4. Mixed strategy and dominance

Jianfei Shen

School of Economics, The University of New South Wales, Sydney 2052, Australia

Exercise 1.

Nash Equilibrium

Player 2

Player 1		E $[z]$	F $[1-z]$
	A $[x]$	5, 4	5, 4
	B $[y]$	8, 3	1, 9
	C $[1-x-y]$	3, 6	7, 2

Matrix A-1

As you can see from Matrix A-1, there is no pure Nash equilibrium. To find the mixed Nash equilibria, we suppose Player 1 plays

$$\begin{cases} A & \text{with probability } x, \\ B & \text{with probability } y, \\ C & \text{with probability } 1-x-y, \end{cases}$$

that is, we suppose that Player 1's mixed strategy is

____

Similarly, we can assume that Player 2 takes the following mixed strategy [according to Matrix A-1]:

____

Jianfei Shen: jianfei.shen@unsw.edu.au

Step 1 If Player 2 plays a mixed strategy, then he must be indifferent between E or F; hence Player 2 must get the same payoffs from playing E and F. His payoff from playing E is

$$\underline{\hspace{6cm}};$$

his payoff from playing F is

$$\underline{\hspace{6cm}}$$

These two payoffs are equal, so we know $x = 1 - 5y/2$.

Step 2 According to the above equation, there are three cases:

- $x = 0$. In this case, Player 1 does not play A.
- $x \in (0, 1)$, which means that $y \in \big(0, 2/5\big)$. In this case, Player 1 plays A and B randomly [he may or may not play C; we have no idea at this stage].
- $x = 1$, which means that $y = 0$. In this case, Player 1 plays A deterministically.

Step 3 If $x = 1$, then A is Player 1's best response against to Player 2's mixed strategy $(z, 1 - z)$; thus,

$$\begin{cases} A \text{ is better than } B \Rightarrow \underline{\hspace{6cm}} \\ A \text{ is better than } C \Rightarrow \underline{\hspace{6cm}} \end{cases}$$

i.e., $1/2 \leqslant z \leqslant 4/7$.

Step 4 Check by yourself that

$$\left\{\big(A,\ z \cdot E + (1 - z) \cdot F\big) \,\middle|\, \frac{1}{2} \leqslant z \leqslant \frac{4}{7}\right\}$$

is a set of Nash equilibria.

Step 5 If $x = 0$, then $y = 2/5$ and $1 - x - y = 1 - 0 - 2/5 = 3/5$; that is, Player 1 plays B with probability $2/5$, and plays C with probability $3/5$. In this case, Player 1 must get the same payoffs from playing B and C, i.e.,

$$8 \cdot z + 1 \cdot (1 - z) = 3 \cdot z + 7 \cdot (1 - z),$$

which solves for $z = 6/11$.

While getting

$$8 \times \frac{6}{11} + 1 \times \left(1 - \frac{6}{11}\right) = \frac{53}{11}$$

from playing B or C randomly, Player 1 can guarantee 5 if he plays A deterministically. Since $5 > 53/11$, we know that $x = 0$ is impossible.

Step 6 We now check the last possibility: $0 < x < 1$. In this case, Player 1 gets the same payoffs from A and B [notice that in this case both x and y are greater than 0]. Hence,

$$5 \cdot z + 5 \cdot (1 - z) = 8 \cdot z + 1 \cdot (1 - z),$$

which solves for $z = 4/7$.

If Player 2 adopts the mixed strategy

$$\left(\frac{4}{7}E + \frac{3}{7}F\right),$$

Player 1's expected payoff from C is

$$3 \times \frac{4}{7} + 7 \times \frac{3}{7} = \frac{33}{7} < 5,$$

which means that Player 1 will not use C, or equivalently,

$$1 - x - y = 0.$$

Therefore, $x + y = 1$. The proceeding equation with $x = 1 - \frac{5}{2}y$ yield

$$x = 1 \quad \text{and} \quad y = 0.$$

A contradiction [remember that in this step we assume $0 < x < 1$ and $0 < y < 1$].

Step 7 From Step 1—Step 6, we conclude that the set of Nash equilibria in this game is

---

Undominated Strategy

All Nash equilibria are undominated strategies.

