9. Repeated games

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Exercise 1 (Prisoners' Dilemma)**.**

Player 2

		L	R
Player 1	T	3, 3	0, 5
	B	5, 0	1, 1

Prisoners' Dilemma

Step 1 In an isolated interaction, (B, R) is the strictly dominant strategy equilibrium.

Step 2 To find the minimum discount factor δ, we use the *grim trigger strategies* [notice that $\min\max u_i\,(s_1, s_2) = 1$, $\forall\ i = 1, 2$]:

- Cooperate in the first period and to continue to do so in every subsequent period as long as both players have previously cooperate,
- while playing B and R in all other circumstances.

Step 3 Given the grim trigger strategies, if Player 1 cooperates, his flow of payoffs is

Time	1	2	3	$\cdots$
Payoff	3	3	3	$\cdots$

but if he deviates at time 1, his flow of payoffs is

[By the *one-shot deviation principle*, we need only to consider such a deviation.]

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Time	1	2	3	4	⋯
Payoff					⋯

Step 4 Hence, the condition is

$$3 + 3 \cdot \delta + 3 \cdot \delta^2 + \cdots \geqslant \rule{5cm}{0.4pt}$$

that is,

$$\frac{3}{1-\delta} \geqslant 5 + \frac{\delta}{1-\delta},$$

which solves for

$$\delta \geqslant \frac{1}{2}.$$

Theorem 0.1. *If* $\delta \in (0, 1)$*, then*

$$a + a \cdot \delta + a \cdot \delta^2 + a \cdot \delta^3 + \cdots = \frac{a}{1-\delta}.$$

Proof. Let

$$X = a + a \cdot \delta + a \cdot \delta^2 + a \cdot \delta^3 + \cdots,$$

then

$$X \cdot \delta = a \cdot \delta + a \cdot \delta^2 + a \cdot \delta^3 + \cdots.$$

Thus,

$$\begin{aligned} X - X \cdot \delta &= (1-\delta) \cdot X \\ &= a \end{aligned}$$

as $a \cdot \delta^\infty \to 0$ since $\delta < 1$. The above equation solves for

$$X = \frac{a}{1-\delta}.$$

□

Exercise 2 (Repeated Twice)**.**

Step 1 There are two pure strategy Nash equilibria of the one-shot game:

$$(B, R) \quad \text{and} \quad (C, D).$$

Step 2 Any SPNE involves playing either of these two pure strategy Nash equilibria in the second period.

		Player 2		
		L	R	D
	T	3, 3	0, 5	−20, −10
Player 1	B	5, 0	1, 1	−10, −10
	C	−10, −20	−10, −10	2, 2

Step 3 With Step 1 and 2, you now should know whether (T, L) is possible in the first stage. [*Hint*: Consider Player 2's strategy: play L in the first stage; play D in the second stage if Player 1 played T in the first stage, otherwise play R. Does Player 1 have incentive to cooperate at the first stage?]

Exercise 3 (Repeated Three Times)***.*** *Hint*: Use the Nash equilibrium at the second and third stage.