). Hence, \( \mathbb{R}^1 \setminus \{f\} \) is open, and so \( \{f\} \) is closed. This proves that \( \overline{\{f\}} = \{f\} \). \( \square \)

**Exercise 44.** For \( f \in \mathbb{R}^1 \) and \( \varepsilon > 0 \), let

\[
V(f, \varepsilon) = \{ g \in \mathbb{R}^1 : |g(x) - f(x)| < \varepsilon, \ \text{for each} \ x \in \mathbb{I} \}.
\]

Verify that the sets \( V(f, \varepsilon) \) form a nhood base at \( f \), making \( \mathbb{R}^1 \) a topological space.
PROOF. Denote $V_f = \{ V(f, \varepsilon) : \varepsilon > 0 \}$. We verify the following properties.

(V-a) If $V(f, \varepsilon) \in V_f$, then $|f(x) - f(x)| = 0 < \varepsilon$; that is, $f \in V(f, \varepsilon)$.

(V-b) Let $V(f, \varepsilon_1), V(f, \varepsilon_2) \in V_f$. Let $\varepsilon_3 = \min\{\varepsilon_1, \varepsilon_2\}$. If $g \in V(f, \varepsilon_3)$, then

$$|g(x) - f(x)| < \varepsilon_3 = \min\{\varepsilon_1, \varepsilon_2\}, \quad \text{for all } x \in I.$$

Hence, $V(f, \varepsilon_3) \subset V(f, \varepsilon_1) \cap V(f, \varepsilon_2)$.

(V-c) For an arbitrary $V(f, \varepsilon) \in V_f$, pick $V(f, \varepsilon/2) \in V_f$. For each $g \in V(f, \varepsilon/2)$, pick $V(g, \varepsilon/2) \in V_f$. If $h \in V(g, \varepsilon/2)$, then $|h(x) - g(x)| < \varepsilon/2$ for all $x \in I$. Hence

$$|h(x) - f(x)| \leq |h(x) - g(x)| + |g(x) - f(x)| < \varepsilon;$$

that is, $V(g, \varepsilon/2) \subset V(f, \varepsilon)$. □

**Exercise 45.** Compare the topologies defined in 1 and 3.

PROOF. It is evident that for every $U(f, F, \delta) \in \mathcal{B}_F$, there exists $V(f, \delta) \in V_f$ such that $V(f, \delta) \subset U(f, F, \delta)$. Hence, the topology in 1 is weaker than in 3 by Hausdorff criterion. □

### 2.3 Bases and Subbases

5D. No Axioms for Subbase

**Exercise 46.** Any family of subsets of a set $X$ is a subbase for some topology on $X$ and the topology which results is the smallest topology containing the given collection of sets.

PROOF. Let $\mathcal{S}$ be a family of subsets of $X$. Let $\tau(\mathcal{S})$ be the intersection of all topologies containing $\mathcal{S}$. Such topologies exist, since $2^X$ is one such. Also $\tau(\mathcal{S})$ is a topology. It evidently satisfies the requirements “unique” and “smallest.”

The topology $\tau(\mathcal{S})$ can be described as follows: It consists of $\emptyset$, $X$, all finite intersections of the $\mathcal{S}$-sets, and all arbitrary unions of these finite intersections. To verify this, note that since $\mathcal{S} \subset \tau(\mathcal{S})$, then $\tau(\mathcal{S})$ must contain all the sets listed. Conversely, because $\bigcup$ distributes over $\bigcap$, the sets listed actually do from a topology containing $\mathcal{S}$, and which therefore contains $\tau(\mathcal{S})$. □

5E. Bases for the Closed Sets

**Exercise 47.** $\mathcal{F}$ is a base for the closed sets in $X$ iff the family of complements of members of $\mathcal{F}$ is a base for the open sets.
Proof. Let \( G \) be an open set in \( X \). Then \( G = X \sim E \) for some closed subset \( E \). Since \( E = \bigcap_{F \in \mathcal{F}} F \), we obtain

\[
G = X \sim \left( \bigcap_{F \in \emptyset \subset \mathcal{F}} F \right) = \bigcup_{F \in \emptyset \subset \mathcal{F}} F^c.
\]

Thus, \( \{F^c : F \in \mathcal{F}\} \) forms a base for the open sets. The converse direction is similar. \( \square \)

Exercise 48. \( \mathcal{F} \) is a base for the closed sets for some topology on \( X \) iff (a) whenever \( F_1 \) and \( F_2 \) belong to \( \mathcal{F} \), \( F_1 \cup F_2 \) is an intersection of elements of \( \mathcal{F} \), and (b) \( \bigcap_{F \in \mathcal{F}} F = \emptyset \).

Proof. If \( \mathcal{F} \) is a base for the closed sets for some topology on \( X \), then (a) and (b) are clear. Suppose, on the other hand, \( X \) is a set and \( \mathcal{F} \) a collection of subsets of \( X \) with (a) and (b). Let \( \mathcal{T} \) be all intersections of subcollections from \( \mathcal{F} \). Then any intersection of members of \( \mathcal{T} \) certainly belongs to \( \mathcal{T} \), so \( \mathcal{T} \) satisfies (F-a). Moreover, if \( \mathcal{F}_1 \subset \mathcal{F} \) and \( \mathcal{F}_2 \subset \mathcal{F} \), so that \( \bigcap_{E \in \mathcal{F}_1} E \) and \( \bigcap_{F \in \mathcal{F}_2} F \) are elements of \( \mathcal{T} \), then

\[
\left( \bigcap_{E \in \mathcal{F}_1} E \right) \cup \left( \bigcap_{F \in \mathcal{F}_2} F \right) = \bigcap_{E \in \mathcal{F}_1} \bigcap_{F \in \mathcal{F}_2} (E \cup F).
\]

But by property (a), the union of two elements of \( \mathcal{F} \) is an intersection of elements of \( \mathcal{F} \), so \( \left( \bigcap_{E \in \mathcal{F}_1} E \right) \cup \left( \bigcap_{F \in \mathcal{F}_2} F \right) \) is an intersection of elements of \( \mathcal{F} \), and hence belongs to \( \mathcal{T} \). Thus \( \mathcal{T} \) satisfies (F-b). Finally, \( \emptyset \in \mathcal{T} \) by (b) and \( X \in \mathcal{T} \) since \( X \) is the intersection of the empty subcollection from \( \mathcal{F} \). Hence \( \mathcal{T} \) satisfies (F-c). This completes the proof that \( \mathcal{T} \) is the collection of closed sets of \( X \). \( \square \)
3

NEW SPACES FROM OLD

3.1 Subspaces

3.2 Continuous Functions

7A. Characterization of Spaces Using Functions

Exercise 49. The characteristic function of $A$ is continuous iff $A$ is both open and closed in $X$.

Proof. Let $\mathbf{1}_A : X \to \mathbb{R}$ be the characteristic function of $A$, which is defined by

$$
\mathbf{1}_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{if } x \notin A.
\end{cases}
$$

First suppose that $\mathbf{1}_A$ is continuous. Then, say, $\mathbf{1}_A^{-1}\left((1/2, 2)\right) = A$ is open, and $\mathbf{1}_A^{-1}\left((-1, 1/2)\right) = X \setminus A$ is open. Hence, $A$ is both open and closed in $X$.

Conversely, suppose that $A$ is both open and closed in $X$. For any open set $U \subset \mathbb{R}$, we have

$$
\mathbf{1}_A^{-1}(U) = \begin{cases} 
A & \text{if } 1 \in U \text{ and } 0 \notin U \\
X \setminus A & \text{if } 1 \notin U \text{ and } 0 \in U \\
\emptyset & \text{if } 1 \notin U \text{ and } 0 \notin U \\
X & \text{if } 1 \in U \text{ and } 0 \in U.
\end{cases}
$$

Then $\mathbf{1}_A$ is continuous.

Exercise 50. $X$ has the discrete topology iff whenever $Y$ is a topological space and $f : X \to Y$, then $f$ is continuous.

Proof. Let $Y$ be a topological space and $f : X \to Y$. It is easy to see that $f$ is continuous if $X$ has the discrete topology, so we focus on the sufficiency
direction. For any $A \subset X$, let $Y = \mathbb{R}$ and $f = 1_A$. Then by Exercise 49 $A$ is open. □

7C. Functions Agreeing on A Dense Subset

**Exercise 51.** If $f$ and $g$ are continuous functions from $X$ to $\mathbb{R}$, the set of points $x$ for which $f(x) = g(x)$ is a closed subset of $X$. Thus two continuous maps on $X$ to $\mathbb{R}$ which agree on a dense subset must agree on all of $X$.

**Proof.** Denote $A = \{x \in X : f(x) \neq g(x)\}$. Take a point $y \in A$ such that $f(y) > g(y)$ (if it is not true then let $g(y) > f(y)$). Take an $\varepsilon > 0$ such that $f(y) - \varepsilon > g(y) + \varepsilon$. Since $f$ and $g$ are continuous, there exist nhoods $U_1$ and $U_2$ of $y$ such that $f[U_1] \subset (-\varepsilon + f(y), \varepsilon + f(y))$ and $g[U_2] \subset (-\varepsilon + g(y), \varepsilon + g(y))$. Let $U = U_1 \cap U_2$. Then $U$ is a nhood of $y$ and for every $z \in U$ we have

$$f(z) - g(z) > |f(y) - \varepsilon| - |g(y) + \varepsilon| \geq 0.$$ 

Hence, $U \subset A$; that is, $U$ is open, and so $\{x \in X : f(x) = g(x)\} = X \setminus U$ is closed.