Exercise 2. See the solution.

Exercise 3.

The Game in Matrix C-1

Step 1 For Player 1, B is strictly dominated. We delete B and this produce the following Matrix C-3.

Player 1 \ Player 2	L	R
T	2, 2	2, 2
C	2, 3	$\frac{1}{2}$, 0
B	0, 0	1, 1

Matrix C-1

Player 1 \ Player 2	L	R
T	2, 2	2, 2
C	3, 3	$\frac{1}{2}$, 0
B	0, 0	1, 1

Matrix C-2

Player 1 \ Player 2	L $[y]$	R $[1-y]$
T $[x]$	2, 2	2, 2
C $[1-x]$	2, 3	$\frac{1}{2}$, 0

Matrix C-3

Step 2 In the game Matrix C-3, we can find the pure Nash equilibria easily. Now let Player 1's mixed strategy be $x \cdot T + (1-x) \cdot C$, and Player 2's mixed strategy be $y \cdot L + (1-y) \cdot R$.

Step 3 As usually, Player 1 is indifferent between T or C, so

$$2 = 2y + \frac{1}{2}(1-y) \Rightarrow y = 1,$$

which means that if Player 2 plays L with certainty, then Player 1 is indifferent between T or C [you can see this from the Matrix C-3]. Hence, given Player 2's strategy L, Player 1's best response is

$$x \cdot T + (1-x)C, \quad x \in [0, 1] .$$

Step 4 We also need to check that given Player 1's mixed strategy $x \cdot T + (1-x)C$, $x \in [0, 1]$, Player 2's best response is L. This is true because

$$2x + 3(1-x) = 3 - x \geqslant 2x.$$

Step 5 With the same logic, you can find the following set of Nash equilibria:

$$\left\{ \left(T,\ y \cdot L + (1-y)R\right) \middle| y \in [0, 1] \right\} .$$

The Game in Matrix C-2

Step 1 Delete the strictly dominated strategy B, and we get Matrix C-4.

Player 2

Player 1		L $[y]$	R $[1-y]$
	T $[x]$	2, 2	2, 2
	C $[1-x]$	3, 3	$\frac{1}{2}$, 0

Matrix C-4

Step 2 The pure Nash equilibria are (T, R) and (C, L).

Step 3 Consider the mixed strategies [sometimes I do not write the mixed strategies explicitly for simplicity. You can find them from the corresponding matrix]. There is a simple way to find the mixed Nash equilibria. For Player 2, he would like to play a mixed strategy $(y \cdot L + (1-y) \cdot R)$ if and only if $x = 1$.

Step 4 For Player 1, he would like to play T with certainty if and only if

$$2 \geqslant 3y + \frac{1}{2}\left(1-y\right) \Rightarrow y \leqslant \frac{3}{5}.$$

Step 5 We need not to consider any of Player 1's mixed strategy since if $x < 1$, Player 1's best response is L, and if Player 2's strategy is L, Player 1's best response is C.

Step 6 We thus know that the set of Nash equilibria is

$$\left\{\left(T,\ y \cdot L + \left(1-y\right) \cdot R\right) \middle| y \in \left[0, \frac{3}{5}\right]\right\} \text{ and } \{(C, L)\}.$$

Exercise 4.

Step 1 If Player 1 uses the following mixed strategy

$$x \cdot A + y \cdot B + \left(1-x-y\right) \cdot C,$$

where $x > 0$, and $1-x-y > 0$. We first suppose that $x \leqslant 1-x-y$, then we can rewrite his mixed strategy as [Consider the case of $x \geqslant 1-x-y$ by yourself]

$$2x \cdot \left(\frac{1}{2}A + \frac{1}{2}C\right) + y \cdot B + \left(1-2x-y\right) \cdot C.$$

For simplicity, we denote the above mixed strategy as CM.

