3. DOMINANCE

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Exercise 1.

Game 1

Dominant and Dominated Strategy.

Definition 0.1 (Dominated Strategy)***.*** A strategy s_1 is strictly dominated by s_1' for Player 1 if and only if

$$u_1(s_1, s_2) < u_1(s_1', s_2) \quad \text{for all } s_2 \in S_2.$$

Step 1 Let's consider the game from the perspective of Player 1. Take a look at Matrix A-1. Each of Player 1's payoffs in the *top* row is lower than the corresponding payoff in the same column of the bottom row ($2 < 6$ and $1 < 4$). This shows that T is strictly dominated by B.

		Player 2	
		L	R
Player 1	T	2	1
	$\boldsymbol{B}$	**6**	**4**

Matrix A-1

		Player 2	
		$\boldsymbol{L}$	R
Player 1	T	**4**	0
	B	**5**	2

Matrix A-2

Step 2 Similar reasoning can be applied to the strategy choices of Player 2 to show that R is strictly dominated by L for him ($0 < 4$ and $2 < 5$). [See Matrix A-2].

Dominant-Strategy Equilibrium.

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Definition 0.2 (Dominant-Strategy Equilibrium)**.** In a dominant-strategy equilibrium every player in the game chooses their dominant strategy. A game will only have a dominant-strategy equilibrium if all the players have a dominant strategy.

Step 1 To see if the game has a dominant-strategy equilibrium we need to check whether *both* players have a dominant strategy.

Step 2 From the previous part we know that Player 1's dominant strategy is B, and Player 2's dominant strategy is L.

Step 3 In a dominant-strategy equilibrium all the players pick their dominant strategies. Hence, the dominant-strategy equilibrium in this game is (B, L).

Other Equilibria. The first principle of rational behavior is that players should not choose a strategy if there exists an alternative strategy that raises her payoffs against all possible strategies of his opponent.

■ Game 2

Weakly Dominant and Weakly Dominated Strategy.

Definition 0.3 (Weakly Dominant Strategy)**.** In a two-player game the payoffs to a player from choosing a weakly dominant strategy are

(a) at least as high as those from choosing any other strategy in response to any strategy the other player chooses and
(b) higher than those from choosing any other strategy in response to at least one strategy of the other player.

Definition 0.4 (Weakly Dominated Strategy)**.** A strategy s_1 is weakly dominated by s_1' for Player 1 if and only if

$$\begin{cases} u_1\left(s_1, s_2\right) \leqslant u_1\left(s_1', s_2\right), & \text{for all } s_2 \in S_2 \\ u_1\left(s_1, s_2\right) < u_1\left(s_1', s_2\right), & \text{for at least one } s_2 \in S_2. \end{cases}$$

A weakly dominated strategy s_i is never strictly better than s_i', and under some environment which is strictly worse than s_i'.

Step 1 From Matrix B-1 we know that for Player 1, T is weakly dominated by B because

$$\begin{cases} u_1\left(T, L\right) = 1 = 1 = u_1\left(B, L\right), & \text{if } s_2 = L \\ u_1\left(T, R\right) = 1 < 1 = u_1\left(B, R\right), & \text{if } s_2 = R. \end{cases}$$

		Player 2	
		L	R
Player 1	T	1	0
	$\mathbf{B}$	**1**	**1**

Matrix B-1

		Player 2	
		L	$\mathbf{R}$
Player 1	T	1	**1**
	B	0	**1**

Matrix B-2

Step 2 From Matrix B-2 we know that for Player 2, L is weakly dominated by R because

$$\begin{cases} u_2(T,L) = 1 = 1 = u_1(T,R), & \text{if } s_1 = T \\ u_1(B,L) = 1 < 1 = u_1(B,R), & \text{if } s_1 = B. \end{cases}$$

> *Remark.* Unlike a strictly dominated strategy, a strategy that is only weakly dominated cannot be ruled out based solely on principles of rationality.

Nash Equilibrium. Name the Nash equilibria in this game:

________________________________.

		Player 2	
		L	R
Player 1	T	$\boxed{1}$, $\boxed{1}$	0, $\boxed{1}$
	B	$\boxed{1}$, 0	$\boxed{1}$, $\boxed{1}$

Matrix: B-1

Strict Equilibrium.

> ***Definition 0.5*** (Strict Equilibrium)***.*** A pair of players' strategies that are the *only* best reply to each other.

Step 1 (T, L) is not a strict equilibrium because, e.g., given L, T is not the only best reply of Player 1: B is a best reply against T for Player 1, too. Similarly, given T, L is not Player 2's unique best reply.

Step 2 However, (B, R) is a strict equilibrium:

- Given R, Player 1's only best reply is B since $u_1(B, R) = 1 > 0 = u_1(T, R)$;
- Given B, Player 2's only best reply is R since $u_2(B, R) = 1 > 0 = u_2(B, L)$.

Admissible Strategy.

> *Remark.* In an admissible equilibrium, no player uses a *weakly* dominated strategy. A *strictly* dominated strategy can be deleted safely.

Step 1 As we have shown in Game 1, T is (weakly) dominated by B, and L is weakly dominated by R; hence, (T, L) is *not* a admissible equilibrium.

Step 2 There is no strategy (weakly) dominates B for Player 1, and there is no strategy (weakly) dominates R; hence, (B, R) is an admissible equilibrium.

■ Game 3

Dominant and Dominated Strategy. In this game [see Matrix C-1], Player 1's strategy T is strictly dominated by ____________, and M is strictly dominated by ____________.