Now suppose that $D := \{x \in X : f(x) = g(x)\}$ is dense. Take an arbitrary $x \in X$. Since $f$ and $g$ are continuous, for each $n \in \mathbb{N}$, there exist nhoods $V_f$ and $V_g$ such that $|f(y) - f(x)| < 1/n$ for all $y \in V_f$ and $|g(y) - g(x)| < 1/n$ for all $y \in V_g$. Let $V_n = V_f \cap V_g$. Then there exists $x_n \in V_n \cap D$ with $|f(x_n) - f(x)| < 1/2n$ and $|g(x_n) - g(x)| < 1/2n$. Since $f(x_n) = g(x_n)$, we have

$$|f(x) - g(x)| \leq |f(x) - f(x_n)| + |f(x_n) - g(x_n)| = |f(x) - f(x_n)| + |g(x_n) - g(x)| < 1/n.$$ 

Therefore, $f(x) = g(x)$. □

7E. Range Immaterial

**Exercise 52.** If $Y \subset Z$ and $f : X \to Y$, then $f$ is continuous as a map from $X$ to $Y$ iff $f$ is continuous as a map from $X$ to $Z$.

**Proof.** Let $f : X \to Z$ be continuous. Let $U$ be open in $Y$. Then $U = Y \cap V$ for some $V$ which is open in $Z$. Therefore,

$$f^{-1}(U) = f^{-1}(Y \cap V) = f^{-1}(Y) \cap f^{-1}(V) = X \cap f^{-1}(V) = f^{-1}(V)$$

is open in $X$, and so $f$ is continuous as a map from $X$ to $Y$.

Conversely, let $f : X \to Y$ be continuous and $V$ be open in $Z$. Then $f^{-1}(V) = f^{-1}(Y \cap V)$. Since $Y \cap V$ is open in $Y$ and $f$ is continuous from $X$ to $Y$, the set $f^{-1}(Y \cap V)$ is open in $X$ and so $f$ is continuous as a map from $X$ to $Z$. □
7G. Homeomorphisms within the Line

- **Exercise 53.** Show that all open intervals in $\mathbb{R}$ are homeomorphic.

**Proof.** We have

- $(a, b) \sim (0, 1)$ by $f_1(x) = (x - a)/(b - a)$.
- $(a, \infty) \sim (1, \infty)$ by $f_2(x) = x - a + 1$.
- $(1, \infty) \sim (0, 1)$ by $f_3(x) = 1/x$.
- $(-\infty, -a) \sim (a, \infty)$ by $f_4(x) = -x$.
- $(-\infty, \infty) \sim (-\pi/2, \pi/2)$ by $f_5(x) = \arctan x$.

Therefore, by compositing, every open interval is homeomorphic to $(0, 1)$. □

- **Exercise 54.** All bounded closed intervals in $\mathbb{R}$ are homeomorphic.

**Proof.** $[a, b] \sim [0, 1]$ by $f(x) = (x - a)/(b - a)$. □

- **Exercise 55.** The property that every real-valued continuous function on $X$ assumes its maximum is a topological property. Thus, $\mathbb{I} := [0, 1]$ is not homeomorphic to $\mathbb{R}$.

**Proof.** Every continuous function assumes its maximum on $[0, 1]$; however, $x^2$ has no maximum on $\mathbb{R}$. Therefore, $\mathbb{I} \not\sim \mathbb{R}$. □

7K. Semicontinuous Functions

- **Exercise 56.** If $f_a$ is a lower semicontinuous real-valued function on $X$ for each $a \in A$, and if $\sup_a f_a(x)$ exists at each $x \in X$, then the function $f(x) = \sup_a f_a(x)$ is lower semicontinuous on $X$.

**Proof.** For an arbitrary $a \in \mathbb{R}$, we have $f(x) \leq a$ iff $f_a(x) \leq a$ for all $a \in A$. Hence,

$$\{x \in X : f(x) \leq a\} = \bigcap_{a \in A} \{x \in X : f_a(x) \leq a\},$$

and so $f^{-1}(-\infty, a]$ is closed; that is, $f$ is lower semicontinuous. □

- **Exercise 57.** Every continuous function from $X$ to $\mathbb{R}$ is lower semicontinuous. Thus the supremum of a family of continuous functions, if it exists, is lower semicontinuous. Show by an example that "lower semicontinuous" cannot be replaced by "continuous" in the previous sentence.

**Proof.** Suppose that $f : X \to \mathbb{R}$ is continuous. Since $(-\infty, x]$ is closed in $\mathbb{R}$, the set $f^{-1}(-\infty, x]$ is closed in $X$; that is, $f$ is lower semicontinuous.

To construct an example, let $f : [0, \infty) \to \mathbb{R}$ be defined as follows:
\[ f_n(x) = \begin{cases} 
  nx & \text{if } 0 \leq x \leq 1/n \\
  1 & \text{if } x > 1/n .
\end{cases} \]

Then
\[ f(x) = \sup_n f_n(x) = \begin{cases} 
  0 & \text{if } x = 0 \\
  1 & \text{if } x > 0 .
\end{cases} \]

and \( f \) is not continuous. \( \square \)

**Exercise 58.** The characteristic function of a set \( A \) in \( X \) is lower semicontinuous iff \( A \) is open, upper semicontinuous iff \( A \) is closed.

**Proof.** Observe that
\[ 1_A^{-1}(-\infty, a] = \begin{cases} 
  \emptyset & \text{if } a < 0 \\
  X \setminus A & \text{if } 0 \leq a < 1 \\
  X & \text{if } a \geq 1.
\end{cases} \]

Therefore, \( 1_A \) is LSC iff \( A \) is open. Similarly for the USC case. \( \square \)

**Exercise 59.** If \( X \) is metrizable and \( f \) is a lower semicontinuous function from \( X \) to \( \mathbb{I} \), then \( f \) is the supremum of an increasing sequence of continuous functions on \( X \) to \( \mathbb{I} \).

**Proof.** Let \( d \) be the metric on \( X \). First assume \( f \) is nonnegative. Define
\[ f_n(x) = \inf_{z \in X} \{ f(z) + nd(x, z) \} . \]

If \( x, y \in X \), then \( f(z) + nd(x, z) \leq f(z) + nd(y, z) + nd(x, y) \). Take the inf over \( z \) (first on the left side, then on the right side) to obtain \( f_n(x) \leq f_n(y) + nd(x, y) \). By symmetry,
\[ |f_n(x) - f_n(y)| \leq nd(x, y); \]

hence, \( f_n \) is uniformly continuous on \( X \). Furthermore, since \( f \geq 0 \), we have
\[ 0 \leq f_n(x) \leq f(x) + nd(x, x) = f(x) . \]

By definition, \( f_n \) increases with \( n \); we must show that \( \lim_n f_n \) is actually \( f \).

Given \( \varepsilon > 0 \), by definition of \( f_n(x) \) there is a point \( z_n \in X \) such that
\[ f_n(x) + \varepsilon > f(z_n) + nd(x, z_n) \quad (3.1) \]

since \( f \geq 0 \). But \( f_n(x) + \varepsilon \leq f(x) + \varepsilon \); hence \( d(x, z_n) \to 0 \). Since \( f \) is LSC, we have \( \liminf_n f(z_n) \geq f(x) \) (Ash, 2009, Theorem 8.4.2); hence
\[ f(z_n) > f(x) - \varepsilon \quad \text{ev.} \quad (3.2) \]

By (3.1) and (3.2),
\[ f_n(x) > f(z_n) - \varepsilon + nd(x, z_n) \geq f(z_n) - \varepsilon > f(x) - 2\varepsilon . \]
for all sufficiently large \( n \). Thus, \( f_n(x) \to f(x) \).

If \( |f| \leq M < \infty \), then \( f + M \) is LSC, finite-valued, and nonnegative. If \( 0 \leq g_n \to (f + M) \), then \( f_n = (g_n - M) \uparrow f \) and \( |f_n| \geq M \). \( \square \)

7M. \( C(X) \) and \( C^*(X) \)

\[ \textbf{Exercise 60.} \text{ If } f \text{ and } g \text{ belong to } C(X), \text{ then so do } f + g, \ f \cdot g \text{ and } a \cdot f, \text{ for } a \in \mathbb{R}. \text{ If, in addition, } f \text{ and } g \text{ are bounded, then so are } f + g, \ f \cdot g \text{ and } a \cdot f. \]

\textbf{Proof.} We first do \( f + g \). Since \( f, g \in C(X) \), for each \( x \in X \) and each \( \varepsilon > 0 \), there exist nhoods \( U_1 \) and \( U_2 \) of \( x \) such that \( f[U_1] \subset (-\varepsilon/2 + f(x), \varepsilon/2 + f(x)) \) and \( g[U_2] \subset (-\varepsilon/2 + g(x), \varepsilon/2 + g(x)) \). Let \( U = U_1 \cap U_2 \). Then \( U \) is a nhood of \( x \), and for every \( y \in U \), we have

\[ ||f(y) + g(y)| - |f(x) + g(x)|| | f(y) - f(x) | = |g(y) - g(x)| < \varepsilon; \]

that is, \( f + g \) is continuous.

We then do \( a \cdot f \). We suppose that \( a > 0 \) (all other cases are similar). For each \( x \in X \) and \( \varepsilon > 0 \), there exists nhood \( U \) of \( x \) such that \( f[U] \subset (-\varepsilon/a + f(x), \varepsilon/a + f(x)) \). Then \( (a \cdot f)[U] \subset (-\varepsilon + a \cdot f(x), \varepsilon + a \cdot f(x)) \). So \( a \cdot f \in C(X) \).