		Player 2		
		D	E	F
Player 1	A $[x]$	1, 2	3, 0	0, 3
	B $[y]$	1, 1	2, 2	2, 0
	C $[1-x-y]$	1, 2	0, 3	3, 0

Step 2 Because $\frac{1}{2}A + \frac{1}{2}C$ is dominated by B, we know immediately that CM is dominated by

$$\big(2x+y\big) \cdot B + \big(1-2x-y\big) \cdot C.$$

Step 3 Step 1 and Step 2 thus show that any of Player 1's mixed strategies putting positive probabilities on A ad C are dominated.

Step 4 There are two cases under which Player 1's mixed strategy

$$x \cdot A + y \cdot B + \big(1-x-y\big) \cdot C$$

cannot be rewritten as CM: $x = 0$ or $1-x-y = 0$. We are not interested in the former case [$x = 0$] because it is not part of a Nash equilibrium [remember that $x \geqslant \frac{1}{3}$ in any Nash equilibria].

Step 5 Thus we need only consider the case of $1-x-y = 0$, which means that Player 1's mixed strategies are

$$x \cdot A + (1-x) \cdot B, \quad x \geqslant \frac{1}{3}.$$

Step 6 Note that $1-x-y = 0$ implies that

$$y = 1-x.$$

Combining this equation with the following equation

$$y \leqslant 2-3x,$$

we have

$$1-x \leqslant 2-3x,$$

which solves for

$$x \leqslant \frac{1}{2}.$$

Step 7 We have proved through Step 1—Step 6 that for Player 1, if his strategy is undominated and is part of Nash equilibrium, then his strategy must be

$$x \cdot A + (1-x) \cdot B, \quad \frac{1}{3} \leqslant x \leqslant \frac{1}{2}.$$

Our final step is to show that the above strategies are really undominated.

Step 8 Suppose that there exist a strategy

$$a \cdot A + b \cdot B + (1-a-b) \cdot C$$

which dominates $x \cdot A + (1-x) \cdot B$, where a, b are probabilities satisfying $a+b \leqslant 1$. We need to solve the following system of inequalities:

$$\begin{cases} 3a + 2b \geqslant 2 + x, & \text{if } s_2 = E \\ 3 - 3a - b \geqslant 2 - 2x, & \text{if } s_2 = F \\ \text{At least one of the above two inequalities holds strictly,} & \\ a + b \leqslant 1, & \\ 0 \leqslant a \leqslant 1, & \\ 0 \leqslant b \leqslant 1. & \end{cases}$$

There is no solution for the above system of inequalities [see Figure 0.1]. This proves that

$$x \cdot A + (1-x) \cdot B, \quad \frac{1}{3} \leqslant x \leqslant \frac{1}{2}.$$

are undominated.

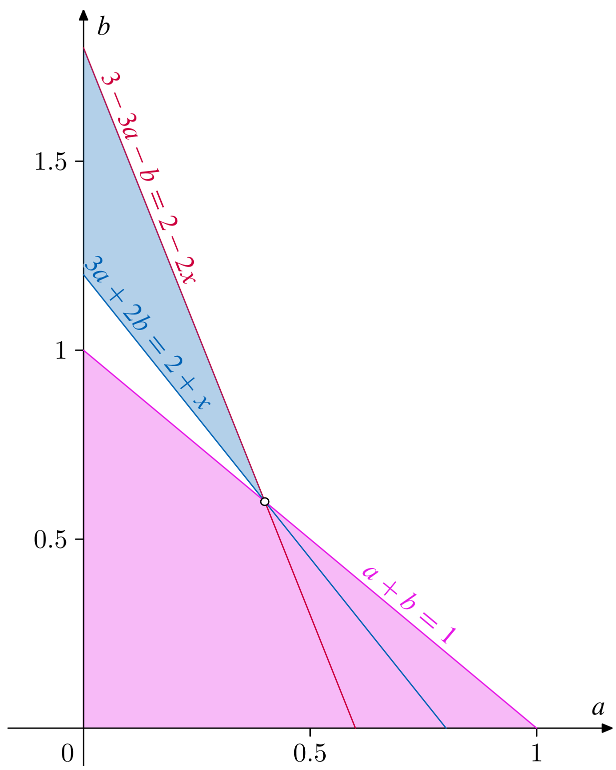

Figure 0.1: There is no solution ($x = 0.4$)