Does Player 2 have strictly dominated strategy? ____________

		Player 2		
		L	C	R
Player 1	T	0 1	9 0	2 3
	M	5 9	7 3	1 7
	B	7 5	10 10	3 5

Matrix C-1

Nash Equilibrium. We can delete Player 1's strictly dominated strategies, and this produces the Matrix C-2.

■ Game 4

		Player 2		
		L	C	R
Player 1	B	7 5	10 10	3 5

Matrix C-2

Dominant and Dominated Strategy. There is no (weakly or strictly) dominated strategy. For example,

$$TT \text{ is undominated by } \begin{cases} T, & \text{if } s_2 = \underline{\hspace{2cm}} \\ M, & \text{if } s_2 = \underline{\hspace{2cm}} \\ B, & \text{if } s_2 = \underline{\hspace{2cm}} \end{cases},$$

$$C \text{ is undominated by } \begin{cases} L, & \text{if } s_1 = \underline{\hspace{2cm}} \\ R, & \text{if } s_1 = \underline{\hspace{2cm}} \end{cases}$$

		Player 2		
		L	C	R
Player 1	TT	2 3	8 2	10 6
	T	3 0	4 5	6 4
	M	5 4	6 1	2 5
	B	4 5	2 3	5 2

Nash Equilibrium.

$$\text{Player 1's best reply is } \begin{cases} \underline{\hspace{2cm}}, & \text{if } s_2 = L \\ \underline{\hspace{2cm}}, & \text{if } s_2 = C \\ \underline{\hspace{2cm}}, & \text{if } s_2 = R \end{cases},$$

$$\text{Player 2's best reply is } \begin{cases} \underline{\hspace{2cm}}, & \text{if } s_1 = TT \\ \underline{\hspace{2cm}}, & \text{if } s_1 = T \\ \underline{\hspace{2cm}}, & \text{if } s_1 = M \\ \underline{\hspace{2cm}}, & \text{if } s_1 = B \end{cases}$$

Therefore, the NE in this game is ________________.

Strict Equilibrium and Admissible Equilibrium. Given $s_2 = R$, Player 1's unique best reply is TT; given $s_1 = TT$, Player 2's unique best reply is R; thus, (TT, R) is a strict equilibrium. (TT, R) is also an admissible strategy.

■ Game 5

Player 2

		L	C	R
Player 1	T	4 3	2 7	0 4
	B	5 5	5 -1	-4 -2

Matrix E-1

Dominant and Dominated Strategy.

- Player 1 has no dominant strategy:

$$\begin{cases} T \text{ is Player 1's best reply,} & \text{if } s_2 = \underline{\hspace{2cm}} \\ B \text{ is Player 1's best reply,} & \text{if } s_2 = \underline{\hspace{2cm}}. \end{cases}$$

- For Player 2, R is (strictly) dominated by C:

$$\begin{cases} u_2\,(T, R) = \underline{\hspace{2cm}} < \underline{\hspace{2cm}} = u_2\,(T, C)\,, & \text{if } s_1 = T \\ u_2\,(B, R) = \underline{\hspace{2cm}} < \underline{\hspace{2cm}} = u_2\,(B, C)\,, & \text{if } s_1 = B. \end{cases}$$

Nash Equilibrium. If we delete all of the strictly dominated strategies, then Matrix E-1 becomes as Matrix E-2 [please complete this matrix and find all of the Nash equilibria].

Player 2

Player 1		

Matrix E-2

Exercise 2.

> *Remark.* In a 3-players game, we usually let Player 1 choose between rows, Player 2 choose between columns, and Player 3 choose between matrixes. In each cell, the upper left entry is Player 1's payoff, the middle entry is Player 2's payoff, and the lower right entry is Player 3's payoff.

Dominant and Dominated Strategy.

- Player 1 has no dominated strategy:
  - If $s_2 = R$ and $s_3 = A$, then Player 1's best reply is T [See the left matrix in Figure 0.1];
  - However, if $s_2 = R$ and $s_3 = B$, then Player 1's best reply is B [See the right matrix in Figure 0.1].

	L	R
$\boldsymbol{T}$		**1** 0 1
B		**0** 0 1

A

	L	R
T		0 0 0
$\boldsymbol{B}$		1 0 0

B

Figure 0.1: Player 1 has no dominant strategy

- For Player 2, R is strictly dominated by L. See Figure 0.2.

	$\boldsymbol{L}$	R
T	1 **1** 1	1 0 1
B	1 **1** 1	0 0 1

A

	$\boldsymbol{L}$	R
T	1 **1** 0	0 0 0
B	0 **1** 0	1 0 0

B

Figure 0.2: Player 2 has a strictly dominant strategy

- For Player 3, B is strictly dominated by A. For example, if $s_1 = T$ and $s_2 = R$, then Player 3's best reply is A [See Figure 0.3]. You'd better to check Player 3's other payoff pairs.

	L	**R**
T		1 0 1
B		

A

	L	**R**
T		1 0 0
B		

B

Figure 0.3: Player 3 has a strictly dominant strategy

Nash Equilibrium. We can delete Player 2's strategy R and Player 3's strategy A safely. In this case, the game becomes as in Figure 0.4. In this reduced game, Player 1 is indifferent between T and B, so he can take any mixed strategy $x \cdot T + (1 - x) \cdot B, \quad x \in [0, 1]$ Therefore, the set of Nash equilibria in this game is

$$NE = \left\{ \underline{\hspace{10cm}} \right\}.$$

	L
T	1 1 1
B	1 1 1

A

Figure 0.4: The reduced game