Finally, to do \( f \cdot g \), we first show that \( f^2 \in C(X) \) whenever \( f \in C(X) \). For each \( x \in X \) and \( \varepsilon > 0 \), there is a nhood \( U \) of \( x \) such that \( f[U] \subset (-\sqrt{f} + f(x), \sqrt{f} + f(x)) \). Then \( f^2[U] \subset (-\varepsilon + f^2(x), \varepsilon + f^2(x)) \), i.e., \( f^2 \in C(X) \).

Since \( f(x) \cdot g(x) = \frac{1}{4} \left[ (f(x) + g(x))^2 - (f(x) - g(x))^2 \right] \),

we know that \( f \cdot g \in C(X) \) from the previous arguments. \( \square \)

\[ \textbf{Exercise 61.} \text{ } C(X) \text{ and } C^*(X) \text{ are algebras over the real numbers.} \]

\textbf{Proof.} It follows from the previous exercise that \( C(X) \) is a vector space on \( \mathbb{R} \). So everything is easy now. \( \square \)

\[ \textbf{Exercise 62.} \text{ } C^*(X) \text{ is a normed linear space with the operations of addition and scalar multiplication given above and the norm } \| f \| = \sup_{x \in X} |f(x)|. \]

\textbf{Proof.} It is easy to see that \( C^*(X) \) is a linear space. So it suffices to show that \( \| \cdot \| \) is a norm on \( C^*(X) \). We focus on the triangle inequality. Let \( f, g \in C^*(X) \).

Then for every \( x \in X \), we have \( |f(x) + g(x)| \leq |f(x)| + |g(x)| \leq \| f \| + \| g \| ; \)

hence, \( \| f + g \| \leq \| f \| + \| g \| \). \( \square \)
3.3 Product Spaces, Weak Topologies

8A. Projection Maps

**Exercise 63.** The $\beta$th projection map $\pi_\beta$ is continuous and open. The projection $\pi_1: \mathbb{R}^2 \to \mathbb{R}$ is not closed.

**Proof.** Let $U_\beta$ be open in $X_\beta$. Then $\pi_\beta^{-1}(U_\beta)$ is a subbasis open set of the Tychonoff topology on $\prod_{\alpha} X_\alpha$, and so is open. Hence, $\pi_\beta$ is continuous.

Take an arbitrary basis open set $U$ in the Tychonoff topology. Denote $I := \{1, \ldots, n\}$. Then

$$U = \prod_{\alpha} U_\alpha,$$

where $U_\alpha$ is open in $X_\alpha$ for every $\alpha \in A$, and $U_{\alpha_j} = X_{\alpha_j}$ for all $j \notin I$. Hence,

$$\pi_\beta(U) = \begin{cases} U_\beta & \text{if } \beta = \alpha_i \text{ for some } i \in I \\ X_\beta & \text{otherwise.} \end{cases}$$

That is, $\pi_\beta(U)$ is open in $X_\beta$ in both case. Since any open set is a union of basis open sets, and since functions preserve unions, the image of any open set under $\pi_\beta$ is open.

![Figure 3.1. $\pi_1(F) = (0, \infty)$](image)

Finally, let $F = \text{epi}(1/x)$. Then $F$ is closed in $\mathbb{R}^2$, but $\pi_1(F) = (0, \infty)$ is open in $\mathbb{R}$; that is, $\pi_1$ is not closed. See Figure 3.1.

**Exercise 64.** Show that the projection of $\mathbb{I} \times \mathbb{R}$ onto $\mathbb{R}$ is a closed map.

**Proof.** Let $\pi: \mathbb{I} \times \mathbb{R} \to \mathbb{R}$ be the projection. Suppose $A \subset \mathbb{I} \times \mathbb{R}$ is closed, and suppose $y_0 \in \mathbb{R} \setminus \pi(A)$. For every $x \in \mathbb{I}$, since $(x, y_0) \notin A$ and $A$ is closed, we find a basis open subset $U(x) \times V(x)$ of $\mathbb{I} \times \mathbb{R}$ that contains $(x, y_0)$, and $[U(x) \times V(x)] \cap A = \emptyset$. The collection $\{U(x) : x \in \mathbb{I}\}$ covers $\mathbb{I}$, so finitely many of them cover $\mathbb{I}$ by compactness, say $U(x_1), \ldots, U(x_n)$ do. Now define $V = \ldots$
Let $X \in \mathcal{X}$, and note that $V$ is an open nhood of $y_0$, and $V \cap \pi[A] = \emptyset$. So $\pi[A]$ is closed; that is, $\pi$ is closed. See Lee (2011, Lemma 4.35, p. 95) for the Tube Lemma.

Generally, if $\pi: X \times Y \to X$ is a projection may where $Y$ is compact, then $\pi$ is a closed map. \hfill \Box

8B. Separating Points from Closed Sets

**Exercise 65.** If $f_\alpha$ is a map (continuous function) of $X$ to $X_\alpha$ for each $\alpha \in A$, then \{ $f_\alpha : \alpha \in A$ \} separates points from closed sets in $X$ if \{ $f_\alpha^{-1}[V] : \alpha \in A$, $V$ open in $X_\alpha$ \} is a base for the topology on $X$.

**Proof.** Suppose that \{ $f_\alpha^{-1}[V] : \alpha \in A$, $V$ open in $X_\alpha$ \} consists of a base for the topology on $X$. Let $B$ be closed in $X$ and $x \notin B$. Then $x \in X \setminus B$ and $X \setminus B$ is open in $X$. Hence there exists $f_\alpha^{-1}[V]$ such that $x \in f_\alpha^{-1}[V] \subset X \setminus B$; that is, $f_\alpha(x) \in V$. Since $V \cap f_\alpha[B] = \emptyset$, i.e., $f_\alpha[B] \subset X_\alpha \setminus V$, and $X_\alpha \setminus V$ is closed, we get $f_\alpha[B] \subset X_\alpha \setminus V$. Thus, $f_\alpha(x) \notin f_\alpha[B]$.

Next assume that \{ $f_\alpha : \alpha \in A$ \} separates points from closed sets in $X$. Take an arbitrary open subset $U \subset X$ and $x \in U$. Then $B := X \setminus U$ is closed in $X$, and hence there exists $\alpha \in A$ such that $f_\alpha(x) \notin f_\alpha[B]$. Then $f_\alpha(x) \in X_\alpha \setminus f_\alpha[B]$ and, since $X_\alpha \setminus f_\alpha[B]$ is open in $X_\alpha$, there exists an open set $V$ of $X_\alpha$ such that $f_\alpha(x) \in V \subset X_\alpha \setminus f_\alpha[B]$. Therefore,

\[
x \in f_\alpha^{-1}[V] \subset f_\alpha^{-1}[X_\alpha \setminus f_\alpha[B]] = X \setminus f_\alpha^{-1}[f_\alpha[B]] \\
\subset X \setminus f_\alpha^{-1}[f_\alpha[B]] \\
\subset X \setminus \overline{B} \\
= U.
\]

Hence, \{ $f_\alpha^{-1}[V] : \alpha \in A$, $V$ open in $X_\alpha$ \} is a base for the topology on $X$. \hfill \Box

8D. Closure and Interior in Products

Let $X$ and $Y$ be topological spaces containing subsets $A$ and $B$, respectively. In the product space $X \times Y$:

**Exercise 66.** $(A \times B)^o = A^o \times B^o$.

**Proof.** Since $A^o \subset A$ is open in $A$ and $B^o \subset B$ is open in $B$, the set $A^o \times B^o \subset A \times B$ is open in $A \times B$; hence, $A^o \times B^o \subset (A \times B)^o$.

For the converse inclusion, let $x = (a, b) \in (A \times B)^o$. Then there is an basis open set $U_1 \times U_2$ such that $x \in U_1 \times U_2 \subset A \times B$, where $U_1$ is open in $A$ and $U_2$ is open in $B$. Hence, $a \in U_1 \subset A$ and $b \in U_2 \subset B$; that is, $a \in A^o$ and $b \in B^o$. Then $x \in A^o \times B^o$. \hfill \Box
Exercise 67. \( \overline{A} \times \overline{B} = \overline{A} \times \overline{B} \).

**Proof.** See Exercise 68. \( \square \)

Exercise 68. Part 2 can be extended to infinite products, while part 1 can be extended only to finite products.

**Proof.** Assume that \( y = (y_\alpha) \in \prod A_\alpha \); we show that \( y_\alpha \in \overline{A}_\alpha \) for each \( \alpha \); that is, \( y \in \prod \overline{A}_\alpha \). Let \( y_\alpha \in U_\alpha \), where \( U_\alpha \) is open in \( Y_\alpha \); since \( y \in \pi_\alpha^{-1}(U_\alpha) \), we must have

\[
\emptyset \neq \pi_\alpha^{-1}(U_\alpha) \cap \prod A_\alpha = (U_\alpha \cap A_\alpha) \times \left( \prod_{\beta \neq \alpha} A_\beta \right),
\]

and so \( U_\alpha \cap A_\alpha \neq \emptyset \). This proves \( y_\alpha \in \overline{A}_\alpha \). The converse inclusion is established by reversing these steps: If \( y \in \prod A_\alpha \), then for any open nhood

\[
B := U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \left( \prod \{ Y_\beta : \beta \neq \alpha_1, \ldots, \alpha_n \} \right),
\]

each \( U_{\alpha_i} \cap A_{\alpha_i} \neq \emptyset \) so that \( B \cap \prod A_\alpha \neq \emptyset \). \( \square \)

Exercise 69. \( \text{Fr}(A \times B) = [\overline{A} \times \text{Fr}(B)] \cup [\text{Fr}(A) \times \overline{B}] \).

**Proof.** We have

\[
\text{Fr}(A \times B) = \overline{A} \times \overline{B} \cap (X \times Y) \setminus (A \times B)
= (\overline{A} \times \overline{B}) \cap [(X \times Y) \setminus (A^o \times B^o)]
= (\overline{A} \times \overline{B}) \cap [(X \times (Y \setminus B^o)) \cup ((X \setminus A^o) \times Y)]
= [\overline{A} \times \text{Fr}(B)] \cup [\text{Fr}(A) \times \overline{B}].
\]

Exercise 70. If \( X_\alpha \) is a nonempty topological space and \( A_\alpha \subset X_\alpha \), for each \( \alpha \in A \), then \( \prod A_\alpha \) is dense in \( \prod X_\alpha \) iff \( A_\alpha \) is dense in \( X_\alpha \), for each \( \alpha \).

**Proof.** It follows from Exercise 68 that

\[
\prod \overline{A}_\alpha = \prod \overline{A}_\alpha;
\]

that is, \( \prod A_\alpha \) is dense in \( \prod X_\alpha \) iff \( A_\alpha \) is dense in \( X_\alpha \), for each \( \alpha \). \( \square \)

8E. Miscellaneous Facts about Product Spaces

Let \( X_\alpha \) be a nonempty topological space for each \( \alpha \in A \), and let \( X = \prod X_\alpha \).

Exercise 71. If \( V \) is a nonempty open set in \( X \), then \( \pi_\alpha(V) = X_\alpha \) for all but finitely many \( \alpha \in A \).

**Proof.** Let \( T_\alpha \) be the topology on \( X_\alpha \) for each \( \alpha \in A \). Let \( V \) be an arbitrary open set in \( X \). Then \( V = \bigcup_{k \in K} B_k \), where for each \( k \in K \) we have \( B_k = \bigwedge_{\alpha \in A} E_{\alpha k} \),
and for each $\alpha \in A$ we have $E_{a_k} \in \mathcal{T}_a$ while

$$A_k := \{\alpha \in A : E_{a_k} \neq X_\alpha\}$$

is finite. Then $\bigcap_{k \in K} A_k$ is finite. If $\alpha_0 \notin \bigcap_{k \in K} A_k$, then there exists $k_0 \in K$ such that $E_{a_0} = X_{\alpha_0}$. Then

$$\pi^{-1}_{\alpha_0}(B_{k_0}) = \pi^{-1}_{\alpha_0} \left( \bigotimes_{a \in A} E_{a_k} \right) = X_{\alpha_0},$$

and so $X_{\alpha_0} = \pi^{-1}_{\alpha_0}(B_{k_0}) \subset \pi^{-1}_{\alpha_0}(V)$ implies that $\pi^{-1}_{\alpha_0}(V) = X_{\alpha_0}$. \hfill \square

**Exercise 72.** If $b_\alpha$ is a fixed point in $X_\alpha$, for each $\alpha \in A$, then $X'_{\alpha_0} = \{x \in X : x_\alpha = b_\alpha$ whenever $\alpha \neq \alpha_0\}$ is homeomorphic to $X_{\alpha_0}$.

**Proof.** Write an element in $X'_{\alpha_0}$ as $(x_{\alpha_0}, b_{-\alpha_0})$. Then consider the mapping $(x_{\alpha_0}, b_{-\alpha_0}) \mapsto x_{\alpha_0}$. \hfill \square

**8G. The Box Topology**

Let $X_\alpha$ be a topological space for each $\alpha \in A$.

**Exercise 73.** In $\prod X_\alpha$, the sets of the form $\prod U_\alpha$, where $U_\alpha$ is open in $X_\alpha$ for each $\alpha \in A$, form a base for a topology.

**Proof.** Let $\mathcal{B} := \{\prod U_\alpha : \alpha \in A, U_\alpha \text{ open in } X_\alpha\}$. Then it is clear that $\prod X_\alpha \in \mathcal{B}$ since $X_\alpha$ is open for each $\alpha \in A$. Now take any $B_1, B_2 \in \mathcal{B}$, with $B_1 = \prod U_\alpha^1$ and $B_2 = \prod U_\alpha^2$. Let

$$p = (p_1, p_2, \ldots) \in B_1 \cap B_2 = \prod \left( U_\alpha^1 \cap U_\alpha^2 \right).$$

Then $p_\alpha \in U_\alpha^1 \cap U_\alpha^2$, and so there exists an open set $B_\alpha \subset X_\alpha$ such that $p_\alpha \in B_\alpha \subset U_\alpha^1 \cap U_\alpha^2$. Hence, $\prod B_\alpha \in \mathcal{B}$ and $p \in B \subset B_1 \cap B_2$. \hfill \square

**8H. Weak Topologies on Subspaces**

Let $X$ have the weak topology induced by a collection of maps $f_\alpha : X \to X_\alpha$, for $\alpha \in A$.

**Exercise 74.** If each $X_\alpha$ has the weak topology given by a collection of maps $g_{\alpha \lambda} : X_\alpha \to Y_{\alpha \lambda}$, for $\lambda \in \Lambda_\alpha$, then $X$ has the weak topology given by the maps $g_{\alpha \lambda} \circ f_\alpha : X \to Y_{\alpha \lambda}$ for $\alpha \in A$ and $\lambda \in \Lambda_\alpha$.

**Proof.** A subbase for the weak topology on $X_\alpha$ induced by $\{g_{\alpha \lambda} : \lambda \in \Lambda_\alpha\}$ is

$$\left\{ g_{\alpha \lambda}^{-1}(U_{\alpha \lambda}) : \lambda \in \Lambda_\alpha, U_{\alpha \lambda} \text{ open in } Y_{\alpha \lambda} \right\}.$$
Then a subbasic open set in \( X \) for the weak topology on \( X \) induced by \( \{ f_\alpha : \alpha \in A \} \) is
\[
\left\{ f_\alpha^{-1}(g_{a\lambda}^{-1}(U_{a\lambda})) : \alpha \in A, \lambda \in A_\alpha, U_{a\lambda} \text{ open in } Y_{a\lambda} \right\}.
\]
Since \( f_\alpha^{-1}(g_{a\lambda}^{-1}(U_{a\lambda})) = (g_{a\lambda} \circ f_\alpha)^{-1}(U_{a\lambda}) \), we get the result. \( \square \)

**Exercise 75.** Any \( B \subset X \) has the weak topology induced by the maps \( f_\alpha | B \).

**Proof.** As a subspace of \( X \), the subbase on \( B \) is
\[
\left\{ B \cap f_\alpha^{-1}(U_\alpha) : \alpha \in A, U_\alpha \text{ open in } X_\alpha \right\}.
\]
On the other hand, \( (f_\alpha | B)^{-1}(U_\alpha) = B \cap f_\alpha^{-1}(U_\alpha) \) for every \( \alpha \in A \) and \( U_\alpha \) open in \( X_\alpha \). Hence, the above set is also the subbase for the weak topology induced by \( \{ f_\alpha | B : \alpha \in A \} \). \( \square \)

### 3.4 Quotient Spaces

#### 9B. Quotients versus Decompositions

**Exercise 76.** The process given in 9.5 for forming the topology on a decomposition space does define a topology.

**Proof.** Let \( (X, \mathcal{T}) \) be a topological space; let \( \mathcal{D} \) be a decomposition of \( X \). Define
\[
\mathcal{F} \subset \mathcal{D} \text{ is open in } \mathcal{D} \iff \bigcup \{ F : F \in \mathcal{F} \} \text{ is open in } X.
\] (3.3)

Let \( \mathcal{G} \) be the collection of open sets defined by (3.3). We show that \( (\mathcal{D}, \mathcal{G}) \) is a topological space.

- Take an arbitrary collection \( \{ \mathcal{F}_i \}_{i \in I} \subset \mathcal{G} \); then \( \bigcup \{ F : F \in \mathcal{F}_i \} \) is open in \( X \) for each \( i \in I \). Hence, \( \bigcup_{i \in I} \mathcal{F}_i \in \mathcal{G} \) since
\[
\bigcup_{F \in \bigcup_{i \in I} \mathcal{F}_i} F = \bigcup_{i \in I} \left( \bigcup_{F \in \mathcal{F}_i} F \right)
\]
is open in \( X \).

- Let \( \mathcal{F}_1, \mathcal{F}_2 \in \mathcal{G} \); then \( \bigcup_{E \in \mathcal{F}_1} E \) and \( \bigcup_{F \in \mathcal{F}_2} F \) are open in \( X \). Therefore, \( \mathcal{F}_1 \cap \mathcal{F}_2 \in \mathcal{G} \) since
\[
\bigcup_{F \in \mathcal{F}_1 \cap \mathcal{F}_2} F = \left( \bigcup_{E \in \mathcal{F}_1} E \right) \cap \left( \bigcup_{F \in \mathcal{F}_2} F \right)
\]
is open in \( X \).

- \( \emptyset \in \mathcal{G} \) since \( \bigcup \emptyset = \emptyset \) is open in \( X \); finally, \( \mathcal{D} \in \mathcal{G} \) since \( \bigcup \mathcal{D} = X \). \( \square \)
EXERCISE 77. The topology on a decomposition space $D$ of $X$ is the quotient topology induced by the natural map $P : X \to D$. (See 9.6.)

PROOF. Let $\mathcal{D}$ be the decomposition topology of $D$, and let $\mathcal{T}_P$ be the quotient topology induced by $P$. Take an open set $F \in \mathcal{D}$; then $\bigcup_{F \in \mathcal{F}} F$ is open in $X$. Hence,

$$P^{-1}(\mathcal{F}) = P^{-1}\left(\bigcup_{F \in \mathcal{F}} F\right) = \bigcup_{F \in \mathcal{F}} P^{-1}(F) = \bigcup_{F \in \mathcal{F}} F$$

is open in $X$, and so $\mathcal{F} \in \mathcal{T}_P$. We thus proved that $\mathcal{D} \subseteq \mathcal{T}_P$.

Next take an arbitrary $F \in \mathcal{T}_P$. By definition, we have $P^{-1}(F) = \bigcup_{F \in \mathcal{F}} F$ is open in $X$. But then $\mathcal{F} \in \mathcal{D}$.

We finally prove Theorem 9.7 (McCleary, 2006, Theorem 4.18): Suppose $f : X \to Y$ is a quotient map. Suppose $\sim$ is the equivalence relation defined on $X$ by $x \sim x'$ if $f(x) = f(x')$. Then the quotient space $X/\sim$ is homeomorphic to $Y$.

By the definition of the equivalence relation, we have the diagram:

$$\begin{align*}
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{P} & & \downarrow{h} \\
X/\sim & & Y
\end{array}
\end{align*}$$

Define $h : X/\sim \to Y$ by letting $h([x]) = f(x)$. It is well-defined. Notice that $h \circ P = f$ since for each $x \in X$ we obtain

$$(h \circ P)(x) = h(P(x)) = h([x]) = f(x).$$

Both $f$ and $P$ are quotient maps so $h$ is continuous by Theorem 9.4. We show that $h$ is injective, subjective and $h^{-1}$ is continuous, which implies that $h$ is a homeomorphism. If $h([x]) = h([x'])$, then $f(x) = f(x')$ and so $x \sim x'$; that is, $[x] = [x']$, and $h$ is injective. If $y \in Y$, then $y = f(x)$ since $f$ is surjective and $h([x]) = f(x) = y$ so $h$ is surjective. To see that $h^{-1}$ is continuous, observe that since $f$ is a quotient map and $P$ is a quotient map, this shows $P = h^{-1} \circ f$ and Theorem 9.4 implies that $h^{-1}$ is continuous. \qed
4

CONVERGENCE

4.1 INADEQUACY OF SEQUENCES

10B. Sequential Convergence and Continuity

→ EXERCISE 78. Find spaces $X$ and $Y$ and a function $F: X \to Y$ which is not continuous, but which has the property that $F(x_n) \to F(x)$ in $Y$ whenever $x_n \to x$ in $X$.

PROOF. Let $X = \mathbb{R}^n$ and $Y = \mathbb{R}$. Define $F: \mathbb{R}^n \to \mathbb{R}$ by letting $F(f) = \sup_{x \in \mathbb{R}} |f(x)|$. Then $F$ is not continuous: Let

$$E = \{ f \in \mathbb{R}^n : f(x) = 0 \text{ or } f(x) = 0 \text{ only finitely often} \},$$

and let $g \in \mathbb{R}^n$ be the function which is 0 everywhere. Then $g \in \overline{E}$. However, $0 \in F[\overline{E}]$ since $F(g) = 0$, and $\overline{F[\overline{E}]} = \{1\}$. □

10C. Topology of First-Countable Spaces

Let $X$ and $Y$ be first-countable spaces.

→ EXERCISE 79. $U \subset X$ is open iff whenever $x_n \to x \in U$, then $(x_n)$ is eventually in $U$.

PROOF. If $U$ is open and $x_n \to x \in U$, then $x$ has a nhood $V$ such that $x \in V \subset U$. By definition of convergence, there is some positive integer $n_0$ such that $n \geq n_0$ implies $x_n \in V \subset U$; hence, $(x_n)$ is eventually in $U$.

Conversely, suppose that whenever $x_n \to x \in U$, then $(x_n)$ is eventually in $U$. If $U$ is not open, then there exists $x \in U$ such that for every nhood $V$ of $x$ we have $V \cap (X \sim U) \neq \emptyset$. Since $X$ is first-countable, we can pick a countable nhood base $\{V_n : n \in \mathbb{N}\}$ at $x$. Replacing $V_n = \bigcap_{i=1}^{n} V_i$ where necessary, we may assume that $V_1 \supset V_2 \supset \cdots$. Now $V_n \cap (X \sim U) \neq \emptyset$ for each $n$, so we can pick $x_n \in V_n \cap (X \sim U)$. The result is a sequence $(x_n)$ contained in $X \sim U$. The result is a sequence $(x_n)$ contained in $X \sim U$.
which converges to \( x \in U \); that is, \( x_n \to x \) but \( (x_n) \) is not eventually in \( U \). A contradiction.

**Exercise 80.** \( F \subset X \) is closed iff whenever \( (x_n) \) is contained in \( F \) and \( x_n \to x \), then \( x \in F \).

**Proof.** Let \( F \) be closed; let \( (x_n) \) be contained in \( F \) and \( x_n \to x \). Then \( x \in \bar{F} = F \).

Conversely, assume that whenever \( (x_n) \) is contained in \( F \) and \( x_n \to x \), then \( x \in F \). It follows from Theorem 10.4 that \( x \in \bar{F} \) with the hypothesis; therefore, \( \bar{F} \subset F \), i.e., \( F = \bar{F} \) and so \( F \) is closed.

**Exercise 81.** \( f : X \to Y \) is continuous iff whenever \( x_n \to x \) in \( X \), then \( f(x_n) \to f(x) \) in \( Y \).

**Proof.** Suppose \( f \) is continuous and \( x_n \to x \). Since \( f \) is continuous at \( x \), for every nhond \( V \) of \( f(x) \) in \( Y \), there exists a nhond \( U \) of \( x \) in \( X \) such that \( f(U) \subset V \). Since \( x_n \to x \), there exists \( n_0 \) such that \( n \geq n_0 \) implies that \( x_n \in U \). Hence, for every nhond \( V \) of \( f(x) \), there exists \( n_0 \) such that \( n \geq n_0 \) implies that \( f(x_n) \in V \); that is, \( f(x_n) \to f(x) \).

Conversely, let the criterion hold. Suppose that \( f \) is not continuous. Then there exists \( x \in X \) and a nhond \( V \) of \( f(x) \), such that for every nhond base \( U_n, n \in \mathbb{N} \), of \( x \), there is \( x_n \in U_n \) with \( f(x_n) \notin V \). By letting \( U_1 \supset U_2 \supset \cdots \), we have \( x_n \to x \) and so \( f(x_n) \to f(x) \); that is, eventually, \( f(x_n) \) is in \( V \). A contradiction.

### 4.2 Nets

**11A. Examples of Net Convergence**

**Exercise 82.** In \( \mathbb{R}^\mathbb{R} \), let

\[
E = \left\{ f \in \mathbb{R}^\mathbb{R} : f(x) = 0 \text{ or } 1, \text{ and } f(x) = 0 \text{ only finitely often} \right\}.
\]

and \( g \) be the function in \( \mathbb{R}^\mathbb{R} \) which is identically 0. Then, in the product topology on \( \mathbb{R}^\mathbb{R} \), \( g \in \bar{E} \). Find a net \( (f_\lambda) \) in \( E \) which converges to \( g \).

**Proof.** Let \( \mathcal{U}_g = \{ U(g, F, \varepsilon) : \varepsilon > 0, F \subset \mathbb{R} \text{ a finite set} \} \) be the nhond base of \( g \).

Order \( \mathcal{U}_g \) as follows:

\[ U(g, F_1, \varepsilon_1) \leq U(g, F_2, \varepsilon_2) \iff U(g, F_2, \varepsilon_2) \subset U(g, F_1, \varepsilon_1) \iff F_1 \subset F_2 \text{ and } \varepsilon_2 \leq \varepsilon_1. \]

Then \( \mathcal{U}_g \) is a directed set. So we have a net \( (f_{F, \varepsilon}) \) converging to \( g \).
11B. Subnets and Cluster Points

**Exercise 83.** Every subnet of an ultranet is an ultranet.

**Proof.** Take an arbitrary subset $E \subset X$. Let $(x_\lambda)$ be an ultranet in $X$, and suppose that $(x_\lambda)$ is residually in $E$, i.e., there exists some $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies that $x_\lambda \in E$. If $(x_{\lambda_\mu})$ is a subnet of $(x_\lambda)$, then there exists some $\mu_0$ such that $\lambda_{\mu_0} \geq \lambda_0$. Then for every $\mu \geq \mu_0$, we have $\lambda_\mu \geq \lambda_0$, and so $\mu \geq \mu_0$ implies that $x_{\lambda_\mu} \in E$; that is, $(x_{\lambda_\mu})$ is residually in $E$. □

**Exercise 84.** Every net has a subnet which is an ultranet.


**Exercise 85.** If an ultranet has $x$ as a cluster point, then it converges to $x$.

**Proof.** Let $(x_\lambda)$ be an ultranet, and $x$ be a cluster point of $(x_\lambda)$. Let $U$ be a nhood of $x$. Then $(x_\lambda)$ lies in $U$ eventually since for any $\lambda_0$ there exists $\lambda \geq \lambda_0$ such that $x_\lambda \in U$. □

11D. Nets Describe Topologies

**Exercise 86.** Nets have the following four properties:

a. if $x_\lambda = x$ for each $\lambda \in \Lambda$, then $x_\lambda \to x$,

b. if $x_\lambda \to x$, then every subnet of $(x_\lambda)$ converges to $x$,

c. if every subnet of $(x_\lambda)$ has a subnet converging to $x$, then $(x_\lambda)$ converges to $x$,

d. (Diagonal principal) if $x_\lambda \to x$ and, for each $\lambda \in \Lambda$, a net $(x_{\lambda_\mu})_{\mu \in M_\lambda}$ converges to $x_\lambda$, then there is a diagonal net converging to $x$; i.e., the net $(x_{\lambda_\mu})_{\lambda \in \Lambda, \mu \in M_\lambda}$, ordered lexicographically by $\Lambda$, then by $M_\lambda$, has a subnet which converges to $x$.

**Proof.** (a) If the net $(x_\lambda)$ is trivial, then for each nhood $U$ of $x$, we have $x_\lambda \in U$ for all $\lambda \in \Lambda$. Hence, $x_\lambda \to x$.

(b) Let $(x_{\varphi(\mu)})_{\mu \in M}$ be a subnet of $(x_\lambda)$. Take any nhood $U$ of $x$. Then there exists $\lambda_0 \in \Lambda$ such that $\lambda \geq \lambda_0$ implies that $x_\lambda \in U$ since $x_\lambda \to x$. Since $\varphi$ is cofinal in $\Lambda$, there exists $\mu_0 \in M$ such that $\varphi(\mu_0) \geq \lambda_0$; since $\varphi$ is increasing, $\mu \geq \mu_0$ implies that $\varphi(\mu) \geq \varphi(\mu_0) \geq \lambda_0$. Hence, there exists $\mu_0 \in M$ such that $\mu \geq \mu_0$ implies that $x_{\varphi(\mu)} \in U$; that is, $x_{\varphi(\mu)} \to x$.

(c) Suppose by way of contradiction that $(x_\lambda)$ does not converge to $x$. Then there exists a nhood $U$ of $x$ such that for any $\lambda \in \Lambda$, there exists some $\varphi(\lambda) \geq \lambda$ with $x_{\varphi(\lambda)} \notin U$. Then $(x_{\varphi(\lambda)})$ is a subnet of $(x_\lambda)$, but which has no converging subnets.
(d) Order \( \{(\lambda, \mu) : \lambda \in \Lambda, \mu \in M_\lambda\} \) as follows:

\[
(\lambda_1, \mu_1) \leq (\lambda_2, \mu_2) \iff \lambda_1 \leq \lambda_2, \text{ or } \lambda_1 = \lambda_2 \text{ and } \mu_1 \leq \mu_2.
\]

Let \( \mathcal{U} \) be the neighborhood system of \( x \) which is ordered by \( U_1 \subseteq U_2 \) iff \( U_2 \subseteq U_1 \) for all \( U_1, U_2 \in \mathcal{U} \). Define

\[
\Gamma = \{(\lambda, U) : \lambda \in \Lambda, U \in \mathcal{U} \text{ such that } x^\lambda \in U \}.
\]

Order \( \Gamma \) as follows: \( (\lambda_1, U_1) \leq (\lambda_2, U_2) \) iff \( \lambda_1 \leq \lambda_2 \) and \( U_2 \subseteq U_1 \). For each \( (\lambda, U) \in \Gamma \) pick \( \mu_\lambda \in M_\lambda \) so that \( x^\lambda \mu_\lambda \in U \) for all \( \mu \geq \mu_\lambda \) (such a \( \mu_\lambda \) exists since \( x^\lambda \mu \to x^\lambda \) and \( x^\lambda \in U \)). Define \( \varphi : (\lambda, U) \mapsto x^\lambda \mu_\lambda \) for all \( (\lambda, U) \in \Gamma \). It is now easy to see that this subnet converges to \( x \).

4.3 Filters

12A. Examples of Filter Convergence

\[\text{Exercise 87. Show that if a filter in a metric space converges, it must converge to a unique point.}\]

\[\text{Proof. Suppose a filter } \mathcal{F} \text{ in a metric space } (X, d) \text{ converges to } x, y \in X. \text{ If } x \neq y, \text{ then there exists } r > 0 \text{ such that } B(x, r) \cap B(y, r) = \emptyset. \text{ But since } \mathcal{F} \to x \text{ and } \mathcal{F} \to y, \text{ we must have } B(x, r) \in \mathcal{F} \text{ and } B(y, r) \in \mathcal{F}. \text{ This contradicts the fact that the intersection of every two elements in a filter is nonempty. Thus, } x = y.\]

12B. Ultrafilters: Uniqueness

\[\text{Exercise 88. If a filter } \mathcal{F} \text{ is contained in a unique ultrafilter } \mathcal{F}', \text{ then } \mathcal{F} = \mathcal{F}'.\]

\[\text{Proof. We first show: Every filter } \mathcal{F} \text{ on a non-empty set } X \text{ is the intersection of the family of ultrafilters which include } \mathcal{F}.\]

Let \( E \) be a set which does not belong to \( \mathcal{F} \). Then for each set \( F \in \mathcal{F} \) we cannot have \( F \subset E \) and hence we must have \( F \cap E^c \neq \emptyset \). So \( \mathcal{F} \cup \{E^c\} \) generates a filter on \( X \), which is included in some ultrafilter \( \mathcal{F}_E \). Since \( E^c \in \mathcal{F}_E \) we must have \( E \notin \mathcal{F}_E \). Thus \( E \) does not belong to the intersection of the set of all ultrafilters which include \( \mathcal{F} \). Hence this intersection is just the filter \( \mathcal{F} \) itself.

Now, if \( \mathcal{F} \) is contained in a unique ultrafilter \( \mathcal{F}' \), we must have \( \mathcal{F} = \mathcal{F}' \). \[\square\]
12D. Nets and Filters: The Translation Process

- **Exercise 89.** A net \((x_\lambda)\) has \(x\) as a cluster point iff the filter generated by \((x_\lambda)\) has \(x\) as a cluster point.

**Proof.** Suppose \(x\) is a cluster point of the net \((x_\lambda)\). Then for every nhood \(U\) of \(x\), we have \(x_\lambda \in U\) i.o. But then \(U\) meets every \(B_{\lambda_0} := \{x_\lambda : \lambda \geq \lambda_0\}\), the filter base of the filter \(\mathcal{F}\) generated by \((x_\lambda)\); that is, \(x\) is a cluster point of \(\mathcal{F}\). The converse implication is obvious. \(\square\)

- **Exercise 90.** A filter \(\mathcal{F}\) has \(x\) as a cluster point iff the net based on \(\mathcal{F}\) has \(x\) as a cluster point.

**Proof.** Suppose \(x\) is a cluster point of \(\mathcal{F}\). If \(U\) is a nhood of \(x\), then \(U\) meets every \(F \in \mathcal{F}\). Then for an arbitrary \((p, F) \in A_\mathcal{F}\), pick \(q \in F \cap U\) so that \((q, F) \in A_\mathcal{F}\), \((q, F) \geq (p, F)\), and \(P(p, F) = p \in U\); that is, \(x\) is a cluster point of the net based on \(\mathcal{F}\).

Conversely, suppose the net based on \(\mathcal{F}\) has \(x\) as a cluster point. Let \(U\) be a nhood of \(x\). Then for every \((p_0, F_0) \in A_\mathcal{F}\), there exists \((p, F) \geq (p_0, F_0)\) such that \(p \in U\). Then \(F_0 \cap U \neq \emptyset\), and so \(x\) is a cluster point of \(\mathcal{F}\). \(\square\)

- **Exercise 91.** If \((x_{\lambda, \mu})\) is a subnet of \((x_\lambda)\), then the filter generated by \((x_{\lambda, \mu})\) is finer than the filter generated by \((x_\lambda)\).

**Proof.** Suppose \((x_{\lambda, \mu})\) is a subnet of \((x_\lambda)\). Let \(\mathcal{F}_{\lambda, \mu}\) is the filter generated by \((x_{\lambda, \mu})\), and \(\mathcal{F}_\lambda\) be the filter generated by \((x_\lambda)\). Then the base generating \(\mathcal{F}_{\lambda, \mu}\) is the sets \(B_{\lambda, \mu} = \{x_{\lambda, \mu} : \mu \geq \mu_0\}\), and the base generating \(\mathcal{F}_\lambda\) is the sets \(B_{\lambda} = \{x_\lambda : \lambda \geq \lambda_0\}\). For each such a \(B_{\lambda, 0}\), there exists \(\mu_0\) such that \(\lambda_{\mu_0} \geq \lambda_0\); that is, \(B_{\lambda, 0} \subset B_{\lambda_0}\). Therefore, \(\mathcal{F}_\lambda \subset \mathcal{F}_{\lambda, \mu}\). \(\square\)

- **Exercise 92.** The net based on an ultrafilter is an ultranet and the filter generated by an ultranet is an ultrafilter.

**Proof.** Suppose \(\mathcal{F}\) is an ultrafilter. Let \(E \subset X\) and we assume that \(E \in \mathcal{F}\). Pick \(p \in E\). If \((q, F) \geq (p, E)\), then \(q \in E\); that is, \(P(p, F) \in E\) ev. Hence, the net based on \(\mathcal{F}\) is an ultranet.

Conversely, suppose \((x_\lambda)\) is an ultranet. Let \(E \subset X\) and we assume that there exists \(\lambda_0\) such that \(x_\lambda \in E\) for all \(\lambda \geq \lambda_0\). Then \(B_{\lambda_0} = \{x_\lambda : \lambda \geq \lambda_0\} \subset E\) and so \(E \in \mathcal{F}\), where \(\mathcal{F}\) is the filter generated by \((x_\lambda)\). Hence, \(\mathcal{F}\) is an ultrafilter. \(\square\)

- **Exercise 93.** The net based on a free ultrafilter is a nontrivial ultranet. Hence, assuming the axiom of choice, there are nontrivial ultranets.

**Proof.** Let \(\mathcal{F}\) be a free ultrafilter, and \((x_\lambda)\) be the net based on \(\mathcal{F}\). It follows from the previous exercise that \((x_\lambda)\) is an ultranet. If \((x_\lambda)\) is trivial, i.e., \(x_\lambda = x\) for some \(x \in X\) and all \(\lambda \in A_\mathcal{F}\), then for all \(F \in \mathcal{F}\), we must have \(F = \{x\}\). But then \(\bigcap \mathcal{F} = \{x\} \neq \emptyset\); that is, \(\mathcal{F}\) is fixed. A contradiction.
Now, for instance, the Frechet filter $\mathcal{F}$ on $\mathbb{R}$ is contained in some free ultrafilter $\mathcal{G}$ by Example (b) when the Axiom of Choice is assumed. Hence, the net based on $\mathcal{G}$ is a nontrivial ultranet. □
5

SEPARATION AND COUNTABILITY

5.1 The Separation Axioms

13B. $T_0$- and $T_1$-Spaces

**Exercise 94.** Any subspace of a $T_0$- or $T_1$-space is, respectively, $T_0$ or $T_1$.

**Proof.** Let $X$ be a $T_0$-space, and $A \subset X$. Let $x$ and $y$ be distinct points in $A$. Then, say, there exists an open nhood $U$ of $x$ such that $y \notin U$. Then $U \cap A$ is relatively open in $A$, contains $x$, and $y \notin A \cap U$. The $T_1$ case can be proved similarly. $\square$

**Exercise 95.** Any nonempty product space is $T_0$ or $T_1$ iff each factor space is, respectively, $T_0$ or $T_1$.

**Proof.** If $X_\alpha$ is a $T_0$-space, for each $\alpha \in A$, and $x \neq y$ in $\prod X_\alpha$, then for some coordinate $\alpha$ we have $x_\alpha \neq y_\alpha$, so there exists an open set $U_\alpha$ containing, say, $x_\alpha$ but not $y_\alpha$. Now $\pi^{-1}_\alpha(U_\alpha)$ is an open set in $\prod X_\alpha$ containing $x$ but not $y$. Thus, $\prod X_\alpha$ is $T_0$.

Conversely, if $\prod X_\alpha$ is a nonempty $T_0$-space, pick a fixed point $b_\alpha \in X_\alpha$, for each $\alpha \in A$. Then the subspace $B_\alpha := \{ x \in \prod X_\alpha : x_\beta = b_\beta \text{ unless } \beta = \alpha \}$ is $T_0$, by **Exercise 94**, and is homeomorphic to $X_\alpha$ under the restriction to $B_\alpha$ of the projection map. Thus $X_\alpha$ is $T_0$, for each $\alpha \in A$. The $T_1$ case is similar. $\square$

13C. The $T_0$-Identification

For any topological space $X$, define $\sim$ by $x \sim y$ iff $[x] = [y]$.

**Exercise 96.** $\sim$ is an equivalence relation on $X$.

**Proof.** Straightforward. $\square$

**Exercise 97.** The resulting quotient space $X/\sim = \bar{X}$ is $T_0$. 

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CHAPTER 5 SEPARATION AND COUNTABILITY

Proof. We first show that $X$ is $T_0$ iff whenever $x \neq y$ then $\overline{\{x\}} \neq \overline{\{y\}}$. If $X$ is $T_0$ and $x \neq y$, then there exists an open nhood $U$ of $x$ such that $y \not\in U$; then $y \notin \{x\}$. Since $y \in \overline{\{y\}}$, we have $\overline{\{x\}} \neq \overline{\{y\}}$. Conversely, suppose that $x \neq y$ implies that $\overline{\{x\}} \neq \overline{\{y\}}$. Take any $x \neq y$ in $X$ and we show that there exists an open nhood of one of the two points such that the other point is not in $U$. If not, then $y \not\in \overline{\{x\}}$; since $\overline{\{x\}}$ is closed, we have $\overline{\{y\}} \subset \overline{\{x\}}$; similarly, $\overline{\{x\}} \subset \overline{\{y\}}$. A contradiction.

Now take any $\overline{\{x\}} \neq \overline{\{y\}}$ in $X/\sim$. Then $\overline{\{x\}} = \overline{\{y\}} \neq \overline{\{x\}} = \{y\}$. Hence, $X/\sim$ is $T_0$.

13D. The Zariski Topology

For a polynomial $P$ in $n$ real variables, let $Z(P) = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : P(x_1, \ldots, x_n) = 0\}$. Let $\mathcal{P}$ be the collection of all such polynomials.

Exercise 98. $\{Z(P) : P \in \mathcal{P}\}$ is a base for the closed sets of a topology (the Zariski topology) on $\mathbb{R}^n$.

Proof. Denote $Z := \{Z(P) : P \in \mathcal{P}\}$. If $Z(P_1)$ and $Z(P_2)$ belong to $Z$, then $Z(P_1) \cup Z(P_2) = Z(P_1 \cdot P_2) \in Z$ since $P_1 \cdot P_2 \in \mathcal{P}$. Further, $\bigcap_{P \in \mathcal{P}} Z(P) = \emptyset$ since there are $P \in \mathcal{P}$ with $Z(P) = \emptyset$ (for instance, $P = 1 + X_1^2 + \cdots + X_n^2$).

It follows from Exercise 48 that $Z$ is a base for the closed sets of the Zariski topology on $\mathbb{R}^n$.

Exercise 99. The Zariski topology on $\mathbb{R}^n$ is $T_1$ but not $T_2$.

Proof. To verify that the Zariski topology is $T_1$, we show that every singleton set in $\mathbb{R}^n$ is closed (by Theorem 13.4). For each $(x_1, \ldots, x_n) \in \mathbb{R}^n$, define a polynomial $P \in \mathcal{P}$ as follows:

$$P = (X_1 - x_1)^2 + \cdots + (X_n - x_n)^2.$$  

Then $Z(P) = \{(x_1, \ldots, x_n)\}$; that is, $\{(x_1, \ldots, x_n)\}$ is closed.

To see the Zariski topology is not $T_2$, consider the $\mathbb{R}$ case. In $\mathbb{R}$, the Zariski topology coincides with the cofinite topology (see Exercise 100). It is well known that the cofinite topology is not Hausdorff (Example 13.5(a)).

Exercise 100. On $\mathbb{R}$, the Zariski topology coincides with the cofinite topology; in $\mathbb{R}^n$, $n > 1$, they are different.

Proof. On $\mathbb{R}$, every $Z(P)$ is finite. So on $\mathbb{R}$ every closed set in the Zariski topology is finite since every closed set is an intersection of some subfamily of $Z$. However, if $n > 1$, then $Z(P)$ can be infinite: for example, consider the polynomial $X_1X_2$ (let $X_1 = 0$, then all $X_2 \in \mathbb{R}$ is a solution).
13H. Open Images of Hausdorff Spaces

EXERCISE 101. Given any set \( X \), there is a Hausdorff space \( Y \) which is the union of a collection \( \{ Y_x : x \in X \} \) of disjoint subsets, each dense in \( Y \).

PROOF. □

5.2 Regularity and Complete Regularity

THEOREM 5.1 (Dugundji 1966). a. Let \( P : X \to Y \) be a closed map. Given any subset \( S \subseteq Y \) and any open \( U \) containing \( P^{-1}(S) \), there exists an open \( V \supseteq S \) such that \( P^{-1}(V) \subseteq U \).

b. Let \( P : X \to Y \) be an open map. Given any subset \( S \subseteq Y \), and any closed \( A \) containing \( P^{-1}S \), there exists a closed \( B \supseteq S \) such that \( P^{-1}(B) \subseteq A \).

PROOF. It is enough to prove (a). Let \( V = Y \setminus P(X \setminus U) \). Then

\[
P^{-1}(S) \subseteq U \implies X \setminus U \subseteq X \setminus P^{-1}(S) = P^{-1}(Y \setminus S) \]
\[
\implies P(X \setminus U) \subseteq P[P^{-1}(Y \setminus S)]
\]
\[
\implies Y \setminus P[P^{-1}(Y \setminus S)] \subseteq V.
\]

Since \( P[P^{-1}(Y \setminus S)] \subseteq Y \setminus S \), we obtain

\[
S = Y \setminus (Y \setminus S) \subseteq Y \setminus P[P^{-1}(Y \setminus S)] \subseteq V;
\]

that is, \( S \subseteq V \). Because \( P \) is closed, \( V \) is open in \( Y \). Observing that

\[
P^{-1}(V) = X \setminus P^{-1}[P(X \setminus U)] \subseteq X \setminus (X \setminus U) = U
\]

completes the proof. □

THEOREM 5.2 (Theorem 14.6). If \( X \) is \( T_3 \) and \( f \) is a continuous, open and closed map of \( X \) onto \( Y \), then \( Y \) is \( T_2 \).

PROOF. By Theorem 13.11, it is sufficient to show that the set

\[
A := \{(x_1, x_2) \in X \times X : f(x_1) = f(x_2)\}
\]

is closed in \( X \times X \). If \( (x_1, x_2) \notin A \), then \( x_1 \notin f^{-1}[f(x_2)] \). Since a \( T_2 \)-space is \( T_1 \), the singleton set \( \{x_2\} \) is closed in \( X \); since \( f \) is closed, \( \{f(x_2)\} \) is closed in \( Y \); since \( f \) is continuous, \( f^{-1}[f(x_2)] \) is closed in \( X \). Because \( X \) is \( T_3 \), there are disjoint open sets \( U \) and \( V \) with

\[
x_1 \in U, \text{ and } f^{-1}[f(x_2)] \subseteq V.
\]
Since \( f \) is closed, it follows from Theorem 5.1 that there exists open set \( W \subset Y \) such that \( \{f(x_2)\} \subset W \), and \( f^{-1}(W) \subset V \); that is,

\[
f^{-1}[f(x_2)] \subset f^{-1}(W) \subset V.
\]

Then \( U \times f^{-1}(W) \) is a nhhood of \((x_1, x_2)\). We finally show that \([U \times f^{-1}(W)] \cap A = \emptyset\). If there exists \((y_1, y_2) \in A\) such that \((y_1, y_2) \in U \times f^{-1}(W)\), then \( y_1 \in f^{-1}[f(y_2)] \subset f^{-1}(W) \); that is, \( y_1 \in U \times f^{-1}(W) \). However, \( U \cap V = \emptyset \) and \( f^{-1}(W) \subset V \) imply that \( U \cap f^{-1}(W) = \emptyset \). A contradiction. \( \square \)

**Definition 5.3.** If \( X \) is a space and \( A \subset X \), then \( X/A \) denotes the quotient space obtained via the equivalence relation whose equivalence classes are \( A \) and the single point sets \( \{x\}, x \in X \sim A \).

**Theorem 5.4.** If \( X \) is \( T_3 \) and \( Y \) is obtained from \( X \) by identifying a single closed set \( A \) in \( X \) with a point, then \( Y \) is \( T_2 \).

**Proof.** Let \( A \) be a closed subset of a \( T_3 \)-space \( X \). Then \( X \sim A \) is an open subset in both \( X \) and \( X/A \) and its two subspace topologies agree. Thus, points in \( X \sim A \subset X/A \) are different from \([A]\) and have disjoint nhoods as \( X \) is Hausdorff. Finally, for \( x \in X \sim A \), there exist disjoint open nhoods \( V(x) \) and \( W(A) \). Their images, \( f(V) \) and \( f(W) \), are disjoint open nhoods of \( x \) and \([A]\) in \( X/A \), because \( V = f^{-1}[f(V)] \) and \( W = f^{-1}[f(W)] \) are disjoint open sets in \( X \). \( \square \)

### 5.3 Normal Spaces

**15B. Completely Normal Spaces**

**Exercise 102.** \( X \) is completely normal iff whenever \( A \) and \( B \) are subsets of \( X \) with \( A \cap \overline{B} = \overline{A} \cap B = \emptyset \), then there are disjoint open sets \( U \supset A \) and \( V \supset B \).

**Proof.** Suppose that whenever \( A \) and \( B \) are subsets of \( X \) with \( A \cap \overline{B} = \overline{A} \cap B = \emptyset \), then there are disjoint open sets \( U \supset A \) and \( V \supset B \). Let \( Y \subset X \), and \( C, D \subset Y \) be disjoint closed subsets of \( Y \). Hence,

\[
\emptyset = \overline{C} \cap \overline{D} = [\overline{C} \cap Y] \cap [\overline{D} \cap Y] = \overline{C} \cap [\overline{D} \cap Y].
\]

Since \( D \subset \overline{D} \), we have \( \overline{C} \cap D = \emptyset \). Similarly, \( C \cap \overline{D} = \emptyset \). Hence there are disjoint open sets \( U' \) and \( V' \) in \( X \) such that \( C \subset U' \) and \( D \subset V' \). Let \( U = U' \cap Y \) and \( V = V' \cap Y \). Then \( U \) and \( V \) are open in \( Y \), \( C \subset U \), and \( D \subset V \); that is, \( Y \) is normal, and so \( X \) is completely normal.

Now suppose that \( X \) is completely normal and consider the subspace \( Y := X \setminus (\overline{A} \cap \overline{B}) \). We first show that \( A, B \subset Y \). If \( A \nsubseteq Y \), then there exists \( x \in A \) with \( x \notin Y \); that is, \( x \in \overline{A} \cap \overline{B} \). But then \( x \in A \cap \overline{B} \). A contradiction. Similarly for \( B \). In the normal space \( Y \), we have
Therefore, there exist disjoint open sets \( U \supset \text{cl}_Y(A) \) and \( V \supset \text{cl}_Y(B) \). Since \( A \subset \text{cl}_Y(A) \) and \( B \subset \text{cl}_Y(B) \), we get the desired result.

**Exercise 103.** Why can’t the method used to show every subspace of a regular space is regular be carried over to give a proof that every subspace of a normal space is normal?

**Proof.** In the first proof, if \( A \subset Y \subset X \) is closed in \( Y \) and \( x \in Y \setminus A \), then there must exists closed set \( B \) in \( X \) such that \( x \notin B \). This property is not applied if \( \{x\} \) is replaced a general closed set \( B \) in \( Y \).

**Exercise 104.** Every metric space is completely normal.

**Proof.** Every subspace of a metric space is a metric space; every metric space is normal Royden and Fitzpatrick (2010, Proposition 11.7).

5.4 Countability Properties

16A. First Countable Spaces

**Exercise 105.** Every subspace of a first-countable space is first countable.

**Proof.** Let \( A \subset X \). If \( x \in A \), then \( V \) is a nhhood of \( x \) in \( A \) iff \( V = U \cap A \), where \( U \) is a nhhood of \( x \in X \) (Theorem 6.3(d)).

**Exercise 106.** A product \( \times \alpha X_\alpha \) of first-countable spaces is first countable iff each \( X_\alpha \) is first countable, and all but countably many of the \( X_\alpha \) are trivial spaces.

**Proof.** If \( \times \alpha X_\alpha \) is first-countable, then each \( X_\alpha \) is first countable since it is homeomorphic to a subspace of \( \times \alpha X_\alpha \). If the number of the family of untrivial sets \( \{X_\alpha\} \) is uncountable, then for \( x \in \times \alpha X_\alpha \) the number of nhhood bases is uncountable.

**Exercise 107.** The continuous image of a first-countable space need not be first countable; but the continuous open image of a first-countable space is first countable.

**Proof.** Let \( X \) be a discrete topological space. Then any function defined on \( X \) is continuous.

Now suppose that \( X \) is first countable, and \( f \) is a continuous open map of \( X \) onto \( Y \). Pick an arbitrary \( y \in Y \). Let \( x \in f^{-1}(y) \), and \( \mathcal{U}_x \) be a countable nhhood base of \( x \). If \( W \) is a nhhood of \( y \), then there is a nhhood \( V \) of \( x \) such that
\( f(V) \subset W \) since \( f \) is continuous. So there exists \( U \in \mathcal{U}_x \) with \( f(U) \subset W \). This proves that \( \{ f(U) : U \in \mathcal{U}_x \} \) is a nhood base of \( y \). Since \( \{ f(U) : U \in \mathcal{U}_x \} \) is \( \Box \)
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COMPACTNESS

6.1 COMPACT SPACES

17B. Compact Subsets

EXERCISE 108. A subset \( E \) of \( X \) is compact iff every cover of \( E \) by open subsets of \( X \) has a finite subcover.

REMARK (Lee 2011, p. 94). To say that a subset of a topological space is compact is to say that it is a compact space when endowed with the subspace topology. In this situation, it is often useful to extend our terminology in the following way. If \( X \) is a topological space and \( A \subset X \), a collection of subsets of \( X \) whose union contains \( A \) is also called a cover of \( A \); if the subsets are open in \( X \) we sometimes call it an open cover of \( A \). We try to make clear in each specific situation which kind of open cover of \( A \) is meant: a collection of open subsets of \( A \) whose union is \( A \), or a collection of open subsets of \( X \) whose union contains \( A \).

PROOF. The “only if” part is trivial. So we focus on the “if” part. Let \( \mathcal{U} \) be an open cover of \( E \), i.e., \( U = \bigcup \{ U : U \in \mathcal{U} \} \). For every \( U \in \mathcal{U} \), there exists an open set \( V_U \) in \( X \) such that \( U = V_U \cap E \). Then \( \{ V_U : U \in \mathcal{U} \} \) is an open cover of \( E \), i.e., \( U \subset \bigcup \{ V_U : U \in \mathcal{U} \} \). Then there exists a finite subcover, say \( V_{U_1}, \ldots, V_{U_n} \) of \( \{ V_U : U \in \mathcal{U} \} \), such that \( E \subset \bigcup_{i=1}^n V_{U_i} \). Hence, \( E = \bigcup_{i=1}^n (V_{U_i} \cap E) \); that is, \( E \) is compact.

EXERCISE 109. The union of a finite collection of compact subsets of \( X \) is compact.

PROOF. Let \( A \) and \( B \) be compact, and \( \mathcal{U} \) be a family of open subsets of \( X \) which covers \( A \cup B \). Then \( \mathcal{U} \) covers \( A \) and there is a finite subcover, say, \( U_1^A, \ldots, U_m^A \) of \( A \); similarly, there is a finite subcover, say, \( U_1^B, \ldots, U_n^B \) of \( B \). But then \( \{ U_1^A, \ldots, U_m^A, U_1^B, \ldots, U_n^B \} \) is an open subcover of \( A \cup B \), so \( A \cup B \) is compact.
References


